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On elastic-plastic wave propagation; transmission of elastic-plastic boundaries

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THE PURPOSE of the present paper is two-fold. Thus, in the first instance, a general approach to the governing equations of one-dimensional elastic-plastic wave propagations (in the absence of strain rate and lateral inertia effects) is presented via the Bergman integral operation method. Termination of the Bergman series is shown to occur in a simple manner for certain multi-parameter non-linear constitutive laws of significance. Secondly, there is a discussion of the important concept of "elastic-plastic boundary". In particular, the propagation of the elastic-plastic boundary for a semi-infinite medium subjected to a monotonically increasing and then monotonically decreasing load at its open end is investigated.

Cel niniejszej pracy jest dwojaki; po pierwsze, przedstawia ogólne podejście do równań rządzących jednowymiarową propagacją fal sprężysto-plastycznych (z pominięciem wpływu prędkości odkształceń i bezwładności poprzecznej) na podstawie metody operatorów całkowych Bergmana. Na przykładzie pewnych wieloparametrowych nieliniowych równań konstytutywnych pokazano proces zakończania szeregów Bergmana. Po drugie, praca zawiera dyskusję ważnego pojęcia "granicy sprężysto-plastycznej". W szczególności zbadano propagację tej granicy dla ośrodka półnieskończonego poddanego działaniu obciążeń, które najpierw monotonicznie wzrastają, a następnie również monotonicznie maleją.

Настоящая работа имеет двойную цель. Во-первых, представлен общий подход к уравнениям описывающим одномерное распространение упруго-пластических волн (с пренебрежением влияния скорости деформаций и поперечной инерции), опираясь на метод интегральных операторов Бергмана. На примере некоторых многопараметрических нелинейных определяющих уравнений показан процесс окончания рядов Бергмана. Во-вторых, работа содержит обсуждение важного понятия „упруго-пластический предел". В частности исследовано распространение этого предела для полубесконечной среды, подвергнутой действию нагрузок, которые сначала монотонно возрастают, а затем тоже монотонно убывают.

1. Introduction

THE PHENOMENON of plastic deformation arises naturally when materials are subjected to large disturbances such as those produced by the detonation of explosives. Its study is of importance, in particular, in connection with the construction of impact-resistant structures. The present paper is concerned with one-dimensional wave propagation in elastic-plastic materials; strain-rate and lateral inertia effects are neglected. A general discussion on the hodograph system governing the plastic region is presented via the Bergman integral operator approach (BERGMAN [1]). It is shown that termination of the Bergman series after two terms occurs for precisely those non-linear stress-strain laws generated by ROGERS and CLEMENTS [2] via Baecklund transformations. For these multi-parameter constitutive laws, the hodograph equations are readily integrated. Such stress-strain laws and their application in non-linear elasticity have been discussed recently by CEKIRGE and VARLEY [3], KAZAKIA and VARLEY [4], ERINGEN and SUHUBI [5] and ROGERS [6]. Thus, a similar reduction in analysis is available, in principle, for the analysis of elastic-plastic wave propagation problems (see COURANT and FRIEDRICHS [7, p. 246]).

An important concept associated with elastic-plastic wave propagation is that of the "elastic-plastic (*E-P*)" boundary, that is, the boundary of irreversible deformation separating the elastic region and the elastic-plastic region in which the deformations are beyond the elastic limit. The determination of the propagation of this boundary has been discussed by several authors, such as LEE [8], SKOBEEV [9], CLIFTON and BODNER [10] and BEVILACQUA [11]. The second part of this paper concerns the derivation of the *E-P* boundary for a semi-infinite medium modelled, in part, by a Bell law, when subjected to monotonically increasing and then decreasing loads. Numerical results are presented for aluminium.

2. The hodograph equations

A Lagrangian formulation is adopted wherein both the material coordinate X and the spatial coordinate x are referred to the same fixed Cartesian system. Thus,

$$x = x(X, t)$$

denotes the position of a typical particle at time t so that

$$x = X+\bar{u},$$

where $\bar{u}$ is the particle displacement in the deformation. The strain and particle velocity at the point x at time t are

$$e = \frac{\partial x}{\partial X}-1, \quad u = \frac{\partial x}{\partial t}$$

whence,

$$\frac{\partial e}{\partial t} = \frac{\partial u}{\partial X}. \tag{2.1}$$

The equation of motion in one-dimensional wave propagations is

$$\frac{\partial T}{\partial X} = \varrho_0 \frac{\partial u}{\partial t}, \tag{2.2}$$

where T and ϱ_0 are, respectively, the stress and density of the undeformed state. When the dynamic response of the medium is isotropic and homogeneous with respect to the undeformed state,

$$T = T(e), \tag{2.3}$$

Eq. (2.2) can be written as

$$a^2(e) \frac{\partial e}{\partial X} = \frac{\partial u}{\partial t}, \tag{2.4}$$

where

$$a^2(e) = \frac{1}{\varrho_0} \frac{dT}{de}. \tag{2.5}$$

Hence, the governing equations (2.1), (2.4) may be written in the convenient matrix from

(2.6) $$\Omega_t = \begin{Bmatrix} 0 & a^2(T) \\ \varrho_0^{-1} & 0 \end{Bmatrix} \Omega_X, \quad \Omega = \begin{Bmatrix} T \\ u \end{Bmatrix}.$$

