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Difference and finite-element methods for the dynamical problem of thermodiffusion in an elastic solid

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IN THE PAPER for the dynamical problem of thermodiffusion in an elastic solid with the homogeneous Dirichlet boundary the difference and Galerkin method, particularity the economic scheme and alternating direction finite-element methods, which are very efficient in numerical practice are considered. The errors estimates of these methods are given. Moreover, the well posed of the considered problem in a Sobolev space for the certain regions is proved.

Dla dynamicznego problemu termodyfuzji w ciele sprężystym z jednorodnymi warunkami brzegowymi Dirichleta rozpatruje się metodę różnic i metodę Galerkina, w szczególności schematy ekonomiczne i metodę elementów skończonych typu naprzemiennych kierunków, które są bardzo wygodne przy ich realizacji na maszynach cyfrowych. Podane zostały oszacowania błędów zbieżności tych metod. Ponadto wykazana jest poprawność rozważanego problemu dla pewnych obszarów w przestrzeniach Soboleva.

Для динамической задачи термодиффузии в упругом теле, с однородными краевыми условиями Дирихле, рассматриваются методы сеток и Галеркина, в частности экономные схемы и схемы метода конечных элементов типа переменных направлений, которые очень пригодн ыпри их реализации на вычислительных цифровых машинах. Даются оценки погрешностей сходимости этих методов. Кроме этого показана корректность рассматриваемой задачи для некоторых областей в пространствах Соболева.

LET US consider the dynamical problem (1.1)–(1.7) of thermodiffusion in an elastic solid with the homogeneous Dirichlet boundary and initial conditions in the region $\Omega\times(0, T)$, where $\Omega\subset R^3$. This problem has been formulated by J. S. Podstrigač (see W. NOWACKI [1] and the references there). G. Fichera has proved the existence and uniqueness of the solution to this problem using the Laplace transform when the boundary $\delta\Omega$ of Ω is C^∞-smooth (see [2]).

In this paper we prove that this problem is well posed in a Sobolev space for certain regions with a piece-wise smooth boundary (see theorem 2.1 and 2.2). Next we deal with the difference and finite-element methods applied to this problem. We consider the implicit difference methods which approximate our problem and are convergent with an error $0(\tau^2+h^2)$ if $\Gamma_h\subset\delta\Omega$ and $0(\tau^2+h^{1/2})$ otherwise; here τ, h_i are the steps of the time and space grid, $h=\max\{h_1, h_2\ h_3\}$ and Γ_h is the boundary of the set grid (see theorem 3.1 and 3.2).

If Ω is a rectangular parallelepiped we consider an economical scheme (see [3, 4]) which is unconditionally stable and convergent with an error $0(\tau^2+h^2)$ (see theorem 4.1 and 4.2).

The second part of the paper deals with the discrete Galerkin methods with "viscosity". An error estimate in this case (see theorem 5.1) is given.

If Ω is a rectangular parallelpiped, we construct the alternating direction Galerkin methods (finite-element methods, see [5]) which are very efficient in numerical practice. Convergence with an error $0(\tau^2+h)$ (see theorem 6.1) is proved.

1. The differential problem

The following system of partial differential equations is considered

$$(1.1)\qquad G\sum_{j=1}^{3} D_j^2 u_i+(\lambda+G)\sum_{j=1}^{3} D_i D_j u_j-p_0 D_i\theta-p_\mu D_i\mu-\varrho D_0^2 u_i = F_i(x,t),\quad i=1,2,3,$$

$$(1.2)\qquad K\sum_{j=1}^{3} D_j^2\theta-cD_0\theta-dD_0\mu-\sum_{j=1}^{3} D_0 D_j u_j=f(x,t),$$

$$(1.3)\qquad D\sum_{j=1}^{3} D_j^2\mu-bD_0\mu-dD_0\theta-p_\mu\sum_{j=1}^{3} D_0 D_j u_j=g(x,t),$$

for $(x,t)\in Q_T=\Omega\times(0,T)$, where Ω is a bounded subset of R^3 with a boundary $\delta\Omega$; G, λ, ϱ, p_θ, p_μ, K, c, d, D and b are given constants; F_i, f and g, $i=1,2,3$, are given real functions:

$$x=(x_1,x_2,x_3),\qquad D_0=\partial/\partial t,\qquad D_i=\partial/\partial x_i.$$

We associate with the system (1.1)–(1.3) the following boundary conditions:

$$(1.4)\qquad u_i(x,t)=0,\quad i=1,2,3,$$

$$(1.5)\qquad \theta(x,t)=0,\quad \mu(x,t)=0,$$

for $x\in\delta\Omega$, $t\in[0,T]$ and the initial conditions

$$(1.6)\qquad u_i(x,0)=u_{i,0}(x),\quad D_0u_i(x,0)=u_i'(x),\quad i=1,2,3,$$

$$(1.7)\qquad \theta(x,0)=\theta_0(x),\quad \mu(x,0)=\mu_0(x).$$

We shall say that Ω satisfies the condition S if there exists a function $y:\Omega\to R^3$ such that $y\in C^2(\bar{\Omega})$ and $S=y(\Omega)$ is a ball or a parallelepiped (see [6], p. 130).

2. A priori estimate

Denote by $(.,.)$ and $\|.\|$ the inner product and the norm in the space $L^2(\Omega)$. Let $H^1(\Omega)$ be the known Sobolev space and $H_0^1(\Omega)$ be a subspace $H^1(\Omega)$ of functions which vanish on the boundary $\delta\Omega$. Recall that $Q_T=\Omega\times(0,T)$. Denote by $H^{k,j}(Q_T)$ a Sobolev space of functions from $L^2(Q_T)$ which have generalized derivatives up to the order k with respect to x_i, $i=1,2,3$, and up to the order j with respect to t. By $H^k(Q_T)$ we mean the space $H^{k,k}(Q_T)$. At last let $\nabla u=(D_1u,D_2u,D_3u)$ and $\|\nabla u\|^2=\sum_{j=1}^{3}\|D_iu\|^2$.

Assume that

$$(2.1)\qquad \begin{gathered} F_i, f, g\in L^2(Q_T),\quad u_{i0}\in H_0^1(\Omega),\\ u_i',\theta_0,\mu_0\in L^2(\Omega)\quad \text{for}\quad i=1,2,3.\end{gathered}$$

THOEREM 2.1. Assume that

$$(2.2)\qquad G>0,\quad G+\lambda>0,\quad \varrho>0,\quad c>0,\quad b>0,\quad d^2<cb.$$

If the components of the solution of the problem (1.1)–(1.7) belong to the following spaces: $u_i \in H^2(Q_T)$, $i = 1, 2, 3$, θ and $\mu \in H^{2,1}(Q_T)$, then

$$(2.3) \quad \sum_{i=1}^{3} \{||D_0 u_i(t)||^2 + ||\nabla u_i(t)||^2\} + \int_0^t [||\nabla\theta(\xi)||^2 + ||\nabla\mu(\xi)||^2] d\xi + ||\theta(t)||^2 + ||\mu(t)||^2$$
$$\leqslant M\Big\{\sum_{i=1}^{3} \int_0^t ||F_i(\xi)||^2 d\xi + \int_0^t [||f(\xi)||^2 + ||g(\xi)||^2] d\xi + \sum_{i=1}^{3} \{||D_0 u_i(0)||^2 + ||\nabla u_i(0)||^2\}$$
$$+ ||\theta(0)||^2 + ||\mu(0)||^2\Big\},$$

where $t \in [0, T]$ and M is a positive constant independent of the solution and the data functions.

