On the theory of large, non-isothermic, elastic-plastic and elastic-visco-plastic deformations

Th. LEHMANN (BOCHUM)

IN THE description of elastic-plastic or elastic-viscoplastic deformations we have to distinguish between the description as a thermo-mechanical process and the corresponding description by means of thermodynamic equations. In some previous papers this was pointed out with regard to large non-isothermic elastic-plastic deformations. The aim of this paper is to extend these considerations to large non-isothermic elastic-viscoplastic bodies.

W opisie odkształceń sprężysto-plastycznych lub sprężysto-lepkoplastycznych odróżniać należy charakterystykę procesu termomechanicznego od odpowiedniego opisu za pomocą równań termodynamicznych. We wcześniejszych pracach wskazano na tę różnicę w odniesieniu do przypadku dużych nieizotermicznych odkształceń sprężysto-plastycznych. Celem niniejszej pracy jest rozszerzenie tych rozważań na przypadek nieizotermicznych odkształceń ciał sprężysto-lepkoplastycznych.

В описании упруго-пластических или упруго-вязкопластических деформаций следует различать характеристику термомеханического процесса от соответствующего описания при помощи термодинамических уравнений. В более ранних работах указана эта разница по отношению к случаю больших, неизотермических, упруго-пластических деформаций. Целью настоящей работы является расширение этих рассуждений на случай неизотермических деформаций упруго-вязкопластических тел.

1. Introduction

THIS PAPER deals with the phenomenological theory of elastic-plastic and elastic-viscoplastic bodies. The processes inside the lattice and at the border of the crystal grains may be taken as the physical background. But in this paper the connection between this physical background and the macroscopic behaviour of the material shall not be discussed. In this respect the reader may be referred to the literature (e.g. [1 to 13]).

For simplicity we shall restrict ourselves to so-called elementary processes, i.e. to processes which are homogeneous throughout the body. In this case we do not need the thermic and mechanical field equations. We may focus our considerations on the thermo-mechanical process of one element.

In some recent papers [14 to 17] it has been shown how the usual description of non-isothermic elastic-plastic deformations as a thermo-mechanical process corresponds to the description by means of thermodynamic state equations. The aim of this paper is to extend these considerations to non-isothermic elastic-viscoplastic bodies. For this purpose, after some general remarks, we will briefly repeat the results of the above papers. Then we will discuss the theory from the rate-independent plastic behaviour to the rate-dependent viscoplastic behaviour. We shall see that there is an important difference from the thermodynamical point of view.

2. Definitions and some basic assumptions

We consider the bodies as classical continua and introduce a body-fixed coordinate system ξ^i deforming together with the body. The metric of this coordinate system in the undeformed state (time t_0) is

$$(2.1) \qquad g_{ik}(\xi^r, t_0) = \overset{\circ}{g}_{ik}(\xi^r).$$

In the deformed state the metric is

$$(2.2) \qquad g_{ik} = g_{ik}(\xi^r, t).$$

All quantities will be related to the metric of the coordinate system ξ^i in the deformed state.

The deformation of the body can be measured by the metric transformation tensor

$$(2.2') \qquad q_k^i = \overset{\circ}{g}^{ir} g_{rk}$$

or its inverse

$$(2.2'') \qquad (q^{-1})_k^i = g^{ir} \overset{\circ}{g}_{rk}.$$

The strain rate is defined by

$$(2.3) \qquad d_k^i = \frac{1}{2} g^{ir}(\dot{g})_{rk} = -\frac{1}{2} g_{kr}(\dot{g})^{ri} = \frac{1}{2}(q^{-1})_r^i(\dot{q})_{.k}^r = -\frac{1}{2}\dot{q}_r^i(q^{-1})_{.k}^r,$$

where (˙) denotes the partial derivation with respect to time with ξ^i being held constant.