The hodograph transformation is now introduced wherein the roles of the dependent and independent variables are interchanged. Thus,

$$\frac{\partial}{\partial X}\bigg|_t = \frac{\partial T}{\partial X}\frac{\partial}{\partial T}\bigg|_u + \frac{\partial u}{\partial X}\frac{\partial}{\partial u}\bigg|_T, \qquad \frac{\partial}{\partial t}\bigg|_X = \frac{\partial T}{\partial t}\frac{\partial}{\partial T}\bigg|_u + \frac{\partial u}{\partial t}\frac{\partial}{\partial u}\bigg|_T,$$

so that

$$\frac{\partial}{\partial u}\bigg|_T = \left[\frac{\partial T}{\partial x}\frac{\partial}{\partial t} - \frac{\partial T}{\partial t}\frac{\partial}{\partial x}\right]\Big/ J, \qquad \frac{\partial}{\partial T}\bigg|_n = \left[\frac{\partial u}{\partial t}\frac{\partial}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial}{\partial t}\right]\Big/ J,$$

where $J = \frac{\partial T}{\partial x}\frac{\partial u}{\partial t} - \frac{\partial T}{\partial t}\frac{\partial u}{\partial x}$. Consequently,

$$X_u = -T_t/J, \quad X_T = u_t/J,$$
$$t_u = T_x/J, \quad t_T = -u_x/J$$

and the system (2.6) becomes

(2.7) $$\begin{bmatrix} t \\ X \end{bmatrix}_t = \begin{bmatrix} 0 & 1/\varrho_0 a^2(T) \\ 1/\varrho_0 & 0 \end{bmatrix}\begin{bmatrix} t \\ X \end{bmatrix}_u.$$

Introduction of the new strain measure

(2.8) $$\phi = \frac{1}{\varrho_0}\int\limits_0^T \frac{1}{a}\,dT = \int\limits_0^e a\,de,$$

reduces (2.7) the canonical form

(2.9) $$\begin{bmatrix} t \\ X \end{bmatrix}_\phi = \begin{bmatrix} 0 & K(\phi)^{-1} \\ K(\phi) & 0 \end{bmatrix}\begin{bmatrix} t \\ X \end{bmatrix}_u,$$

where

(2.10) $$K(\phi) = a(T).$$

Hence,

(2.11) $$\frac{\partial}{\partial\phi}\left[K\frac{\partial t}{\partial\phi}\right] = K\frac{\partial^2 t}{\partial u^2}$$

so that, if we set

(2.12) $$t^* = K^{\frac{1}{2}}t,$$

then (2.11) becomes

(2.13) $$\frac{\partial^2 t^*}{\partial\phi^2} - \frac{\partial^2 t^*}{\partial u^2} + Mt^* = 0,$$

where

(2.14) $$M = -(K^{\frac{1}{2}})_{\phi\phi}/K^{\frac{1}{2}}.$$

3. The Bergman series approach

Solutions of (2.13) are sought in the form

$$t^{*(1)} = \sum_{n=0}^{\infty} h_n(\phi) F_n(u+\phi), \tag{3.1}$$

$$t^{*(2)} = \sum_{n=0}^{\infty} g_n(\phi) G_n(u-\phi), \tag{3.2}$$

where

$$F'_n = F_{n-1}, \tag{3.3}$$

$$G'_n = G_{n-1}, \quad n = 1, 2, \ldots. \tag{3.4}$$

It is observed that

$$f = {}^1/_2[\phi - u], \tag{3.5}$$

$$g = {}^1/_2[\phi + u] \tag{3.6}$$

are the Riemann invariants. They are constant respectively along any one α- or β-characteristic which propagate from left to right and right to left, respectively, according to the respective equations

$$dXdt|_\alpha = a(e), \tag{3.7}$$

$$dX/dt|_\beta = -a(e). \tag{3.8}$$

The formal substitution of (3.1) into (2.13) shows that the latter admits such solutions if the recurrence relations

$$2h_{n+1,\phi} + h_{n,\phi\phi} + Mh_n = 0, \quad n = 0, 1, 2, \ldots, \quad h_0 = \text{constant} \tag{3.9}$$

are obtained. If we take $h_0 = 1$, it is seen that (3.9) provides the 'transport equations' for the h_n in the form

$$h_n = -{}^1/_2 \int_{\phi_n}^{\phi} [h_{n-1,\phi\phi} + Mh_{n-1}] d\phi, \quad n = 1, 2, \ldots \tag{3.10}$$

where the ϕ_i are arbitrary constants of integration. Hence, the h_n may be generated once the $T(e)$ constitutive law is specified.

The recurrence relation (3.3) provides, on integration,

$$F_n = \frac{2^n}{n!} \int_0^g (g-g_1)^n F'_0(g) dg_1 \tag{3.11}$$

whence we obtain the formal solution

$$t^{*(1)} = \sum_{n=0}^{\infty} \frac{2^n h_n(\phi)}{n!} \int_0^g (g-g_1)^n F'_0(g_1) dg_1 = \int_0^g F'_0(g_1) \sum_{n=0}^{\infty} \left[\frac{2^n}{n!} h_n(\phi)(g-g_1)^n\right] dg_1, \tag{3.12}$$

where the interchange of the process of summation and integration is valid in the region of uniform convergence of

(3.13) $$\sum_{n=0}^{\infty} \frac{2^n h_n(\phi)}{n!} (g-g_1)^n.$$

In a similar manner, it is seen that (2.13) admits solutions of the form

(3.14) $$t^{*(2)} = \int_0^f G_0'(f) \sum_{n=0}^{\infty} \left[(-1)^n \frac{2^n}{n!} g_n(\phi)(f-f_1)^n \right] df_1$$

valid in region of uniform convergence of

(3.15) $$\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n!} g_n(\phi)(f-f_1)^n$$

and where

(3.16) $$2g_{n+1,\phi} - g_{n,\phi\phi} - Mg_n = 0, \quad n = 0, 1, 2, \ldots, \quad g_0 = \text{constant}$$

so that the 'transport equations' for the g_n are (with $g_0 = 1$)

(3.17) $$g_n = {}^1/_2 \int_{\phi_n}^{\phi} [g_{n,\phi\phi} + Mg_n] d\phi, \quad n = 1, 2, \ldots$$

where again, the ϕ_i are arbitrary constants of integration.