Proof. Let us form the inner products of the equations of the system (1.1)–(1.3) with $-D_0 u_i$ for $i = 1, 2, 3$, and with $-\theta$, $-\mu$, respectively. Next, let us summ up for $i = 1, \ldots, 5$ the expressions obtained and next integrate them with respect to ξ, $\xi \in (0, t)$. Applying the Green formulae we get

$$(2.4) \quad \sum_{i=1}^{3} \{\varrho||D_0 u_i(t)||^2 + I(u_i(t))\} + c||\theta(t)||^2 + b||\mu(t)||^2 + \int_0^t [K||\nabla\theta(\xi)||^2 + D||\nabla\mu(\xi)||^2] d\xi$$
$$+ 2d(\theta(t), \mu(t)) \leqslant \sum_{i=1}^{3} [\varrho||D_0 u_i(0)||^2 + I(u_i(0))] + (c + |d|)||\theta(0)||^2 + (b + |d|)||\mu(0)||^2$$
$$+ 0.5\left\{\sum_{i=1}^{3} \int_0^t \left[\frac{1}{\varepsilon_i} ||F_i(\xi)||^2 + \varepsilon_i ||D_0 u_i(\xi)||^2\right] d\xi + \int_0^t \left[\frac{1}{\varepsilon_4} ||g(\xi)||^2 \frac{1}{\varepsilon_5} ||f(\xi)||^2 + \varepsilon_4 ||\theta(\xi)||^2\right.\right.$$
$$\left.\left. + \varepsilon_5 ||\mu(\xi)||^2\right] d\xi\right\},$$

where

$$I(u_i(t)) = \sum_{j=1}^{3} \sum_{i=1}^{3} \{G(D_j u_i(t), D_j u_i(t)) + (\lambda + G)(D_j u_j(t), D_i u_i(t))\}.$$

It is easy to verify that

$$G \sum_{i=1}^{3} ||\nabla u_i(t)||^2 \leqslant \sum_{i=1}^{3} I(u_i(t)) \leqslant \max\{G, \lambda + G\} \sum_{i=1}^{3} ||\nabla u_i(t)||^2.$$

Using these estimates, the assumptions (2.2) and the Gronwall's lemma we get the inequality (2.3). This completes the proof.

Corollary 2.1. From (2.3) it follows the uniqueness of the solution of the problem (1.1)–(1.7) in the spaces $H^2(Q_T)$ for u_i, $i = 1, 2, 3$ and $H^{2,1}(Q_T)$ for θ and μ.

The obtained estimate (2.3) can be used to prove that our problem is well-posed in the so-called energetic class (see [6], p. 227). We only sketch a proof since it is similar to Ladyzenskaja's idea. We shall use a functional method defined in [6].

Rewrite the problem (1.1)–(1.7) in a form of an operator equation as

$$A\mathbf{u} = \{\mathrm{F}, \mathbf{u}_0, \mathbf{u}'\},$$

where

$$\mathbf{F} = \{F_i\}_{i=1}^5, \quad F_4 = f, \quad F_5 = g, \quad \mathbf{u}_0 = \{u_{i0}\}_{i=1}^3, \quad \mathbf{u}' = \{u'_{0i}\}_{i=1}^5,$$
$$u'_{04} = \theta_0, \quad u'_{05} = \mu_0, \quad \mathbf{u} = \{u_i\}_{i=1}^5, \quad u_4 = \theta, \quad u_5 = \mu.$$

The domain of A has to be a subset of $\prod_{i=1}^{5} L^2(Q_T)$ and the range $R(A) \subset W$, where W is the Hilbert space defined by

$$W = \prod_{i=1}^{5} L^2(Q_T) \times \prod_{i=1}^{3} H_0^1(\Omega) \times \prod_{i=1}^{5} L^2(\Omega)$$

with the inner product

$$(\{\mathbf{F}, \mathbf{u}_0, \mathbf{u}'\}, \{\mathbf{G}, \mathbf{v}_0, \mathbf{v}'\}) = \sum_{i=1}^{5} (F_i, G_i)_{L^2(Q_T)} + \sum_{i=1}^{3} (u_{0i}, v_{0i})_{H^1(\Omega)} + \sum_{i=1}^{5} (u'_{0i}, v'_{0i})_{L^2(\Omega)}.$$

Note that A is a linear and unbounded operator. For definiteness we set $D(A) = H_0^{2,2}(Q_T)$, where $H_0^{2,2}(Q_T)$ is a subspace of $H^{2,2}(Q_T)$ of functions which vanish at $x \in \delta\Omega$ and $t \in (0, T)$. Similarly to [6] (p. 229), it is possible to verify that A can be extended to $\bar{A}$, where $\bar{A}$ is the so-called closure of A.

Using (2.3) one can prove that $\bar{A}$ is invertible and $R(\bar{A}) = W$.

We shall call $\mathbf{u} = A^{-1}\{\mathbf{F}, \mathbf{u}_0, \mathbf{u}'\}$ the generalized solution of (1.1)–(1.7) in the energetic class. Hence we get the following theorem.

THEOREM 2.2 If Ω satisfies the S condition and (2.2) holds, then the problem (1.1)–(1.7) has a unique generalized solution which satisfies the estimate (2.3) for $t \in (0, T)$.

Remark 2.1. It is possible to generalize theorem 2.2 for a region Ω which can be presented in the form $\bigcup_i \Omega_i$ where for each Ω_i there exists a cover Ω_i^ε such that $\Omega_i^\varepsilon \cap \Omega_i$ satisfies the S condition (see [6], p. 131).

3. The implicit differences scheme

In this section we deal with an implicit difference scheme which approximates the problem (1.1)–(1.7). It will be proved that the solution of the difference scheme satisfies an estimate analogous to (2.3). Next we shall show convergence provided the solution of (1.1)–(1.7) is sufficiently smooth or belongs to a certain Sobolev space. To do this the several definitions are needed. Let R_h^3 be a grid on R^3 of the form $R_h^3 = \{x = (i_1 h_1, i_2 h_2, i_3 h_3),\ h_j > 0,\ i_j$ — integers, $j = 1, 2, 3\}$.

By Ω_h we denote the grid set:

$$\Omega_h = \{x : x \in R_h^3 \wedge I_i^{\pm} x \in \bar{\Omega} \wedge I_i^+ I_j^- x \in \bar{\Omega}, \quad i \neq j, \quad i, j = 1, 2, 3\},$$

where

$$I_i^{\pm} x = x \pm e_i h_i, \quad e_i = (\delta_{1i}, \delta_{2i}, \delta_{3i})$$

and δ_{ij} stands for the Kronecker delta.

Let $\bar{\Omega}_h = R_h^3 \cap \bar{\Omega}$ and $\Gamma_h = \bar{\Omega}_h / \Omega_h$.

Finally let ω_τ be a time grid defined by

$$\omega_\tau = \{t = n\tau, \quad n = 0, \ldots, N, \quad N\tau = T\}.$$

The difference quotients are defined as follows

$$\partial_i y(x) = \partial_i y = (I_i^+ y - y)/h_i, \quad \overline{\partial}_i y = (y - I_i^- y)/h_i,$$
$$\tilde{\partial}_i y = (I_i^+ y - I_i^- y)/2h_i,$$
$$\partial_i \overline{\partial}_i y = (I_i^+ y - 2y + I_i^- y)/h_i^2,$$
$$y_t^n = (y^{n+1} - y^n)/\tau, \quad y_{\tilde{t}}^n = (y^{n+1} - y^{n-1})/2\tau,$$
$$y_{t\bar{t}}^n = (y^{n+1} - 2y^n + y^{n-1})/\tau^2, \quad \hat{y}^n = (y^{n+1} + y^{n-1})/2,$$

where

$$I_i^\pm y(x) = y(I_i^\pm x), \quad y^n(x) = y(x, n\tau).$$

The difference scheme approximating the problem (1.1)–(1.7) is of the form

$$G \sum_{j=1}^{3} \partial_j \overline{\partial}_j \hat{v}_i^n + \frac{\lambda + G}{2} \sum_{j=1}^{3} (\partial_i \overline{\partial}_j + \overline{\partial}_i \partial_i) \hat{v}_i^n - p_\theta \tilde{\partial}_i \hat{v}_2^n - p_\mu \tilde{\partial}_i \hat{v}_5^n - \varrho v_{it\bar{t}}^n = F_i^n, \quad i = 1, 2, 3; \tag{3.1}$$

$$K \sum_{j=1}^{3} \partial_j \overline{\partial}_j \hat{v}_4^n - c v_{4\tilde{t}}^n - d v_{5\tilde{t}}^n - p_\theta \sum_{j=1}^{3} \tilde{\partial}_j v_{j\tilde{t}}^n = f^n, \tag{3.2}$$