We assume that the total deformation of the body can be split into an elastic and an inelastic part by setting

$$(2.4) \qquad q_k^i = \overset{\circ}{g}^{im} \overset{*}{g}_{mr} \overset{*}{g}^{rs} g_{sk} = \underset{P}{q}_{.r}^i \underset{E}{q}_k^r.$$

$\overset{*}{g}_{ik}$ belongs to a fictitious intermediate state which is in general incompatible. From Eq. (2.4) we derive an additive splitting of the total strain rate (for details see [17, 18]) according to

$$(2.5) \qquad d_k^i = \mathrm{sym}\, \frac{1}{2} \underset{E}{\{(q^{-1})_r^i(\dot{q})_{.k}^r\}} + \mathrm{sym}\, \frac{1}{2} \underset{P}{\{(q^{-1})_r^i(\dot{q})_{.m}^r} \underset{E}{q_k^m\}} = \underset{E}{d_k^i} + \underset{P}{d_k^i},$$

where sym {...} denotes the symmetrical part of the tensors.

For simplicity we assume further more that the thermo-elastic behaviour of the bodies is isotropic and unaffected by inelastic deformations in the sense that the material constants characterizing the thermo-elastic behaviour are independent of inelastic deformations. Thus we obtain a unique relation between the elastic deformations represented by q_k^i, the Kirchhoff stresses s_k^i and the temperature T:

$$(2.6) \qquad \underset{E}{q_k^i} = \underset{E}{q_k^i}(s_s^r, T), \quad s_k^i = s_k^i(\underset{E}{q_s^r}, T), \quad T = T(s_k^i, \underset{E}{q_s^r}).$$

This function may be transformed into an incremental relation by differentiation with respect to time. This leads to a general expression of the form

$$(2.7) \qquad \underset{E}{d_k^i} = \underset{E}{d_k^i}\{s_k^i, s_k^i|_0, T, \dot{T}; g_{ik}, d_k^i\}.$$

In this formula

$$(2.8) \qquad s_k^i|_0 = (\dot{s})_{,k}^i + d_r^i s_k^r - d_k^r s_s^i$$

denotes the covariant derivation with respect to time, also called the convected or corotational derivation, which is due to Zaremba and Jaumann. From Eq. (2.7) and (2.8) we see that the total strain rate enters the incremental form of the thermo-elastic stress-strain relations. Therefore, the thermo-elastic deformations are not independent of the inelastic deformation occurring at the same time. The same follows from the fact that in the integrated form (2.6) of the thermo-elastic stress-strain relations the stresses and the strain are related to the deformed state of the body.

In many cases we may replace the hyper-elastic behaviour described by Eq. (2.6) and (2.7) by a hypo-elastic one without large error. Then we obtain in the linear case

$$(2.9) \qquad \underset{E}{d_k^i} = \frac{1}{2G}\left\{ s_k^i|_0 - \frac{\nu}{1+\nu} s_r^r|_0 \delta_k^i \right\} + \alpha \dot{T} \delta_k^i.$$

We suppose that inelastic deformations occur if and only if a function

$$(2.10) \qquad \left. \begin{array}{l} F(s_k^i, T; k \ldots, \alpha_k^i \ldots) = 0 \\ \dfrac{\partial F}{\partial s_k^i} s_k^i|_0 + \dfrac{\partial F}{\partial T} \dot{T} > 0 \end{array} \right\} \quad \text{elastic-plastic material,}$$

or if

$$(2.11) \qquad F(s_k^i, T; k \ldots, \alpha_k^i \ldots,) > 0 \quad \text{elastic-viscoplastic material.}$$

The function F represents the yield condition which bounds the domain of pure thermo-elastic behaviour in the 10-dimensional space of stress and temperature. The inequality $(2.10)_2$ is the loading condition. The actual form of the yield condition for a given material is determinated by a set of so-called internal parameters (internal variables), which are scalars or tensors of even order. The current values of the internal parameters depend on the initial state of the material and the history of the thermo-mechanical process.

Further assumptions concerning inelastic behaviour will be specified in connection with different cases.