If there is termination in the Bergman series (3.1) so that

$$g_{N+k} = 0, \quad K > 0,$$

then

(3.18) $$g_{N,\phi\phi/g_N} = (K^{\frac{1}{2}})_{,\phi\phi}/K^{\frac{1}{2}},$$

whence

(3.19) $$g_N = K^{\frac{1}{2}}\left\{\bar{\alpha} + \bar{\beta}\int K^{-1} d\phi\right\},$$

where α, β are arbitrary constants of integration. In particular, if $N = 0$ so that the Bergman series terminates after a single term, then

(3.20) $$K = [\beta\phi + \delta]^2,$$

where β, δ are arbitrary constants of integration. The relation (3.20) leads to the three-parameter (T, e) — constitutive laws (Rogers and Clements [2])

(3.21) $$T = -\lambda[e+\mu]^{-3} - \nu,$$

where $\lambda > 0$ and ν are arbitrary constants. Similarly, the Bergman series (3.2) terminates after a single term in this case. From (2.13) it is seen that

(3.22) $$t^* = [\beta\phi + \delta] = (\phi - u) + G(\phi + u)$$

while (2.9) produces

$$X_u = [\beta\phi+\delta]^2 t_\phi = -\beta\{F(\phi-u)+G(\phi+u)\}+(\beta\phi+\delta)\{F'(\phi-u)+G'(\phi+u)\},$$

whence

(2.23) $$X = \beta\{F^*(\phi-u)-G^*(\phi+u)\}+(\beta\phi+\delta)\{-F(\phi-u)+G(\phi+u)\},$$

where $F \equiv F^{*\prime}$, $G \equiv G^{*1}$.

If $N = 1$, so that the Bergman series terminates after at most two terms, then (3.19)

$$2\left[\bar{\alpha}(K^{\frac{1}{2}})(K^{\frac{1}{2}})_\phi+\bar{\beta}(K^{\frac{1}{2}})_\phi K^{\frac{1}{2}}\int K^{-1}d\phi\right] = (K^{\frac{1}{2}})_{\phi\phi}.$$

In particular, if $\bar{\beta} = 0$, then

(3.24) $$(K^{\frac{1}{2}})_\phi-\bar{\alpha}(K^{\frac{1}{2}})^2-\bar{\delta} = 0,$$

where $\bar{\delta}$ is an arbitrary constant of integration. The Riccati equation (3.24) generates precisely the stress-strain laws treated by CEKIRGE and VARLEY in [3] and associated with the Baecklund transformation method by ROGERS [6].

It may be noted that the solution (3.22), (3.23) of the hodograph equations for the stress-strain law (3.21) is appropriate for the solutilon of the initial value problem

(3.25) $$\phi(X, 0) = 1,$$

(3.26) $$u(X, 0) = V^*(X).$$

In the hodograph plane these conditions become

(3.27) $$t(V, 1) = 0,$$

(3.28) $$X(V, 1) = X^*(V),$$

where X^* and V^* are inverse functions.

Application of (3.22) and the initial condition (3.27) shows that $G(\xi) = -F(2-\xi)$, whence,

(3.29) $$t = \frac{1}{[\beta\phi+\delta]}[F(\phi-u)-F(2-\{\phi+u\})],$$

(3.30) $$X = \beta\left\{\int[F(2-\{\phi+u\})-F(\phi-u)]du\right\}-[\beta\phi+\delta]\,[F(\phi-u)+F(2-\{\phi+u\})].$$

Combination of (3.30) and the initial condition (3.28) shows that

(3.31) $$X_{t=0} = X^*(V) = -2[\beta+\delta]F(1-\mathrm{V}).$$

Thus, the solution of the initial value problem is given by

(3.32) $$t = \frac{1}{2(\beta\phi+\delta)(\beta+\delta)}\{X^*(-1+\phi+u)-X^*(1-\Phi+u)\},$$

(3.33) $$X = \frac{\beta}{2(\beta+\delta)}\int[X^*(1-\phi+u)-X^*(-1+\phi+u)]du$$
$$+(\beta\phi+\delta)\,[X^*(1-\phi+u)+X^*(-1+\phi+u].$$

A further class of solutions of the hodograph equations may be readily generated. Thus, if we set $\phi^* = 2\phi$, then (3.16) becomes

(3.34) $$4g_{n,\phi^*\phi^*}+Mg_n^*-4g_{n+1,\phi^*} = 0, \quad n = 0, 1, 2, \ldots, \quad g_0 = \text{const}.$$

If M is specialised to be of the form

$$M = \frac{4c}{(b-\phi^*)^2} \tag{3.35}$$

then (3.34) becomes

$$g_{n+1,\phi*} = g_{n,\phi*\phi*}+c(b-\phi^*)^{-2}g_n, \quad n = 0, 1, 2, \ldots, \quad g_0 = \text{const}. \tag{3.36}$$

It is assumed, for convenience, that $g_0 = 1$.