$$D \sum_{j=1}^{3} \partial_j \hat{\partial}_j \hat{v}_5 - b v_{5\tilde{t}}^n - d v_{4\tilde{t}}^n - p_\mu \sum_{j=1}^{3} \tilde{\partial}_j v_{j\tilde{t}}^n = g^n \tag{3.3}$$

for $x \in \Omega_h$, $n = 1, \ldots, N-1$, with the difference boundary conditions

$$v_1^n = v_2^n = v_3^n = 0, \quad x \in \Gamma_h, \tag{3.4}$$

$$v_4^n = 0, \quad v_5^n = 0, \quad x \in \Gamma_h, \tag{3.5}$$

and the initial conditions

$$v_i^0(x) = u_{i,0}(x), \quad v_i^1(x) = v_{i,1}(x), \quad x \in \Omega_h, \quad i = 1, 2, 3; \tag{3.6}$$

$$v_4^0(x) = \theta_0(x), \quad v_5^0 = \mu_0, \quad v_4^1(x) = \theta_1(x), \quad v_5^1 = \mu_1, \quad x \in \Omega_h. \tag{3.7}$$

The functions $v_{i,1}(x)$, $\theta_1(x)$, $\mu_1(x)$ can be calculated by

$$v_{i,1}(x) = u_{i,0}(x) + \tau u_i'(x) + \frac{\tau^2}{2} D_0^2 u_i(x, 0), \tag{3.8}$$

$$\theta_1(x) = \theta_0(x) + \tau D_0 \theta(x, 0), \quad \mu_1(x) = \mu_0(x) + \tau D_0 \mu(x, 0). \tag{3.9}$$

The difference problem (3.1)–(3.7) approximates the differential problem (1.1)–(1.7) in the grid points with an error $0(\tau^2 + h^2)$ if $\Gamma_h \subset \delta\Omega$ and $0(\tau^2 + h)$ if $\Gamma_h \not\subset \delta\Omega$, $h = \max(h_1, h_2, h_3)$ provided the solution of (1.1)–(1.7) is sufficiently smooth.

Now let us consider the stability of the scheme (3.1)–(3.7). To this end let us introduce the Hilbert space $H_h = L_h^2(\overline{\Omega}_h)$ of the grid functions defined on $\overline{\Omega}_h$ with the following inner product and the norm

$$(u, v)_h = \sum_{x \in \overline{\Omega}_h} h_1 \times h_2 \times h_3 u(x) \cdot v(x), \quad \|u\|_h^2 = (u, u)_h.$$

Let $\mathring{H}_h$ be a subspace of H_h of the functions which are equal to zero at the grid points of Γ_h. We shall also use the space H_h^1 and H_{0h}^1 which are the difference analogous of H^1 and H_0^1, respectively. The space $H_h^1(\bar{\Omega}_h)$ is the Hilbert space of the grid functions defined on $\bar{\Omega}_h$ with the inner product

$$(u, v)_{1,h} = (u, v)_h + h_1 h_2 h_3 \sum_{i=1}^{3} \sum_{\Omega_h^i} \partial_i u(x) \cdot \partial_i v(x),$$

where Ω_h^i means the set of all points of $\bar{\Omega}_h$ at which ∂_i are defined.

The space H_{0h}^1 differs from H_h^1 since the functions of H_{0h}^1 satisfy the conditions: $u(x) = 0$, $x \in \Gamma_h$.

Let $(B_i y)(x) = -\partial_i \bar{\partial}_i y(x)$, $x \in \Omega_h$ for $y(x) = 0$, $x \in \Gamma_h$.

LEMMA 3.1. The operator $B = \sum_{i=1}^{3} B_i$, $B: \mathring{H}_h \to \mathring{H}_h$ is self-adjoint and positive definite, i.e.

$$B = B^* \geqslant \delta E, \quad \delta > 0,$$

where δ depends only on the diameter of Ω. The proof of lemma 3.1 can be get by using the formulae of summation by parts (see [3], p. 46). In the sequel the Hilbert space H_{hB} will be needed which differs from the space H_h only by the definition of the inner product and the norm, namely

$$(u, v)_B = (Bu, v)_h, \quad ||u||_B^2 = (Bu, u)_h.$$

It is easy to prove that the norm of H_{hB} and H_{0h}^1 are equivalent with the constants independent of h_i. To simplify the further formulae we shall drop the index h.

THEOREM 3.1. If (2.2) holds, then the solution of (3.1)–(3.7) satisfies the inequality

$$(3.10) \quad \max_n \Big\{ \sum_{i=1}^{3} [||v_{it}^n||_{\mathring{H}}^2 + ||v_i^n||_B^2] + \sum_{i=4}^{5} ||v_i^n||_{\mathring{H}}^2 \Big\} + \tau \sum_{n=1}^{N-1} \sum_{i=4}^{5} ||\hat{v}_i^n||_B^2$$
$$\leqslant M \Big\{ \tau \sum_{n=1}^{N-1} \Big[\sum_{i=1}^{3} ||F_{i\tilde{t}}^n||_{B-1}^2 + ||f^n||_{B-1}^2 + ||g^n||_{B-1}^2 \Big]$$
$$+ \sum_{r=0}^{1} \Big\{ \sum_{i=1}^{3} [||F_i^{r+1}||_{B-1}^2 + ||v_i^r||_B^2] + \sum_{i=4}^{5} ||v_i^r||_{\mathring{H}}^2 \Big\} + \sum_{i=1}^{3} ||v_{it}^0||_{\mathring{H}}^2 \Big\},$$

where M is a positive constant independent on the data functions, the grid steps, and the solution of (3.1)–(3.7).

P r o o f. Let us form the inner products in $\mathring{H}$ of (3.1) with $-2\tau v_{i\tilde{t}}^n$, $i = 1, 2, 3$ and (3.2), (3.3) with $-2\tau \hat{v}_i^n$, $i = 4, 5$, respectively, and perform the summation over $i = 1, \ldots, 5$ and $n = 1, \ldots, k-1$. Using the formulae of summation by parts (see [3], p. 46) and the identity

$$2(y^{n+1}, y_t^n) = (y^n, y^n)_t + \tau(y_t^n, y_t^n)$$

we get

$$(3.11)\qquad \sum_{i=1}^{3}\{\varrho\|v_{it}^{k-1}\|^2+I(v_i^k)+I(v_i^{k-1})\}+c[\|v_4^k\|^2+\|v_4^{k-1}\|^2]+b[\|v_5^k\|^2+\|v_5^{k-1}\|^2]$$

$$+\tau\sum_{n=1}^{k-1}\Big\{[2d\{(v_{5\tilde{t}}^n,\hat{v}_4^n)+(v_{4\tilde{t}}^n,\hat{v}_5^n)\}]+\sum_{j=1}^{3}[\|\partial_j\hat{v}_4^n\|^2+\|\partial_j\hat{v}_5^n\|^2]\Big\}=\sum_{i=1}^{3}[I(v_i^0)+I(v_i^1)$$

$$+\varrho\|v_{it}^0\|^2]+c[\|v_4^0\|^2+\|v_4^1\|^2]+b[\|v_5^0\|^2+\|v_5^1\|^2]$$

$$+2\tau\sum_{n=1}^{k-1}\Big\{\sum_{i=1}^{3}(F_i^n,v_{i\tilde{t}}^n)+(f^n,\hat{v}_4^n)+(g^n,\hat{v}_5^n)\Big\},$$

where

$$I(v_i^n)=0.5\sum_{j=1}^{3}\{G(\partial_j v_i^n,\partial_j v_i^n)+(\lambda+G)(\partial_i v_i^n,\partial_j v_i^n)\}.$$

It is easy to prove that

$$(3.12)\qquad \sum_{i=1}^{3}I(v_i^n)\geqslant\frac{G}{2}\sum_{i=1}^{3}\|v_i^n\|_B^2.$$

A simple calculation yields the following estimates

$$(3.13)\qquad 2\tau\sum_{n=1}^{k-1}(F_i^n,v_{i\tilde{t}}^n)\leqslant\varepsilon_1\{\|v_i^k\|_B^2+\|v_i^{k-1}\|_B^2\}+M(\varepsilon_1)\Big\{\|F_i^1\|_{B^{-1}}^2$$

$$+\|F_i^2\|_{B^{-1}}^2+\|v_i^0\|_B^2+\|v_i^1\|_B^2+\tau\sum_{n=2}^{k-1}\|F_{i\tilde{t}}^n\|_{B^{-1}}^2+\tau\sum_{n=1}^{k-2}\|v_i^n\|_B^2\Big\},$$

$$(3.14)\qquad 2(z,\hat{v}_i^n)\leqslant\varepsilon_2\|z\|_{B^{-1}}^2+\frac{1}{\varepsilon_2}\|\hat{v}_i^n\|_B^2,$$

where $\varepsilon_i>0$ and z can be equal to f^n or g^n. Substituting (3.12)–(3.14) in the equation (3.11) we get (3.10) which completes the proof.