3. Some general remarks on the thermodynamics and the description of thermo-mechanical processes

We shall not discuss all aspects of the thermodynamics of irreversible thermo-mechanical processes. We shall restrict ourselves, rather, to some features essential for the elastic-plastic and elastic-viscoplastic elementary processes under consideration. This means, for instance, that we need not analyse whether the applied heat q arises from heat conduction or from heat sources. For the same reason it is unnecessary in our case to introduce the temperature gradient (besides the temperature) and the body-forces (besides the stresses) as (independent or dependent) process variables.

Under these restrictions, we get the following scheme for the description of a thermo-mechanical process:

The first step is to determine the reference frame of the description by specifying the

body-fixed coordinate-system ξ^i and the time scale t, i.e. the space-time base with the metric $g_{ik}(\xi^r, t_0)$ in the initial state[1].

The next step consists in choosing a suitable set of independent (external) process variables. The set of the independent process variables must be complete, i.e. the thermo-mechanical process must be determined uniquely by the history of these process variables. One possible choice is to introduce the stresses s_k^i and the temperature T as independent process variables. But we may replace these quantities (totally or partially) by the con-jugated external process variables, namely the strains (represented by q_k^i) and the applied heat q.

The dependent process variables may be divided into two groups. The first group contains only those quantities additionally required to determine the state of the material. In the case where the (external) independent process variables are, at the same time, thermodynamic state variables (like the stresses and the temperature), this first group contains only the so-called internal parameters (scalars and tensors of even order). If not all (external) independent process variables are state variables, then this first group of independent process variables has to be correspondingly enlarged.

The second group of dependent process variables contains all other quantities not required to fix the state of the material. The number of these quantities is unlimited since we may arbitrarily define new quantities (for example through combination with others).

Under the usual assumptions we may express the functionals which represent the dependent process variables by a system of first-order differential-equations combined with some auxiliary conditions. This leads to the basic scheme of the process description shown in Table 1.

The choice of the independent process variables depends on the kind of process to be described. An isothermic pure shear process, for instance, requires the introduction of the stresses and the temperature as independent process variables, whereas an adiabatic simple shear process is determined by the given strains and the (vanishing) applied heat.

The number of internal parameters is independent of the kind of the process. It de-pends only on the material and on the desired exactness of the description of the material behaviour.

We may define the individual internal parameters in different ways obtaining different sets of internal parameters on the same level of description. But in any case, the different sets have to be equivalent.

The constitutive equations describing the evolution of the metric and the dependent process variables contain on the right side

a) the current values of the independent process variables (s_k^i, T),

b) the current values of the first group of the dependent process variables $(k \ldots, a_k^i \ldots)$,

c) for elastic-plastic bodies furthermore the increments of the independent process variables $(s_k^i|_0, \dot{T})$; for elastic-viscoplastic bodies is this only valid for the elastic strain increments.

If, in the equations of evolution, other quantities are introduced as independent

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[1] In classical mechanics only the metric of the space depends on the process whereas, in relativistic mechanics, the metric of the time depends on the process, too. A special (classical) case with a changing metric of time is the so-called endochronic theory of viscoplasticity introduced by VALANIS [19, 20].

Table 1.

1. Space-time base:
  body-fixed coordinate-system: ξ^i
  time scale: t
  metric: $g_{ik}(\xi^r, t)$
2. Independent process variables: s_k^i, T
3. Dependent process variables:
  1. group: $k \ldots, \alpha_k^i \ldots, A_{ks}^{ir} \ldots$
  (internal variables)
  2. group: $q, \ldots$
4. Constitutive equations:
  yield condition, loading condition etc.
  equations of evolution:
  base:

$$d_k^i = \frac{1}{2} g^{ir} \dot{g}_{rk} = d_k^i \{s_k^i, s_k^i|_0, T, \dot{T}; g_{ik}; k, \ldots, \alpha_k^i \ldots A_{ks}^{ir}\}$$

  internal parameters:

$$\dot{k} = \dot{k}\{\ldots\},$$
$$\alpha_k^i|_0 = \alpha_k^i|_0\{\ldots\}$$

  other dependent variables:

$$\dot{q} = \dot{q}\{\ldots\}$$

process variables instead of the stresses s_k^i and the temperature T, then the increments of these independent process variables take the place of the increments of the stresses and the temperature in the scheme in Table 1. But the current values of the stresses and the temperature can only be replaced by state variables which are unique functions of s_k^i and T. The same is true with respect to other constitutive equations like the yield and loading condition.