The solution of the recurrence relations (3.36) is sought in the form

$$g_n = \sum_{j=0}^{n} a_j \mu_{j,n} n!(b-\phi^*)^{-n+j} \tag{3.37}$$

where the $a_j, j = 0, 1, 2, \ldots, n$, are arbitrary constants. Hence, on substitution, it is seen that constants μ_{jn} obey the recurrence relations

$$[(n+1)(n+1-j)]\mu_{j,n+1} = [(n-j)(n-j+1)+c]\mu_{jn}, \quad j = 0, 1, 2, \ldots, n,$$

$$\mu_{jn} = \begin{cases} 0, & j > n, \\ 1, & j = n = 0. \end{cases} \tag{3.38}$$

Thus,

$$t^{*(2)} = \int_0^f G_0'(f) \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} a_j \mu_{jn}(b-\phi^*)^j \right] (-1)^n 2^n \left[\frac{f-f_1}{b-\phi^*}\right]^n df_1 \tag{3.39}$$

valid in the region of uniform convergence of

$$\sum_{n=0}^{\infty} \left[\sum_{j=0}^{n} a_j \mu_{jn}(b-\phi^*)^j \right] (-1)^n 2^n \left[\frac{f-f_1}{b-\phi^*}\right]^n. \tag{3.40}$$

In particular, if $j = 0$, the $\mu_{0,n}$ obey the same recurrence relations as the $\mu_{0,\nu}$ in the hypergeometric series

$${}_2H_1(\alpha^*, \beta^*; 1; x) = \sum_{\nu=0}^{\infty} \mu_{0,\nu} x^\nu, \tag{3.41}$$

where

$$a^* = {}^1/_2 - ({}^1/_4 - c)^{\frac{1}{2}}, \tag{3.42}$$

$$\beta^* = {}^1/_2 + ({}^1/_4 - c)^{\frac{1}{2}}. \tag{3.43}$$

Thus, if

$$a_j = \begin{cases} 1, & j = 0, \\ 0; & j > 0, \end{cases} \tag{3.44}$$

then the particular solution

$$g_n = n!\, \mu_{0,n}(b-\phi^*)^{-n} \tag{3.45}$$

is obtained, whence

$$(3.46)\qquad \sum_{n=0}^{\infty}\frac{(-1)^n 2^n g_n(\phi)(f-f_1)^n}{n!} = \sum_{n=0}^{\infty}\frac{g_n(\phi)[u-\phi+2f_1]^n}{n!} = {}_2H_1\left(\alpha^*, \beta^*;\ 1;\left[\frac{u-\phi+2f_1}{b-\phi^*}\right]\right).$$

The validity of the associated solution

$$(3.47)\qquad t^{*(2)} = \int_0^f G_0'(f)\,{}_2H_1\left(\alpha^*, \beta^*;\ 1;\left[\frac{(f_1-f)}{b-2\phi}\right]\right)df_1$$

is assured in the region

$$(3.48)\qquad -1 < \frac{2(f_1-f)}{b-2\phi} < +1.$$

Similar results hold for the solutions $t^{*(1)}$ involving the Riemann invariant g.

In view of (3.35),

$$(K^{\frac{1}{2}})_{\phi^*\phi^*} + c(b-\phi^*)K^{\frac{1}{2}} = 0$$

so that

$$(3.49)\qquad K^{\frac{1}{2}} = \gamma_1|b-\phi^*|^{\alpha^*} + \gamma_2|b-\phi^*|^{\beta^*},$$

where α^*, β^* are defined by (3.42), (3.43) and γ_1, γ_2 are arbitrary constants of integration. The specialisation $\alpha^* = 0$, $\beta^* = 1$ generates the case (3.20) leading to the stress-strain law (3.21).

This completes the general discussion. The subsequent work is concerned with the propagation of elastic-plastic boundaries in connection with a semi-infinite medium subjected to increasing and then decreasing loading at the open section.

4. Propagation of elastic-plastic boundaries

When a semi-infinite medium is subjected to a loading at the open section, the pulse propagates as a simple wave, namely $g = 0$. The Riemann invariant at the boundary is assigned as

$$(4.1)\qquad f = F(\alpha),$$

where α is the time measure at the boundary, whilst, through the relations (3.5), (3.6),

$$(4.2)\qquad \phi = F(\alpha) \quad \text{and} \quad u = -F(\alpha).$$

Equation (3.7) then provides

$$(4.3)\qquad t = \alpha + X/a(\alpha),$$

where $a(\alpha)$ is defined by the relations (2.5), (2.8) and (4.2).

The theory of wave propagation is the same for elastic and elastic-plastic materials for a loading in which the stress at the boundary of a semi-infinite medium increases continuously. When the stress at the boundary begins to decrease, the wave propagation differs in elastic-plastic materials. A unique state equation governs both loading and unloading processes in elastic materials. In elastic-plastic materials loading and unloading occur according to different relations, and hysteresis effects cannot be neglected. Fig. 1 shows the dynamical stress-strain relation of polycrystalline metals such as aluminium, zinc, silver etc.. The material is loaded along $OABE$ and plastic deformations start at the elastic limit A. The OA portion of the curve is a straight line and unloading from any point on ABE occurs along a line parallel to OA. CD is the magnitude of strain released during the unloading and OC is the irreversible strain that depends upon the level of stress at which the unloading starts. The reloading from the zero stress level C occurs along CBE.

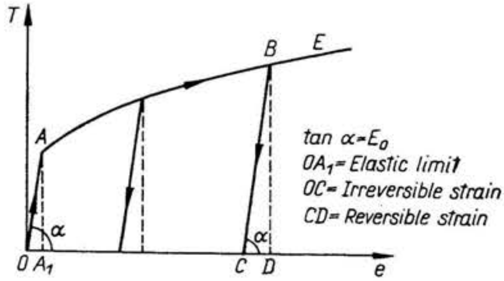

FIG. 1. Stress-strain relation for a polycrystalline material.