Now we are in a position to prove the convergence of the scheme (3.1)–(3.7).

THEOREM 3.2. Let the assumptions (2.2) hold. If the functions

$$D_0^4u_i,\quad D_0D_i^\alpha D_j^\beta u_i\quad(\alpha+\beta\leqslant 4),\quad D_0^2D_i^2u_i,$$

$$D_0D_i^4\mu,\quad D_0^3\theta,\quad D_0^3\mu,\quad D_0D_i^4\theta,\quad i,j=1,2,3$$

are bounded, and the functions $v_{i,0}$, $i=1,2,3,\theta_1,\mu_1$ are defined by (3.8), (3.9) then the following inequality holds

$$(3.15)\qquad \|z\|_U^2\equiv\max_n\Big\{\sum_{i=1}^{3}[\|z_{it}^n\|_H^2+\|z_i^n\|_{H^1}^2]+\sum_{i=4}^{5}\|z_i^n\|_H^2\Big\}+\tau\sum_{n=1}^{N-1}\sum_{i=4}^{5}\|\hat{z}_i^n\|_{H^1}^2\leqslant MQ(\tau,h),$$

where $z_i^n=v_i^n-u_i^n$, $i=1,2,3$, $z_4^n=v_4^n-\theta^n$, $z_5^n=v_5^u-\mu^n$,

$$Q(\tau,h)=\begin{cases}\tau^4+h & \text{if}\quad \Gamma_h\not\subset\delta\Omega,\\ \tau^4+h^4 & \text{if}\quad \Gamma_h\subset\delta\Omega.\end{cases}$$

P r o o f. If $\Gamma_h \subset \delta\Omega$, then theorem 3.2 immediately follows from theorem 3.1 since the approximations error of (1.1)–(1.7) is $0(\tau^2+h^2)$. Hence let $\Gamma_h \not\subset \delta\Omega$. Let us express the solution of (1.1)–(1.7) at the grid points in the form $u_i^n = u_{i\Omega}^n + u_{i\Gamma}^n$, $i = 1, \ldots, 5$, $u_4 = \theta$, $u_5 = \mu$, where

$$u_{i\Gamma}^n = \{0 \quad \text{for} \quad x \in \Omega_h \quad \text{and} \quad u_i^n(x) \quad \text{for} \quad x \in \Gamma_h\}.$$

The functions $u_{i\Omega}^n$ satisfy the system (3.1)–(3.3) with the right-hand side equal to

$$G_i^n = 0(\tau^2+h^2)+\xi^n, \quad i = 1, \ldots, 5$$

where

$$\|\xi^n\|_{B^{-1}} = 0(h^{1/2}).$$

Applying theorem 3.1 for $v_i^n - u_{i\Omega}^n$, $i = 1, \ldots, 5$, and the triangle we get (3.15). Hence theorem 3.1 follows.

R e m a r k 3.1. The analogous results hold for a non-uniform grid (in the space direction) with an error $0(\tau^2+h^2)$ if $\Gamma_h \subset \delta\Omega$.

The scheme of (3.1)–(3.7) is convergent under the assumption that the solution of (1.1)–(1.7) is sufficiently smooth in the classical sense. Such solution exists when the boundary $\delta\Omega$ of Ω is sufficiently smooth, see [2].

Let us now pass to the problem of convergence of the scheme (3.1)–(3.7) under the assumption that the solution of (1.1)–(1.7) belongs to a certain Sobolev space.

THEOREM 3.3. Let (2.2) hold and let the following functions belong to $L^2(Q_T)$:

$$D_0^3 u_i, \quad D_0^2 D^\alpha u_i, \quad D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3},$$

$$D_0^2\theta, \quad D_0^2\mu, \quad D_0^2 D^\alpha\theta, \quad D_0^2 D^\alpha\mu, \quad \alpha = \alpha_1+\alpha_2+\alpha_3 \leqslant 3,$$

where u_i, $i = 1, 2, 3$, θ, μ is the solution of (1.1)–(1.7). Then

$$\|z\|_U^2 = 0(\tau+h^{1/2}),$$

where $\|\cdot\|_U$ is defined in (3.15) and $z_i^n = v_i^n - u_i^n$, $i = 1, 2, 3$, $z_4^n = v_4^n - \theta^n$, $z_5^n = v_5^n - \mu^5$, and v_i^n is the solution of (3.1)–(3.9).

The proof is omitted since the proof technique is similar to the proof of theorem 3.2.

4. The economical scheme

Let Ω be a rectangular parallelepiped. In this case we can approximate the system of (1.1)–(1.7) by an economical scheme with a splitting operator. By an economical scheme (see [3, 4]) we mean a scheme which is unconditionally stable and the total number of arithmetic operations needed to solve this difference scheme is proportional to the total number of the grid points of $\Omega_h \times \omega_\tau$.

An example of such scheme for (1.1)–(1.7) is presented below

$$(4.1) \quad \varrho \prod_{j=1}^{3} (E-\theta\tau^2\partial_j\bar{\partial}_j)v_{i\bar{t}t}^n + G\sum_{j=1}^{3}\partial_j\bar{\partial}_j v_i^n + 0.5(\lambda+G)\sum_{j=1}^{3}(\partial_i\bar{\partial}_j + \bar{\partial}_i\partial_j)v_j^n - p_\theta\tilde{\bar{\partial}}_i v_4^n - p_\mu\partial_i v_5^n = F_i^n, \quad i = 1, 2, 3,$$

$$(4.2)\qquad \prod_{j=1}^{3}(E-\tau A_0^{-1}A\partial_j\bar{\partial}_j)\bar{v}_{\bar{t}}^{n}+\Big(\sum_{j=1}^{3}\tilde{\partial}_j v_{\bar{t}}^{n}\Big)A_0^{-1}\bar{p}-A_0^{-1}A\sum_{j=1}^{3}\partial_j\bar{\partial}_j\bar{v}^{n-1} = A_0^{-1}\bar{G}^n,\quad x\in\Omega_h,\quad n=1,\ldots,N-1,$$

where

$$\theta>0,\quad \bar{v}^n=(v_4^n,v_5^n)^T,\quad G^n=(-f^n,-g^n)^T,\quad \bar{p}=(p_\theta,p_\mu)^T,$$

$$A_0=\begin{pmatrix}c & d\\ d & b\end{pmatrix},\quad A=\begin{pmatrix}K & O\\ O & D\end{pmatrix}$$

with the difference boundary conditions (3.4), (3.1) and the initial conditions (3.6), (3.7).