The corresponding description of thermo-mechanical processes by means of thermo-dynamical state equations is based on the first and second law of thermodynamics. The first law states, under our simplifying assumptions, that the increment of the specific internal energy u is the sum of the increments of the specific mechanical work w and the (specific) applied heat q:

$$(3.1) \qquad \dot{u} = \dot{w} + \dot{q}.$$

The increment of mechanical work is given by

$$(3.2) \qquad \dot{w} = \frac{1}{\overset{\circ}{\varrho}} s_k^i d_i^k$$

and may be split into an elastic and an inelastic part according to

$$(3.3) \qquad \dot{w} = \frac{1}{\overset{\circ}{\varrho}} \underset{E}{s_k^i d_i^k} + \frac{1}{\overset{\circ}{\varrho}} \underset{P}{s_k^i d_i^i} = \underset{E}{\dot{w}} + \underset{I}{\dot{w}}.$$

The increment of elastic work, which can be written in the form

$$(3.4) \qquad \underset{E}{\dot{w}} = \frac{1}{\overset{\circ}{\varrho}} s_k^i (q^{-1})_r^k \underset{E}{q_i^r}|_0,$$

3*

is reversible. The increment of inelastic work has to be split once more into one part $\underset{D}{\dot{w}}$, which is dissipated at once, and into another part $\underset{H}{\dot{w}}$, which is used for changing the internal state. So we obtain

(3.5)
$$\underset{I}{\dot{w}} = \frac{1}{\varrho}\,\underset{P}{s_k^i d_i^k} = \underset{H}{\dot{w}} + \underset{D}{\dot{w}}.$$

On the dissipated mechanical energy the second law of thermodynamic states

(3.6)
$$\underset{D}{\dot{w}} \geqslant 0.$$

In thermodynamic equilibrium states the state of the material elements is a unique function of a closed set of thermodynamic state variables. We introduce as independent thermodynamic state variables:

(3.7)
$$\underset{E}{q_k^i},\ T,\ h\ ...,\ \beta_k^i\ ....$$

The choice of $\underset{E}{q_k^i}$ and T as state variables is based on the fact that in pure thermo-elastic deformations both quantities form a suitable set of thermodynamic state variables. The other state variables (scalars and tensors of even order) are added for the description of the changes of the internal structure of the material. The number of these other state variables must coincide with the number of internal parameters ($k\ ...,\ \alpha_k^i\ ...$) used in the process description. They need not necessarily be identical to those internal parameters, but they must form an equivalent set.

In thermodynamic equilibrium states the specific free energy (Helmholtz-function)

(3.8)
$$\varphi = u - Ts$$

(where s denotes the specific entropy) must be a unique function of the thermodynamic state variables:

(3.9)
$$\varphi = \varphi(\underset{E}{q_k^i},\ T,\ h\ ...,\ \beta_k^i\ ...).$$

With respect to the assumption that the thermo-elastic behaviour is unaffected by inelastic deformations, we may split the specific free energy into one term $\underset{E}{\varphi}$, related to the thermo-elastic part of deformations, and one other term $\underset{H}{\varphi}$, which corresponds to the changes of the internal structure of the material. Thus we obtain

(3.10)
$$\varphi = \underset{E}{\varphi}(\underset{E}{q_k^i},\ T) + \underset{H}{\varphi}(T,\ h\ ...,\ \beta_k^i\ ...).$$

From the relation

(3.11)
$$T\dot{s} = \dot{q} + \underset{D}{\dot{w}}$$

and from Eq. (3.1) and (3.8) we derive

(3.12)
$$\dot{\varphi} = \underset{E}{\dot{w}} - s\dot{T} + \underset{H}{\dot{w}}.$$