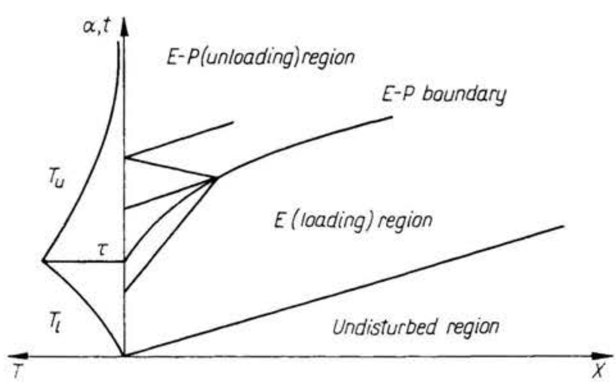

FIG. 2. Loading and unloading regions in the characteristic field.

When the stress, which is beyond the elastic limit, starts to decrease at the boundary, the medium shows an elastic-plastic response. At this stage, the unloading state equation must account for the wave motion. The state equations $T_e = T_e(e)$ and $T_p = T_p(e)$ constitute, respectively, elastic and elastic-plastic regions, see Fig. 2. In the elastic region E, the medium is loaded and

$$\partial T_e/\partial t > 0, \quad \partial e/\partial t > 0; \tag{4.4}$$

in the elastic-plastic region E-P, the medium is unloaded with

(4.5) $$\partial T_p/\partial t < 0, \quad \partial e/\partial t < 0.$$

The regions E and $E = P$ are separated by the $E-P$ boundary $t_p = \psi(X_p)$, which is not known 'a priori'. It is this boundary that is sought. In the region E, the pulse propagates as a simple wave according to the relation (4.3), so that the information at the boundary is carried along the α-characteristics. The $E-P$ boundary starts to propagate when the stress begins to decrease at the boundary at $\alpha = \tau$ (see Fig. 2). At a point X_p, the strain increases continuously until $t_p = \psi(X_p)$ when the $E-P$ boundary is reached. Then, the elastic component of the strain reverses and the same point is then included in the $E-P$ region. The behaviour of the function $t_p = \psi(X_p)$ depends strongly on the boundary conditions assingned at $X = 0$.

Let us proceed to obtain the solution of the equations of motion in the $E-P$ region. The state equation in the $E-P$ region may be written as

(4.6) $$T_p = T_b(X_p)+E_0\,[e-e_b(X_p)],$$

where $T_b(X_p)$ and $e_b(X_p)$ are, respectively, the values of the stress and strain at the time $t_p = \psi(X_p)$ at which the $E-P$ boundary reaches the point X_p; E_0 is the linear elasticity modulus of the material. Through the equations (2.1) and (2.2) the particle velocity and the stress are given as

(4.7) $$u = g(t+X/a_0)+f(t-X/a_0),$$

(4.8) $$T = (E_0/a_0)\,[g(t+X/a_0)-f(t-X/a_0)],$$

for the region governed by the relation (4.6), where $a_0 = \sqrt{E_0/\varrho_0}$, and f and g are unknown functions.

With the parameter α being a natural time measure at $X = 0$, the stress at the boundary is assigned as

(4.9) $$T = \begin{cases} T_l(\alpha), & 0 \leqslant \alpha \leqslant \tau, \\ T_u(l), & \tau < \alpha < \infty, \end{cases}$$

where T_l and T_u are, respectiviely, monotonically increasing and decreasing functions. Namely, at first, when $0 \leqslant \alpha \leqslant \tau$, the medium is loaded and then it is unloaded. At time $\alpha = \tau$ the $E-P$ boundary is formed at the boundary and the initial condition of the $E-P$ boundary is

(4.10) $$X_p(\alpha = \tau) = 0.$$

In this paper, we are concerned with the $E-P$ boundaries across which stress and particle velocity are continuous, but their derivatives are discontinuous [8]. Hence at the $E-P$ boundary

(4.11) $$T_l = T_p \quad \text{and} \quad U_l = U_p,$$

where T_l and U_l are, respectively, the stress and particle velocity at the elastic side of the $E-P$ boundary and T_p and U_p are, respectively, the stress and particle velocity at the elastic-plastic side of the $E-P$ boundary.

It should be noted that the pulse propagates as a simple wave in the elastic region E. Equations (2.3), (2.8) and (4.2) give that

$$U_l = U_l(\alpha). \tag{4.12}$$

The evaluation of the function $U_l(\alpha)$ for a centred simple wave will be given in the second example of Sect. 5. Then, Eqs. (4.7)–(4.9), (4.11) and (4.12) yield

$$U_l(\alpha) = g(t_p+X_p)/a_0)+f(t_p-X_p/a_0), \tag{4.13}$$

$$T_l(\alpha) = (E_0/a_0)\,[g(t_p+X/a_0)-f(t_p-X_p/a_0)], \tag{4.14}$$

and also

$$t_p = \alpha+X_p/a(\alpha). \tag{4.15}$$

At the boundary $X = 0$, the equations (4.8), and (4.9) produce

$$T_u(\alpha) = (E_0/a_0)\,[g(\alpha)-f(\alpha)]. \tag{4.16}$$

By insertion of equations (4.15) and (4.16) into equations (4.13) and (4.14), the unknown function g is eliminated, and hence

$$U_l(\alpha)-\frac{a_0}{E_0}T_u\left(\alpha+\frac{X_p}{a(\alpha)}+\frac{X_p}{a_0}\right) = f\left(\alpha+\frac{X_p}{a(\alpha)}+\frac{X_p}{a_0}\right)+f\left(\alpha+\frac{X_p}{a(\alpha)}-\frac{X_p}{a_0}\right), \tag{4.17}$$

$$\frac{a_0}{E_0}\left[T_l(\alpha)-T_u\left(\alpha+\frac{X_p}{a(\alpha)}+\frac{X_p}{a_0}\right)\right] = f\left(\alpha+\frac{X_p}{a(\alpha)}+\frac{X_p}{a_0}\right)-f\left(\alpha+\frac{X_p}{a(\alpha)}-\frac{X_p}{a_0}\right). \tag{4.18}$$