THEOREM 4.1. Let the assumptions (2.2) hold. If $\theta\geqslant\theta_0(G,\lambda,\varrho)>0$, then the solutions of (4.1), (4.2), (3.4)–(3.7) satisfie the following inequality

$$(4.3)\qquad \|v\|_Q^2\equiv\max_n\Big\{\sum_{i=1}^{3}I_1(v_i^n)+\sum_{i=4}^{5}I_2(v_i^n)\Big\}+\tau\sum_{k=1}^{N-1}\sum_{i=4}^{5}\|\hat{v}_i^n\|_B^2 \leqslant M\Big\{\tau\sum_{n=1}^{N-1}\Big[\sum_{i=1}^{3}\|F_i^n\|^2+\|f^n\|^2+\|g^n\|^2\Big]+\sum_{i=1}^{3}I_1(v_i^0)+\sum_{i=4}^{5}I_2(v_i^0)\Big\},$$

where

$$I_1(v_i^k)=\|v_{it}^{k-1}\|_{\mathring{H}}^2+\|v_i^k\|_B^2+\|v_i^{k-1}\|_B^2+\tau^4\sum_{l<j}\|\partial_l\partial_j v_{it}^{k-1}\|_{\mathring{H}}^2+\tau^6\|\partial_1\partial_2\partial_3 v_t^{k-1}\|_{\mathring{H}}^2;$$

$$I_2(v_i^k)=\|v_i^k\|_{\mathring{H}}^2+\|v_i^{k-1}\|_{\mathring{H}}^2+\sum_{r=0}^{1}\Big\{\sum_{l<j}\tau^2\|\partial_l\partial_j v_i^{k-r}\|^2+\tau^3\|\partial_1\partial_2\partial_3 v_i^{k-r}\|_{\mathring{H}}^2\Big\}.$$

Proof. Multiply (4.2) by A_0. After some transformations we get

$$A_0\bar{v}_{\bar{t}}^n-\sum_{j=1}^{3}A\partial_j\bar{\partial}_j\hat{\bar{v}}^n+\tau^2A_0^{-1}A\sum_{l>j}\partial_l\bar{\partial}_l\partial_j\bar{\partial}_j\bar{v}_{\bar{t}}^n+\tau^3A(A_0^{-1}A)^2\prod_{i=1}^{3}\partial_i\bar{\partial}_i\bar{v}_{\bar{t}}^n+\sum_{j=1}^{3}\tilde{\partial}_j v_{\bar{t}}^n\bar{p}=\bar{G}^n.$$

Let us form the inner products in $\mathring{H}$ of (4.1) with $2\tau v_{\bar{t}}^n$ and next (4.2) with $2\tau\hat{\bar{v}}^n$. Similarly to the proof of theorem 3.1 we summ up the inner products obtained with respect to n, $n=1,\ldots,k-1$. Using the assumption (2.2) and the Gronwall inequality we get (4.3).

THEOREM 4.2. Let the assumptions of theorem 3.2 and 4.1 hold. Besides let the function $D_1^2D_2^2D_3^2u_i$, $i=1,\ldots,5$, $D_0^2D_l^2D_j^2u_i$ $(l<j)$, $i=1,2,3$, $D_0D_l^2D_j^2u_i$ $(l<j)$, $i=4,5$ be bounded; here u_i, $i=1,2,3$, $u_4=\theta$, $u_5=\mu$ are the solutions of the system (1.1)–(1.7). Then $\|z\|_Q=0(\tau^2+h^2)$, where $\|\cdot\|_Q$ is defined in (4.3), $z_i^n=v_i^n-u_i^n$, $i=1,2,3$, $z_4^n=v_4^n-\theta^n$, $z_5^n=v_5^n-\mu^n$; v_i^n is the solution of the problem (4.1), (4.2) and (3.4)–(3.7), and u_i^n, θ^n, μ^n are the solutions of the problem (1.1)–(1.7) taken on the grid.

This theorem directly follows from theorem 4.1 and the approximation of the system (1.1)–(1.7) by the scheme (4.1), (4.2), (3.4)–(3.9) with an error $0(\tau^2+h^2)$.

Remark 4.1. If $n=2$ and Ω is convex, we can construct an analogous scheme with a splitting operator which is unconditionally stable and convergent with an error $0(\tau^2+h^2)$ provided $\Gamma_h\subset\delta\Omega$. Furthermore this can be generalized for $n=3$ when Ω has the form $\Omega=\Omega_2\times(0,l_3)$ and Ω_2 is convex in the plane R_2 (see [7]).

5. Galerkin methods

In this section we consider the Galerkin method with "viscosity" for the problem (1.1)–(1.7). The presence of the term of order $0(\tau^2)$, which is called viscosity, helps to solve a system of algebraic linear equations since the matrix of this system has a simple form.

The Galerkin method is based on the weak form (variational form) of the differential equations. The weak form of (1.1)–(1.7) is as follows

$$(5.1)\quad \varrho\sum_{i=1}^{3}\big(D_0^2u_i(t),v_i\big)+c\big(D_0u_4(t),v_4\big)+d\big(D_0u_5(t),v_4\big)+b\big(D_0u_5(t),v_5\big)$$
$$+d\big(D_0u_4(t),v_5\big)+a_1\big(u(t),v\big)+a_2(u,v)+K\big(\nabla u_4(t),\nabla v_4\big)$$
$$+D\big(\nabla u_5(t),\nabla v_5\big)+a_3(D_0u,v)=-\sum_{i=1}^{5}(F_i,v_i),$$
$$\forall\, v_i\in H_0^1(\Omega),\quad i=1,\ldots,5,\quad 0<t<T,$$

$$(5.2)\quad u_i(0)=u_{0i},\quad i=1,2,3,\quad u_4(0)=\theta_0,\quad u_5(0)=\mu_0,$$

$$(5.3)\quad Du_i(0)=u_i',\quad i=1,2,3,$$

where

$$u_4=\theta,\quad u_5=\mu,\quad F_4=f,\quad F_5=g,$$
$$a_1(u,v)=\sum_{j=1}^{3}\sum_{i=1}^{3}\{G(D_ju_i,D_jv_i)+(\lambda+G)(D_iu_j,D_jv_i)\},$$
$$a_2(u,v)=\sum_{i=1}^{3}\{p_\theta(D_iu_4,v_i)+p_\mu(D_iu_5,v_i)\},$$
$$a_3(D_0u,v)=-\sum_{j=1}^{3}\{p_\theta(D_0u_j(t),D_jv_4)+p_\mu(D_0u_j(t),D_jv_5)\}.$$

Let $\mathcal{M}$ be a m-dimensional subspace of $H_0^1(\Omega)$. Let ω_τ be the grid of the form
$$\omega_\tau=\{t=n\tau,\quad n=0,\ldots,N,\quad N\tau=T\}.$$

The problem (5.1)–(5.3) is approximated by the discrete Galerkin method in the form:

$$(5.4)\quad \varrho\sum_{i=1}^{3}(U^k_{it\bar{t}},v_i)+c(U^k_{4\bar{t}},v_4)+d(U^k_{5\bar{t}},v_4)+b(U^k_{5\bar{t}},v_5)+d(U^k_{4\bar{t}},v_5)+a_1(U^k,v)$$
$$+a_2(\hat{U}^k,v)+\sum_{i=1}^{3}\theta\tau^2(\nabla U^k_{it\bar{t}},\nabla v_i)+K(\nabla\hat{U}^k_4,\nabla v_4)+D(\nabla\hat{U}^k_5,\nabla v_5)+a_3(U^k_{\bar{t}},v)$$
$$=-\sum_{i=1}^{5}(F^k_i,v_i)$$

for
$$\forall\, v_i\in\mathcal{M},\quad k=1,\ldots,N-1;$$

$$(5.5)\quad (U_i^0,v_i)=(u_{0i},v_i),\quad \forall\, v_i\in\mathcal{M},\quad i=1,\ldots,5;$$

$$(5.6)\quad (U_i^1,v_i)=(u_{1i},v_i),\quad \forall\, v_i\in\mathcal{M},\quad i=1,\ldots,5.$$

Here $u_{04}=\theta_0$, $u_{05}=\mu_0$ and u_{1i} are the data functions, which can be calculated from (3.8), (3.9). Here θ is a positive parameter.

Now consider the approximation error of the solutions of (5.4)–(5.6) and (5.1)–(5.3).