On the other hand we obtain from Eq. (3.10)

$$(3.13) \qquad \dot{\varphi} = \frac{\partial \varphi}{\partial q_k^i}\Big|_0 q_k^i\big|_0 + \frac{\partial}{\partial T}(\varphi + \varphi)\dot{T} + \frac{\partial \varphi}{\partial h}\dot{h} + \ldots + \frac{\partial \varphi}{\partial \beta_k^i}\beta_k^i\big|_0 + \ldots$$

By comparing Eqs. (3.12) and (3.13) we get

$$\frac{1}{\overset{\circ}{\varrho}} s_r^i (q^{-1})_k^r = \frac{\partial \varphi}{\partial q_i^k}\Big|_E ,$$

$$(3.14) \qquad s = -\frac{\partial \varphi}{\partial T} = -\frac{\partial}{\partial T}(\varphi + \varphi),$$

$$\dot{w} = \frac{\partial \varphi}{\partial h}\dot{h} + \ldots + \frac{\partial \varphi}{\partial \beta_k^i}\beta_k^i\big|_0 + \ldots$$

For irreversible processes this scheme of description by means of thermodynamic state equations has to be completed by some statements about the entropy production depending on the thermo-mechanical process. Under our assumption we have only to deal with entropy production by dissipated mechanical work in connection with inelastic deformations. Thus we may assume generally

$$(3.15) \qquad \dot{w}_D = \frac{1}{\overset{\circ}{\varrho}} \mu_k^i d_i^k.$$

So finally we get the following basic scheme for describing elastic-plastic or elastic-viscoplastic deformations by means of state equations as shown in Table 2.

Table 2.

thermodynamic state variables:	$q_k^i, T, h, \ldots, \beta_k^i \ldots,$
free specific energy:	$\varphi = \varphi(q_k^i, T) + \varphi(T, h \ldots, \beta_k^i \ldots)$
thermic state equation:	$\frac{1}{\overset{\circ}{\varrho}} s_r^i (q^{-1})_k^r = \frac{\partial \varphi}{\partial q_k^i}\Big|_E$
caloric state equation:	$s = -\frac{\partial \varphi}{\partial T} = -\frac{\partial}{\partial T}(\varphi + \varphi)$
change of internal structure:	$\dot{w}_H = \frac{\partial \varphi}{\partial h}\dot{h} + \ldots + \frac{\partial \varphi}{\partial \beta_k^i}\beta_k^i\big|_0 + \ldots$
entropy production:	$\dot{w}_D = \frac{1}{\overset{\circ}{\varrho}} \mu_k^i d_i^k$
evolution of entropy:	$\dot{q} + \dot{w}_D = T\dot{s} = -T\left(\frac{\dot{\partial \varphi}}{\partial T}\right) = -T\Big\{ \frac{\partial^2 \varphi}{\partial T^2}\dot{T} + \frac{\partial^2 \varphi}{\partial q_k^i \partial T}q_k^i\big|_0 + \frac{\partial^2 \varphi}{\partial h \partial T}\dot{h} + \ldots + \frac{\partial^2 \varphi}{\partial \beta_k^i \partial T}\beta_k^i\big|_0 + \ldots \Big\}$

The specific free energy φ, which determines the non-dissipated work of the thermo-mechanical process, and the quantity μ_k^i, which governs the entropy production are to be specified according to the material behaviour. On the other hand, the complete set of constitutive equations cannot be derived only from the knowledge of φ and μ_k^i. But the knowledge of φ and μ_k^i gives some important information about the frame in which the constitutive equations can be defined.

4. Elastic-plastic bodies

Elementary processes of elastic-plastic bodies may be considered as a sequence of equilibrium states, at least as long as the rate of the deformation is moderate so that the specific free energy is well-defined in each stage of the processes.