Combination of these equations leads to

$$\alpha+\frac{X_p}{a(\alpha)}+\frac{X_p}{a_0} = f^{-1}\;[\psi_1(\alpha)], \tag{4.19}$$

$$\alpha+\frac{X_p}{a(\alpha)}-\frac{X_p}{a_0} = f^{-1}\;[\psi_2(\alpha)], \tag{4.20}$$

where

$$\psi_1(\alpha) = {}^1/_2\left\{u_l(\alpha)+\frac{a_0}{E_0}\left[T_l(\alpha)-2T_u\left(\alpha+\frac{X_p(\alpha)}{a(\alpha)}+\frac{X_p(\alpha)}{a_0}\right)\right]\right\}, \tag{4.21}$$

$$\psi_2(\alpha) = {}^1/_2\left[u_l(\alpha)-\frac{a_0}{E_0}T_l(\alpha)\right], \tag{4.22}$$

and f^{-1} is the inverse function of f. If we set

$$a^* = \psi_2^{-1}\{\psi_1[\alpha, X_p(\alpha)]\} = \varkappa[\alpha, X_p(\alpha)], \tag{4.23}$$

where ψ_1^{-1} is the inverse function of ψ_1, the equation (4.20) becomes

$$\varkappa[\alpha, X_p(\alpha)]+\frac{X_p\{\varkappa[\alpha, X_p(\alpha)]\}}{a\{\varkappa[\alpha, X_p(\alpha)]\}}-\frac{X_p\{\varkappa[\alpha, X_p(\alpha)]\}}{a_0}f^{-1}[\psi_1(\alpha)]. \tag{4.24}$$

Thus, (4.20) and (4.24) provide a single functional equation,

$$X_p(\alpha) = L\{\alpha, \varkappa[\alpha, X_p(\alpha)]\}+K\{\alpha, \varkappa[\alpha, X_p(\alpha)]\}X_p\{\varkappa[\alpha, X_p(\alpha)]\}, \tag{4.25}$$

where

(4.26) $$L[\alpha, X_p(\alpha)] = a_0 a(\alpha)\{\varkappa[\alpha, X_p(\alpha)]-\alpha\}/[a_0+(\alpha)],$$

(4.27) $$K[\alpha, X_p(\alpha)] = a(\alpha)\{a_0-a\{\varkappa[\alpha, X_p(\alpha)]\}\}/a\{\varkappa[\alpha, X_p(\alpha)]\}\,[a_0+a(\alpha)].$$

Equation (4.25) may be written in the form

(4.28) $$X_p(\alpha_n^m) = \{X_p(\alpha_{n-1}^m)-L[\alpha_{u-1}^m, X_p(\alpha_{n-1}^m)]\}/K[\alpha_{n-1}^m, X(\alpha_{n-1}^m)]$$

through which the coordinates of the $E-P$ boundary are obtained from the condition $X_p(\alpha_0^0) = 0$, where the subscripts n are defined by

(4.29) $$\alpha_n^m = \varkappa[\alpha_{n-1}^m, X_p(\alpha_{n-1}^m)].$$

Equation (4.28) gives the coordinates of the $E-P$ boundary when $m = 0$, namely at the discrete values of α. If more boundary points are necessary in any interval $\alpha_{n-1}^0 < \alpha < \alpha_n^0$, then we start by setting

(4.30) $$\alpha_{n-1}^m = \alpha_{n-1}^0+m\Delta\alpha,$$

and equation (4.25) can be written as

(4.31) $$X_p(\alpha_{n-1}^m) = L\{\alpha_{n-1}^m, \varkappa[\alpha_{n-1}^m, X_p(\alpha_{n-1}^{m-1})]\}+K\{\alpha_{n-1}^m, \varkappa[\alpha_{n-1}^m, X_p(\alpha_{n-1}^{m-1})]\}X_p(\alpha_n^{m-1}).$$

Equations (4.28) and (4.31) then determine the space coordinates of the $E-P$ boundary in the $t-X$ plane. It should be noted that the values of Eqs. (4.28) and (3.21) then determine the space coordinates of the $E-P$ boundary in the $t-X$ plane. It should be noted that the value of

(4.32) $$\Delta\alpha = (\alpha_n^0-\alpha_{n-1}^0)/k$$

has to be chosen to satisfy the condition

(4.33) $$|X_p(\alpha_{n-1}^k)-X_p(\alpha_n^0)| < \varepsilon$$

for admissible values of ε. The time coordinate of the $E-P$ boundary can be found through the equation (4.3).

5. Applications

Consider the case in which a material behaves as a linear elastic material until the strain reaches the elastic limit and then obeys a parabolic law of the type given by Bell, namely

(5.1) $$\begin{aligned} T &= E_0 e, && 0 \leqslant e \leqslant e_e, \\ T &= \beta(e+e_b)^{\frac{1}{2}}, && e_e < e, \end{aligned}$$

where e_e is the elastic limit, and β and e_b are the constants defined by BELL [12]. Unloading beyond the elastic limit occurs according to the relation

(5.2) $$T = T_0+E_0(e-e_0),$$

where T_0 and e_0 are, respectively, the values of stress and strain at which unloading starts. The relations (5.1) and (5.2) define polycrystalline materials. For comparison with experiments, the application of the method is considered for aluminium so that $E_0 = 703.1\times 10^3$

kg/cm², $\varrho_0 = 2.8124 \times 10^{-6}$ kg×sec²×cm⁻⁴, $e_e = 0.001$, $\beta = 39.4 \times 10^2$ kg/cm² and $e_b = 0.03084$ (see BELL [12], CRISTESCU [13]).