THEOREM 5.1. Let (2.2) hold and $\theta \geqslant \theta_0(G, \lambda) > 0$. Let the following functions $\nabla D_0^4 u_i$, $\nabla D_0^3 u_i$, $D_0 F_i$, $i = 1, \ldots, 5$ belong to $L^2(Q_T)$ and let z_i^n denote $z_i^n = u_i^n - U_i^n$, where u_i^n, U_i^n are the solutions of (5.1)–(5.3) and (5.4)–(5.6), respectively. Then z^n can be estimated as follows

$$(5.7)\qquad \|z_u^2\| \equiv \max_n\Big\{\sum_{i=1}^{3}[\|z_{it}^{n-1}\|^2+\|\nabla z_i^n\|^2]+\sum_{i=4}^{5}\|z_i^n\|^2\Big\}+\tau\sum_{k=1}^{N-1}\sum_{i=4}^{5}\|\hat{z}_i^k\|^2$$
$$\leqslant M\Big\{\max_n\Big\}\sum_{i=1}^{3}[\|\tilde{u}_i^n\|^2+\|\nabla\tilde{u}_i^n\|^2]+\sum_{i=4}^{5}\|\tilde{u}_i^n\|^2\Big\}+\sum_{i=1}^{3}\Big\{\tau\sum_{k=1}^{N-1}[\|\nabla\tilde{u}_{i\bar{t}}^k\|^2+\|\tilde{u}_{it\bar{t}}^k\|^2]$$
$$+\sum_{r=0}^{1}\|\nabla z_i^r\|^2+\|z_{it}^0\|^2\Big\}+\sum_{i=4}^{5}\Big\{\sum_{k=1}^{N-1}[\|\nabla\tilde{u}_i^k\|^2+\|\tilde{u}_{i\bar{t}}^k\|^2]+\sum_{r=0}^{1}\|z_i^r\|^2\Big\}+\tau^4\Big\},$$

where $u_i^k = u_i^k - \tilde{u}_i$ and $\tilde{u}_i$ is an arbitrary function from $\mathcal{M}$.

In the proof of theorem 5.1 the identities which are listed below will be used.

LEMMA 5.1.

$$(5.8)\qquad 2\tau(y_{t\bar{t}}^k, y_{\bar{t}}^k) = (y_{\bar{t}}^k, y_{\bar{t}}^k)_{\bar{t}};$$

$$(5.9)\qquad (y_t^k, v^k) = (y^k, v^k)_t - (y^{k+1}, v_t^k);$$

$$(5.10)\qquad (y_t^k, v^{k+1}) = (y^k, v^k)_t - (y^k, v_t^k);$$

$$(5.11)\qquad 2(y_{\bar{t}}^k, \hat{y}^k) = (y^k, y^k)_{\bar{t}};$$

$$(5.12)\qquad (y_{\bar{t}}^k, v^k) = (y^k, v^k)_{\bar{t}} - \frac{1}{2}\{(y^{k+1}, v_t^k)+(y^{k-1}, v_{\bar{t}}^k)\};$$

$$(5.13)\qquad (y_{\bar{t}}^k, \hat{v}^k)+(\hat{y}^k, v_{\bar{t}}^k) = (y^k, v^k)_{\bar{t}};$$

$$(5.14)\qquad y^n = \hat{y}^n - \frac{\tau^2}{2}y_{t\bar{t}}^n.$$

There lemma 5.1 can be proved by simple calculations based on the definitions of difference quotients.

P r o o f of the theorem 5.1. It is easy to see that

$$(5.15)\qquad \begin{aligned}&(\nabla u_i)^n = \nabla\hat{u}_i^n + \delta_{0,i}^n, \qquad (D_0^2 u_i)^n = u_{it\bar{t}}^n + \delta_{1i}^n,\\ &(D_0 u_i)^n = u_{i\bar{t}}^n + \delta_{2i}^n,\end{aligned}$$

where

$$\|\delta_{si}^n\|_{L^2} = 0(\tau^2), \quad s = 0, 1, 2, \quad i = 1, \ldots, 5.$$

Substitute in Eq. (5.1) $t = n\tau$ and subtract it from (5.4). Next using (5.15) and summing up for $k = 1, \ldots, n-1$, we get

$$(5.16)\qquad \sum_{k=1}^{n-1}\Big\{\sum_{i=1}^{3}\varrho(z_{it\bar{t}}^k, v_i)+\theta\tau^2\sum_{i=1}^{3}(\nabla z_{it\bar{t}}^k, \nabla v_i)+I_1(z_{\bar{t}}^k, v)+a_1(z^k, v)+a_2(\hat{z}\ , v)$$
$$+K(\nabla\hat{z}_4^k, \nabla v_4)+D(\nabla\hat{z}_5^k, \nabla v_5)+a_3(z_{\bar{t}}^k, v)\Big\} = -\sum_{k=1}^{n-1}\Big\{\sum_{i=1}^{3}[\varrho(\delta_{1i}^k, v_i)-\tau^2\theta(\nabla u_{it\bar{t}}^k, \nabla v_i)]$$
$$+I_1(\delta_2^k, v)+a_2(\delta_{04}^k, v)+a_3(\delta_2^k, v)+K(\nabla\delta_{04}^k, \nabla v_4)+D(\nabla\delta_{05}^k, \nabla v_5)\Big\}$$

for $\forall\, v \in \mathcal{M}$, where

$$I_1(w^k, v) = c(w_4^k, v_4)+d(w_5^k, v_4)+b(w_5^k, v_5)+d(w_4^k, v_5),$$
$$w^k = (w_1^k, \ldots, w_5^k)^T, \quad v = (v_1, \ldots, v_5)^T.$$

Let v_i in (5.16) be equal to $v_i = 2\tau(z_{i\bar{t}}^k+\tilde{u}_{i\bar{t}}^k)$ for $i = 1, 2, 3$ and $v_i = 2\tau(\hat{z}_i^k+\hat{\tilde{u}}_i^k)$ for $i = 4, 5$, where $\tilde{u}^k = u_i^k-\tilde{u}_i$ and $\tilde{u}_i$ is an arbitrary function from $\mathcal{M}$. The terms which appear in the left-hand side of (5.16) are estimated from below.

Let J_0^n denote the first term of the left-hand side of (5.16). From (5.8), (5.9) and the ε — inequality we get

$$\sum_{k=1}^{n-1} J_0^k \geqslant \sum_{i=1}^{3}\Big\{\varrho(1-\varepsilon_1)\|z_{it}^{n-1}\|^2-M\Big\{\|z_{it}^0\|^2+\sum_{r=0}^{1}[\|\tilde{u}_{it}^{n-r}\|^2$$
$$+\|\tilde{u}_{it}^r\|^2]+\tau\sum_{k=1}^{n-2}\|z_{it}^k\|^2+\tau\sum_{k=1}^{n-1}\|\tilde{u}_{it\bar{t}}^k\|^2\Big\}\Big\}. \tag{5.17}$$

The second term in (5.16) is estimated by (5.17) where ϱ is replaced by $\tau^2\theta$ and z, $\tilde{u}$ are replaced by ∇z, $\nabla\tilde{u}$, respectively. The sixth and seventh term of (5.16) are estimated using the ε — inequality

$$2\tau\sum_{k=1}^{n-1}(\nabla\hat{z}_i^k, \nabla z_i^k+\nabla\hat{\tilde{u}}_i^k) \geqslant \tau\sum_{k=1}^{n}\Big\{(2-\varepsilon_2)\|\nabla\hat{z}_i^k\|^2-\frac{1}{\varepsilon_2}\|\nabla\hat{\tilde{u}}_i^k\|^2\Big\}. \tag{5.18}$$

Let us now estimate the other terms of (5.16). To estimate $I_1(z_{\bar{t}}^k, v)$ two inequalities are needed. The first one is (with $\hat{u}_i = \hat{\tilde{u}}_i$ for $i = 4, 5$):

$$4\tau\sum_{k=1}^{n-1}(z_{i\bar{t}}^k, \hat{z}_i^k+\tilde{u}_i^k) \geqslant (1-\varepsilon_3)\{\|z_i^n\|^2+\|z_i^{n-1}\|^2\}-M\Big\{\frac{1}{\varepsilon_3}[\|\hat{u}_i^n\|^2$$
$$+\|\hat{u}_i^{n-1}\|^2+\tau\sum_{k=0}^{n-1}\|\hat{u}_{it}^k\|^2+\tau\sum_{k=0}^{n-2}\|z_i^k\|^2+\sum_{r=0}^{1}[\|z_i^r\|^2+\|\tilde{u}_i^r\|^2]\Big\}. \tag{5.19}$$