We consider a simple example assuming that the specific free energy is given as a function of the state variables q_k^i, T, h, β_k^i in the form
$$\underset{E}{}$$

(4.1)
$$\varphi = \underset{E}{\varphi}(\underset{E}{q_k^i}, T) + \underset{H}{\varphi}(T, h, \beta_k^i) = \underset{E}{\varphi}(\underset{E}{q_k^i}, T) + f(T) + h + g\beta_k^i\beta_i^k.$$

In this formula g denotes a constant with the dimension of a specific energy like the variable h and the function $f(T)$.

Furthermore we shall assume that the dissipation of mechanical energy is given by

(4.2)
$$\underset{D}{\dot{w}} = \frac{1}{\overset{\circ}{\varrho}}\,\xi\,\{s_k^i - c\overset{\circ}{\varrho}g\beta_k^i\}\underset{P}{d_i^k},$$

where $\xi < 1$ and c denote constant numbers. For elastic-plastic bodies the inelastic work is identical with the plastic work:

(4.3)
$$\underset{I}{\dot{w}} = \underset{P}{\dot{w}} = \frac{1}{\overset{\circ}{\varrho}}\,s_k^i\underset{p}{d_i^k}.$$

Thus we obtain from Eq. (3.5) and (4.2)

(4.4)
$$\underset{H}{\dot{w}} = \underset{P}{\dot{w}} - \underset{D}{\dot{w}} = (1-\xi)\underset{P}{\dot{w}} + c\xi g\beta_k^i\underset{p}{d_i^k}.$$

On the other hand, Eq. $(3.14)_3$ leads with the approach in Eq. (4.1) to

(4.5)
$$\underset{H}{\dot{w}} = \frac{\partial\underset{H}{\varphi}}{\partial h}\,\dot{h} + \frac{\partial\underset{H}{\varphi}}{\partial\beta_k^i}\,\beta_k^i|_0 = \dot{h} + 2g\beta_i^k\beta_k^i|_0.$$

Equations (4.4) and (4.5) are compatible, for instance, if we put

(4.6)
$$\dot{h} = (1-\xi)\underset{P}{\dot{w}} \rightarrow h - \overset{\circ}{h} = (1-\xi)\underset{P}{w}$$

and

(4.7)
$$\beta^i_k|_0 = \frac{c\xi}{2} \, d^i_k \rightarrow \beta^i_k|_0 = \zeta d^i_k.$$

From Eq. (4.6) it follows that in our case the plastic work w is equivalent to the thermodynamic state variable h. This is still true if we take ξ as a function of h, i.e. $\xi = \xi(h)$. But it does not hold in the general case when ξ also depends on the other state variables T and β^i_k.

Equation (4.7) shows that only in co-axial deformations (with non-rotating principle axes) and with constant ζ the state variable β^i_k is equivalent to the plastic deformations. Especially if we take ζ as a function of the state variables T, h and β^i_k, which is possible, β^i_k becomes non-equivalent to the plastic deformations.

From the thermodynamical considerations it follows that we may introduce the quantities h and β^i_k defined by Eq. (4.6) and (4.7) or any other equivalent set (for example w and $c\overset{\circ}{\varrho}g\beta^i_k$) as internal variables into the corresponding constitutive equations of the process description. The constitutive equations themselves are not yet determined completely by Eqs. (4.1), (4.2) and (4.6), (4.7) giving only the restrictive frame for the formulation of these relations. We may derive a complete set of constitutive relations, which is compatible with this frame, by the further assumptions:

a) the yield condition may have the general form

(4.8)
$$F = (t^i_k - \alpha^i_k)\,(t^k_i - \alpha^k_i) - k^2(w, T) = 0,$$

where t^i_k denotes the deviator of the Kirchhoff stresses s^i_k;

b) the plastic deformations may obey the so-called normality rule (theory of plastic potential)

(4.9)
$$d^i_k = \lambda \frac{\partial F}{\partial s^k_i} \, ;$$

c) the quantities ξ and ζ are constant.

This leads to the following system of constitutive relations given in Table 3.