The interpretation of the results is faciliated by the use of normalization quantities:

$$a_0 = \sqrt{E_0/\varrho_0} \quad \text{and} \quad T_0 = E_0 e_e. \tag{5.3}$$

Then the normalized quantities, which are shown starred, are written as

$$a^* = a/a_0, \quad u^* = u/a_0, \quad c^* = c/a_0 \quad \text{and} \quad T^* = T/T_0. \tag{5.4}$$

The problem is considered in infinite space and time, so that there is no normalization for length and time. For these variables, thoug artificial, the measures are taken as

$$X^* = X/a_0 \quad \text{and} \quad t^* = t. \tag{5.5}$$

The stars will be dropped in the following calculations.

The main concern in section is to obtain the trajectory of the $E-P$ boundary produced by an impact load. To understand the details of the phenomenon, the following examples are treated:

i) At $X = 0$, the stress is linearly increased and monotonically decreased through a given function, so that

$$\begin{aligned} T_l &= T_m \alpha, & 0 \leqslant \alpha \leqslant 1, \\ T_u &= T_m/\alpha^2, & 1 < \alpha < \infty, \end{aligned} \tag{5.6}$$

where α is the time at the boundary and

$$T_m = (\beta/T_0)(e_m + e_b)^{\frac{1}{2}} \tag{5.7}$$

is the maximum value of the applied stress, $\big(e_m = 0.06$ (TAYLOR [14])$\big)$.

The various regions of the characteristic plane $t-X$ are shown in Fig. 3. The region R_1 is an undisturbed region if the semi-infinite medium is initially at rest. In the region

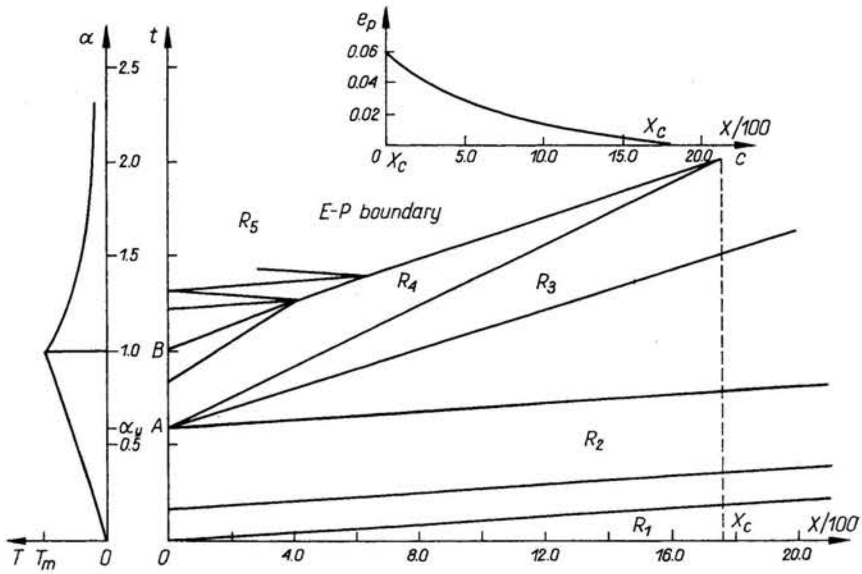

FIG. 3. Trajectory of the *E-P* boundary and distribution of the permanent strains in the medium.

R_2, only elastic waves are present with the wave velocity $a = 1$. The region R_3 is a transition region and the values of strain and stress at the elastic limit are carried at the wave velocities $a_p \leqslant a \leqslant 1$ where a_p is calculated from

$$a = (\sqrt{\beta/2\varrho_0}/a_0)\,(e+e_b)^{-\frac{1}{4}} \tag{5.8}$$

for $e = e_e$. All the characteristics emanate from the point A at which the material passes the elastic limit. The region R_4 is the nonlinear loading region, the disturbances propagate at the wave velocities $a < a_p < 1$. The values of a are found from (5.8) for $e_e < e < e_m$. The region R_5 is the elastic-plastic region, irreversible deformations are present and the deformations propagate at the wave velocities $a = 1$. The $E-P$ boundary separates the regions R_4 and R_5.

In this case

$$\psi_1(\alpha) = \{(\sqrt{8\beta/9\varrho_0}/a_0)\,[e_b^{3/4}-(T_m T_0\,\alpha/\beta)^{3/2}]+T_m T_0\,\alpha/E_0 \\ -2T_m T_0/E_0[\alpha+X_p(\alpha)/a(\alpha)+X_p(\alpha)]^2\}/2, \tag{5.9}$$

$$\psi_2(\alpha) = \{(\sqrt{8\beta/9\varrho_0}/a_0)\,[e_b^{3/4}-(T_m T_0\,\alpha/\beta)^{3/2}]-T_m T_0\alpha/E_0\}/2, \tag{5.10}$$

$$a(\alpha) = (\sqrt{\beta/2\varrho_0}/a_0)(T_m T_0\,\alpha/\beta)^{-\frac{1}{2}}, \tag{5.11}$$

and $\varkappa(\alpha) = \psi_2{}^{-1}\,[\psi_1(\alpha)]$. Hence, the relations (4.28), (4.31) and (4.3) determine the trajectory of the $E-P$ boundary until it reaches the characteristics which separates the regions R_3 and R_4. Therefore the parameter α varies as $\alpha_y < \alpha < 1$, where α_y is the time at the boundary at which the stress exceeds the elastic limit. The initial condition (4.10) for the $E-P$ boundary produces a singularity at $\varkappa(\alpha = 1)$. For the elimination of this singularity, the initial condition of the $E-P$ boundary is written as

$$X_p(\alpha = 1) = \varepsilon, \quad t_p = 1+\varepsilon/a(\alpha = 1), \tag{5.12}$$

where ε has a small value around zero. The numerical example is treated for $\varepsilon = 0.001$. The space ordinate $X_p(\alpha = \alpha_y)$ of the point C is called the "penetration distance", beyond which there are no irreversible deformations and the material has not yielded. The distribution of the permanent strains in the elastic-plastic region can be found, through (5.1), (5.2) and (5.7), when

$$e_p = (T_m T_0/\beta)^2-e_b-(T_m T_0\,\alpha/E_0) \tag{5.13}$$

is plotted against $X_p(\alpha)$ with $\alpha_y \leqslant \alpha \leqslant 1$. This curve is also shown in Fig. 3.