The inequality (5.19) follows from (5.11) and (5.12). The second inequality needed is

$$2\tau\sum_{n=1}^{n-1}\{(z_{5\bar{t}}^k, \hat{z}_4^k+\tilde{u}_4^k)+(z_{4\bar{t}}^k, \hat{z}_5^k+\tilde{u}_5^k)\} \geqslant \sum_{k=n-1}^{n}\Big\{(z_5^k, z_4^k)-0.5\sum_{i=4}^{5}[\varepsilon_4\|z_i^k\|^2+\varepsilon_4^{-1}\|u_i^k\|^2]\Big\}$$
$$-M\Big\{\tau\sum_{i=4}^{5}\Big\{\sum_{k=0}^{n-2}\|z_i^k\|^2+\sum_{k=0}^{n-1}\|\tilde{u}_{it}^k\|^2\Big\}+\sum_{k=0}^{1}\sum_{i=4}^{5}[\|z_i^k\|^2+\|\tilde{u}_i^k\|^2]\Big\}. \tag{5.20}$$

To prove (5.20) it is sufficient to use (5.13), (5.12) and the ε—inequality.

Using (5.19), (5.20) we get

$$\sum_{k=1}^{n-1} I_1(z_{\bar{t}}^k, 2\tau\hat{z}^k+2\tau\tilde{u}^k) \geqslant \sum_{i=4}^{5}\Big\{\sum_{k=n-1}^{n}[(\tilde{d}-\varepsilon_3-\varepsilon_4)\|z_i^k\|^2$$
$$-M\|\tilde{u}_i^k\|^2]-M\Big\{\sum_{k=0}^{1}[\|z_i^k\|^2+\|\tilde{u}_i^k\|^2]+\tau\sum_{k=0}^{n-2}\|z_i^k\|^2+\tau\sum_{k=0}^{n-1}\|\tilde{u}_{it}^k\|^2\Big\}\Big\}, \tag{5.21}$$

where $\tilde{d} = cb-d^2$.

Let us estimate $a_1(z^k, v)$. Once more from (5.14) and (5.12) we have

$$(5.22)\quad \sum_{n=2}^{k-1} a_1(z^k, 2\tau(z_{\bar{t}}^k+\tilde{u}_{\bar{t}}^k)) \geqslant \frac{1}{2}\sum_{i=1}^{3}\Big\{\sum_{r=0}^{1}[(G-\varepsilon_5)\|\nabla z_i^{n-r}\|^2-\tilde{G}\|\nabla z_i^r\|^2]$$
$$-\tilde{G}\tau^2[\|\nabla z_{it}^{n-1}\|^2+\|\nabla z_{it}^0\|^2]-M\Big\{\sum_{r=0}^{1}\|\tilde{u}_{it}^{n-r}\|^2+\|\tilde{u}_{it}^r\|^2-\tau\Big\{\sum_{k=0}^{n-2}\|\nabla z_i^k\|^2$$
$$+\sum_{k=0}^{n-1}\|\nabla\tilde{u}_{it}^k\|^2+\tau^3\sum_{k=0}^{n-2}[\|\nabla z_{it}^k\|^2+\|\nabla\tilde{u}_{it\bar{t}}^k\|^2\Big\}\Big\}\Big\},$$

where $\tilde{G} = \max(G, \lambda+G)$.

It is easy to verify that the other two terms can be estimated using the formulae of summation by part and the ε — inequality as follows

$$(5.23)\quad \sum_{n=1}^{k-1}\{a_2(\hat{z}^k, 2\tau z_{\bar{t}}^k+2\tau\tilde{u}_{\bar{t}}^k)+a_3(z_{\bar{t}}^k, 2\tau(\hat{z}^k+\tilde{u}^k))\}$$
$$\geqslant -\tau\varepsilon_6\sum_{i=1}^{3}\sum_{k=0}^{n-1}\|z_{it}^k\|^2-\tau\varepsilon_7\sum_{i=4}^{5}\sum_{k=1}^{n-1}\|\hat{z}_i^k\|^2-M\tau\Big\{\sum_{i=4}^{5}\sum_{j=1}^{3}\Big\{\sum_{k=1}^{n-1}\|D_j\tilde{u}_i^k\|+\sum_{k=0}^{n-1}\sum_{i=1}^{3}\|\tilde{u}_{it}^k\|^2\Big\}\Big\}.$$

An upper bound for the right-hand side of (5.16) is given by

$$(5.24)\quad \tau\sum_{k=1}^{n-1}\Big\{\sum_{i=1}^{3}\{\varepsilon_8\|z_{i\bar{t}}^k\|^2+M[\|\tilde{u}_{i\bar{t}}^k\|^2+\tau^4\|\nabla u_{it\bar{t}}^k\|^2+\|\nabla\tilde{u}_{i\bar{t}}^k\|^2]$$
$$+\varepsilon_9\|\nabla z_{i\bar{t}}^k\|^2\}+\sum_{i=4}^{5}\{\varepsilon_{10}\|\hat{z}_i^k\|^2+M[\|\tilde{u}_i^k\|^2+\|\nabla\tilde{u}_i^k\|^2]+\varepsilon_{10}\|\nabla\hat{z}_i^k\|^2\}+\tau^4\Big\}.$$

Substituting (5.17), (5.18) and (5.21)–(5.24) in (5.16), taking suitable ε_i, θ larger than $\theta_0(G, \lambda)$ and applying the Gronwall's lemma we obtain (5.7). This completes the proof.

6. Alternating-direction Galerkin (finite-element) methods for rectangular parallelepipeds

By *AD*-Galerkin method we mean the Galerkin method in which the total number of arithmetic operations needed to perform one time step is $0(m)$, where m is the number of unknowns at each time step. This method has been formulated for the parabolic and hyperbolic equations in [5]. Here we extend this method to our problem. We shall use the notations from the Sects. 4 and 5 and the following new ones:

$$\bar{u} = (u_4, u_5)^T, \quad \langle\bar{u},\bar{v}\rangle = \sum_{i=4}^{5}\int_{\Omega} u_i v_i d\Omega, \quad \bar{A} = A_0^{-1}A,$$

where here $\Omega = (0, l_1)\times(0, l_2)\times(0, l_3)$.

Let $\mathscr{M}$ be a m-dimensional subspace of $H_0^1(\Omega)$ such that D_iD_jw $(i<j)$ and $D_1D_2D_3w$ belong to $L^2(\Omega)$ for $w\in\mathscr{M}$. The equation (5.1) is approximated by

$$(6.1)\quad \varrho\sum_{i=1}^{3}\{(U_{it\bar{t}}^k, v_i)+J_0(U_{it\bar{t}}^k, v_i)\}+a_1(U^k, v)+a_2(U^k, v)$$
$$+a_3(U_{\bar{t}}^k, v)+\langle A_0\bar{U}_{\bar{t}}^k, \bar{v}\rangle+\langle A\nabla\hat{\bar{U}}^k, \nabla\bar{v}\rangle+J_1(\bar{U}_{\bar{t}}^k,\bar{v}) = -\sum_{i=1}^{5}(F_i^k, v)$$

for $\forall v_i \in \mathcal{M}$, $k = 1, \ldots, N-1$ with initial conditions (5.5) and (5.6). Here

$$J_0(z, w) = \theta\tau^2(\nabla z, \nabla w) + \theta^2\tau^4 \sum_{l<j} D_l D_j z, D_l D_j w) + \theta^3\tau^6(D_1 D_2 D_3 z, D_1 D_2 D_3 w),$$

$$J_1(\bar{z}, \bar{w}) = \tau^2 \sum_{l<j} \langle A\bar{A} D_l D_j \bar{z}, D_l D_j \bar{w}\rangle + \tau^3 \langle A\bar{A}^2 D_1 D_2 D_3 \bar{z}, D_1 D_2 D_3 \bar{w}\rangle,$$

θ — positive parameter.