This constitutive law of an elastic-plastic body describes a material showing a combination of isotropic and anisotropic (kinematic) hardening which depend on temperature. If the constant c tends to zero, the anisotropic hardening vanishes. For this special case some elementary processes concerning a real material are discussed in [16].

The approach may be extended to more complex hardening laws, for example, by introducing a fourth-order tensor as an additional internal parameter or state variable respectively [14, 15]. On the other hand, we should keep in mind that only under very special assumptions concerning the approaches for the free energy and the entropy production can the plastic work be used as internal parameter as it is usually done. Replacing these assumptions by more general approaches we gain larger freedom for introducing internal parameters without changing the basic scheme of the constitutive relations. Thus we get many possibilities for the definition of constitutive relations for elastic-plastic bodies.

Table 3.

independent process variables:	s_k^i, T
dependent process variables:	
internal parameters:	$\underset{P}{w}, \beta_k^i$
others:	$q \ldots$
constitutive law:	
if the yield condition	$F = (t_k^i - c_{\mathring{\varrho}} g \beta_k^i)(t_i^k - c_{\mathring{\varrho}} g \beta_i^k) - \underset{P}{k^2}(w, T) = 0$
and loading condition	$2(t_k^i - c_{\mathring{\varrho}} g \beta_k^i) t_i^k\vert_0 - \dfrac{\partial k^2}{\partial T} \dot{T} > 0$
are fulfilled, then	

$$d_k^i = \underbrace{\frac{1}{2_v G}\left\{ s_k^i\vert_0 - \frac{v}{1+v} s_r^r\vert_0 \delta_k^i \right\} + \alpha \dot{T} \delta_k^i}_{\substack{d_k^i \\ E}} + \underbrace{\frac{2(t_s^r - c_{\mathring{\varrho}} \beta_s^r) t_r^s\vert_0 - \dfrac{\partial k^2}{\partial T}\dot{T}}{2\zeta c_{\mathring{\varrho}} g k^2 + \dfrac{1}{\mathring{\varrho}}\dfrac{\partial k^2}{\partial w}(t_n^m - c_{\mathring{\varrho}} g \beta_n^m) t_m^n}(t_k^i - c_{\mathring{\varrho}} g \beta_k^i)}_{\substack{d_k^i \\ P}}$$

$$\underset{P}{\dot{w}} = \frac{1}{\mathring{\varrho}} s_k^i \underset{P}{d_i^k} = \frac{1}{\mathring{\varrho}} t_k^i d_i^k$$

$$\beta_k^i\vert_0 = \zeta \underset{P}{d_k^i}$$

$$\underbrace{\dot{q} + \xi \frac{1}{\mathring{\varrho}}(t_k^i - c_{\mathring{\varrho}} g \beta_k^i)\underset{P}{d_i^k}}_{\substack{\dot{w} \\ D}} = \underbrace{-T \cdot \frac{\partial^2 \varphi}{\partial T^2}}_{c_v} \dot{T} - T \frac{\partial^2 \underset{E}{\varphi}}{\partial q_k^i \partial T} \underset{E}{q_k^i}\vert_0$$

otherwise

$$d_k^i = \underset{E}{d_k^i}$$

$$\underset{P}{\dot{w}} = 0$$

$$\beta_k^i\vert_0 = 0$$

$$\dot{q} = c_v \dot{T} - T \frac{\partial^2 \underset{E}{\varphi}}{\partial q_k^i \partial T} \underset{E}{q_k^i}\vert_0$$

5. Elastic-viscoplastic bodies

Now we extend our considerations to elastic-viscoplastic bodies. As opposed to the elastic-plastic behaviour, thermo-mechanical processes in elastic-viscoplastic bodies cannot be considered as a sequence of equilibrium states in any case, not even in the case of elementary processes. Elastic-viscoplastic deformations are substantially connected with non-equilibrium states. This is the most important difference in comparison with elastic-plastic deformations from the thermodynamical point of view. One consequence of this fact is that we may get a continuation of a process without any change in the independent

process variables. This is, for instance, valid in the case of creep with constant stress and temperature or in the case of an adiabatic relaxation at constant strain. In such cases the body tends from a non-equilibrium state to an equilibrium state.