ii) At $X = 0$, the traction is suddenly applied and after the duration time t_d, decreased exponentially, that is, the medium is subjected to an impact load at the boundary,

$$\begin{aligned} T_l &= T_m, & 0 \leqslant \alpha \leqslant t_d, \\ T_u &= T_m \exp[-\lambda(\alpha-t_d)], & t_d < \alpha < \infty, \end{aligned} \tag{5.14}$$

where λ is the unloading rate and $\lambda > 1$.

In this case all the characteristics emanate from the origin, so that a marked difference exists between the features of the $t-X$ plane of this and the previous case. Fig. 4 shown the various regions of the $t-X$ plane. The regions R_1 and R_2 are, respectively, the undisturbed and transition regions. These regions are separated by a shock wave whose velocity is $s = 1$. In the region R_3, the medium is loaded along the nonlinear characteristics.

The region R_4 is the elastic-plastic region. The regions R_3 and R_4 are separated by the $E-P$ boundary which starts from the point B,

(5.15) $$X_B = t_d a(e_m)/[1-a(e_m)], \quad t_B = t_d[2a(e_m)-1]/[a(e)-1)],$$

at which the last loading characteristic intersects the first unloading characteristic. In the region K, which is the constant region, the irreversible strain is equal to its maximum value.

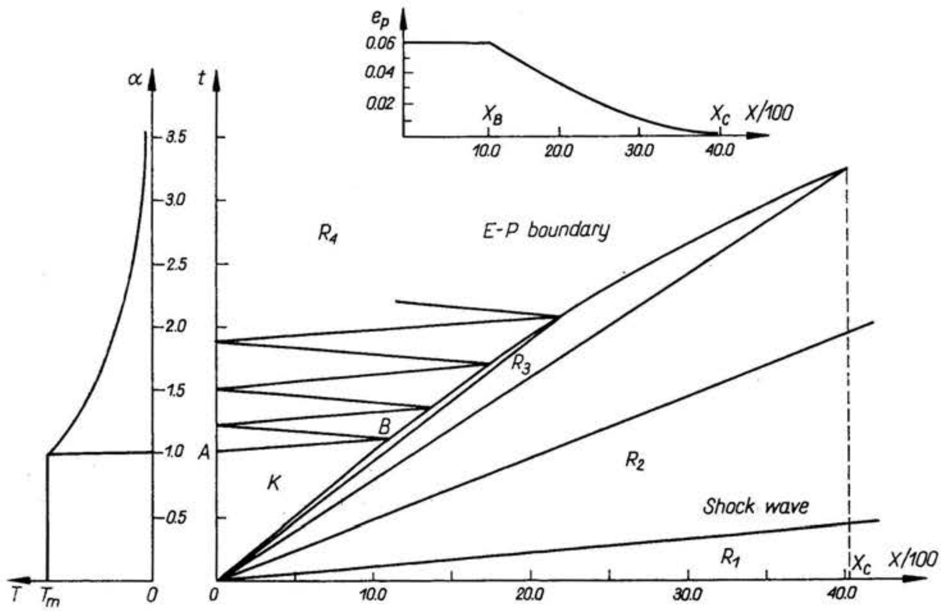

FIG. 4. The characteristic field in the case (ii).

The strain e is chosen as a parameter instead of α, so that

(5.16) $$\psi_1(e) = \{(\sqrt{8\beta/9\varrho_0}/a_0)[e_b^{3/4}-(e+e_b)^{3/4}]+\beta(e+e_b)^{1/2}/E_0 - (2T_m T_0/E_0)\exp\{-\lambda[X_p(e)/a(e)+X_p(e)-r_d)]\}\}/2,$$

(5.17) $$\psi_2(e) = \{(\sqrt{8\beta/9\varrho_0}/a_0)[e_b^{3/4}-(e+e_b)^{3/4}]-\beta(e+e_b)^{1/2}/E_0\}/2,$$

(5.18) $$a(e) = (\sqrt{\beta/2\varrho_0}/a_0)\,(e+e_b)^{-1/4},$$

and

(5.19) $$\varkappa(e) = \psi_2^{-1}[\psi_1(e)].$$

Then, the relations (4.28), (4.31) and

(5.20) $$t_p = X_p(e)/a(e)$$

determine the trajectory of the $E-P$ boundary in the $t-X$ plane. The penetration distance X_c is obtained for $e = e_e$. The numerical results are presented for $\lambda = 1.0$ and $t_d = 1.0$ in Fig. 4 for the same numerical data as in the previous case.

The maximum distance reached by irreversible deformations is termed the penetration distance. Table 1 compares the penetration distances for aluminium in boundary-value problems (i) and (ii) with analogous problems treated by CEKIRGE [15] but with the assumpton of a rigid-plastic material response. It is observed that the penetration distance is greater for elastic-plastic response. This result is to be expected from strain energy considerations.

Table 1. Space and time coordinates of the penetration point in the elastic-plastic and rigid-plastic semi-infinite media.

	Case (i)		Case (ii), $t_d = 0.001$	
	X_p	t_p	X_p	t_p
Elastic-plastic boundary	0.176	2.00	0.16	1.29
Rigid-plastic boundary	0.173	1.973	0.145	1.161

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