Note that for $z^n = u^n - U^n$ where u^n, U^n are the solutions of (5.1)–(5.3) and of (6.1), (5.5), (5.6), it is possible to obtain a similar estimate to (5.7).

We are now in a position to describe the AD-Galerkin method. Let $\mathcal{M}$ be the subspace of $H_0^1(\Omega)$ such that the basis of $\mathcal{M}$ is a tensor product of the functions of one space variable. Let for $i = 1, 2, 3$

$$\{\alpha_{is}(x_i): s = 1, \ldots, N_i\} \subset H_0^1(0, l_i)$$

and let

$$\mathcal{M}_i = \operatorname{span}(\alpha_{i1}, \ldots, \alpha_{iN_i}),$$
$$\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \mathcal{M}_3.$$

Denote

$$(f, g)_i = \int_0^{l_i} fg\,dx_i.$$

The solution U_i^k of (6.1) is sought for in the form

$$U_i^k(x) = \sum_{s,p,q} \xi_{i,spq}^k \alpha_{spq}(x), \quad \text{where} \quad \alpha_{spq}(x) = \alpha_{1s}(x_1)\alpha_{2p}(x_2)\alpha_{3q}(x_3).$$

Let C_i, A_i be the following matrices

$$C_i = \{(\alpha_{ip}, \alpha_{iq})_i\}_{p,q=1}^{N_i}, \quad A_i = \{(D_i\alpha_{ip}, D_i\alpha_{iq})_i\}_{p,q=1}^{N_i}.$$

Let I_i be the $N_i \times N_i$ identity matrix and let $\bar{I}$ be a 2×2 identity matrix. Using these notations we can rewrite (6.1) as follows

$$\prod_{l=1}^{3} (P_l + \theta\tau^2 Q_l)\xi_{i\bar{t}\bar{t}}^k = \phi_i^k, \quad i = 1, 2, 3, \tag{6.2}$$

$$\prod_{l=1}^{3} (P'_l + \tau Q'_l)\bar{\xi}_{\bar{t}}^k = \bar{\phi}^k, \tag{6.3}$$

where

$$\xi_i^k = \{\xi_{i,spq}\}_{s,p,q=1}^{N_1,N_2,N_3}, \quad \phi_{i,spq}^k = -\frac{1}{\varrho}\Big\{\sum_{j=1}^{3} \{G(D_j U_i^k, D_j\alpha_{spq}) +$$
$$+ (\lambda + G)(D_i U_j^k, D_j\alpha_{spq})\} + (p_\theta D_i U_4^k + p_\mu D_i U_5^k, \alpha_{spq}) + (F_i, \chi_{psq})\Big\}$$

for $i = 1, 2, 3$,

$$P_2 = I_1 \otimes C_2 \otimes I_3, \quad Q_2 = I_1 \otimes A_2 \otimes I_3,$$
$$P'_2 = \bar{I}_1 \otimes (\bar{I} \otimes C_2) \otimes I_3, \quad Q'_2 = I_1 \otimes (\bar{A} \otimes A_2) \otimes I_3.$$

The matrices P_i, P'_i, Q_i, Q'_i are defined in a similar way where $i = 1$ and 3;

$$\bar{\xi}^k = \{\bar{\xi}_{spq}^k\}_{s,p,q=1}^{N_1,N_2,N_3}, \quad \bar{\xi}_{spq}^k = \{\xi_{4,spq}^k, \xi_{5,spq}^k\}.$$

The vectors $\bar{\phi}^k$, $\bar{U}^k$, $\bar{\alpha}_{spq}$ are defined similarly; $\bar{G}^k = \{f^k, g^k\}$;

$$\phi_{spq} = -\left\{\left\langle\left(\sum_{j=1}^{3} D_j U_{j\tilde{t}}^k\right) A_0^{-1}\bar{p},\ \bar{\alpha}_{spq}\right\rangle + \langle \bar{A}\nabla\bar{U}^{k-1}, \nabla\bar{\alpha}_{spq}\rangle + \langle A_0^{-1}\bar{G}^k, \bar{\alpha}_{spq}\rangle\right\}.$$

The Eqs. (6.2) and (6.3) are considered under the following initial conditions:

$$(U_i^0, \alpha_{spq}) = (u_{0i}, \alpha_{spq}), \quad i = 1, \ldots, 5 \tag{6.4}$$

where $u_{04} = \theta_0, u_{05} = \mu_0$;

$$(U_i^1, \alpha_{spq}) = (u_{1i}, \alpha_{spq}), \quad i = 1, \ldots, 5, \tag{6.5}$$

where u_{1i} is defined in (5.6).

Let us define a basis of $\mathcal{M}$ which is convenient in numerical calculations. Let π_i denote the grid on $[0, l_i]$ of the form

$$\pi_i = \{x_i : x_i = jh_i, \quad j = 0, \ldots, N_i+1, \quad (N_i+1)h_i = l_i\},$$

and let $w_p(x_i) = (x_i - ph_i)/h_i$.

The functions $\alpha_{ip}(x_i)$ are defined by

$$\alpha_{ip}(x_i) = \begin{cases} w_{p-1}(x_i), & x_i \in [(p-1)h_i, h_i] \\ 1-w_p(x_i), & x_i \in [ph_i, (p+1)h_i] \\ 0, & x_i \in [0, (p-1)h_i] \cup [(p+1)h_i, l_i] \end{cases} \tag{6.6}$$

for $p = 1, \ldots, N_i$.

The matrices C_i and A_i are now tridiagonal. Hence the total number of arithmetic operations needed to solve (6.2) and (6.3) is of order of $N \times N_1 \times N_2 \times N_3$.

THEOREM 6.1. Let the assumptions of theorem 5.1 hold. Besides, let the following functions $D_1 D_2 D_3 u_i$, $D_0^2 D_s D_p u_i$ for $i = 1, \ldots, 5$, $s, p = 1, 2, 3$ belong to $L^2(Q_T)$, where u is the solution of (5.1)–(5.3). Then AD-Galerkin method of (6.2)–(6.5) with the basis (6.6) is convergent if $\tau \to 0$ and $h \to 0$, where $h = \max\{h_1, h_2, h_3\}$. Moreover

$$||z||_U = 0(\tau^2 + h),$$

where $||\cdot||_U$ is defined in (5.7), $z_i^k = u_i^k - U_i^k$, $i = 1, \ldots, 5$; u_i^k, U_i^k are the solutions of (5.1)–(5.3) and (6.1), (5.5), (5.6). It is possible to verity that this theorem follows from the estimate (5.7) which holds for z^k, and from the fact that

$$||u - \tilde{u}||_{H^1(\Omega)} = 0(h)$$

provided $u \in H^2(\Omega)$, where $\tilde{u}$ is the projection of u into $\mathcal{M}$ (see [8]).

References

1. W. NOWACKI, *Thermodiffusion in solids*, Mech. Teoret. Stos., **13**, 2, 143–158, 1975.
2. G. FICHERA, *Uniqueness, existence and estimate of the solution in the dynamical problem of thermodiffusion in an elastic solid*, Arch. of Mech., **26**, 5, 903–920, 1974.
3. A. A. SAMARSKIJ, *Vvedenie v teoriju raznostnych schem*, Nauka, Moskva 1971.

4. E. G. DJAKONOV, *Raznostnye metody rešenija kraevych zadač*, Wyp. 2, Moskovskij Univ., 1973.
5. I. DOUGLAS, Jr., T. DUPONT, *Alternating direction Galerkin methods on rectangles. Numerical solution of partial differential equations, II*, Academic Press, N. York 1972.
6. O. A. LADYZENSKAJA, *Kraevye zadači matematičeskoj fiziki*, Nauka, Moskva 1973.
7. M. DRYJA, *Absolutnaja ustoičvost raznostnych schem*, ŻWMiMF, 15, 966–976, 1975.
8. G. STRANG, G. J. FIX, *An analysis of the finite element method*, Prentice-Hall, Englewood Gliffs, N-Y 1973.

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