In order to establish the constitutive relations for elastic-viscoplastic bodies we adopt the usual assumption that the stresses which produce the inelastic deformations may be
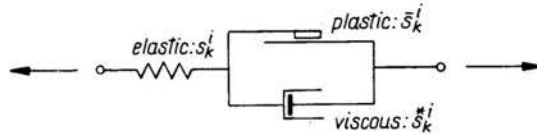
elastic: s_k^i plastic: $\bar{s}_k^i$

viscous: $\overset{*}{s}_k^i$

FIG. 1.

expressed as the sum of the so-called athermal or inviscid stresses $\bar{s}_k^i$ and the exceeding (viscous) stresses $\overset{*}{s}_k^i$:

$$(5.1) \qquad s_k^i = \bar{s}_k^i + \overset{*}{s}_k^i = \bar{s}_k^i + (s_k^i - \bar{s}_k^i).$$

This assumption corresponds to a material model as shown in Fig. 1. From the structure of this model we gather that the total work rate can be split in the following way:

$$(5.2) \qquad \dot{w} = \underset{E}{\dot{w}} + \underbrace{\underset{P}{\dot{w}} + \underset{V}{\dot{w}}}_{\underset{I}{\dot{w}}} = \frac{1}{\overset{\circ}{\varrho}}\underset{E}{s_k^i d_i^k} + \frac{1}{\overset{\circ}{\varrho}}\underset{P}{\bar{s}_k^i d_i^k} + \frac{1}{\overset{\circ}{\varrho}}\underset{P}{\overset{*}{s}_k^i d_i^k}.$$

The viscous part of the work is dissipated totally. Thus we may write

$$(5.3) \qquad \underset{V}{\dot{w}} = \underset{Dv}{\dot{w}}.$$

Regarding the plastic work we have already stated that one part is used for changing the internal state and only the remaining part can be considered to be dissipated. Therefore we have to put

$$(5.4) \qquad \underset{P}{\dot{w}} = \underset{H}{\dot{w}} + \underset{Dp}{\dot{w}}.$$

So we finally obtain

$$(5.5) \qquad \dot{w} = \underset{E}{\dot{w}} + \underset{H}{\dot{w}} + \underbrace{\underset{Dp}{\dot{w}} + \underset{Dv}{\dot{w}}}_{\underset{D}{\dot{w}}}.$$

We assume again that the changes of the internal state of the material can be regarded as a sequence of equilibrium states. Then the specific free energy is well-defined in each stage of the process and we may take over all statements concerning the specific free energy from the theory of elastic-plastic bodies. In doing so, however, we have to observe that into that part $\underset{H}{\dot{w}}$ of the plastic work rate, which is used for changing the internal structure, only the athermal stresses $\bar{s}_k^i$ enter, since only these stresses operate in the plastic mechanism. For the same reason we are only allowed to introduce the athermal stresses $\bar{s}_k^i$ into the statement concerning the dissipated plastic work $\underset{Dp}{\dot{w}}$.

On the other hand, we have to add to the dissipated viscous work $\dot{w} = \dot{w}$ to $\dot{w}$ in
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad D_V \quad\quad V \quad D_P$
order to obtain the total dissipation $\dot{w}$. The dissipated viscous work depends on different
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad D$
mechanisms determining the relation between the (viscous) stresses $\overset{*}{s}{}_k^i$ and the inelastic
strain rate d_k^i. We shall not discuss this problem in detail referring the reader to the
$\quad\quad\quad\quad P$
literature [1 to 13]. We only remark that we may choose two different approaches as first
approximations in the isotropic case, namely,

(5.6)
$$\underset{P}{d_k^i} = \frac{1}{2\eta}\,\overset{*}{t}{}_k^i = \frac{1}{2\eta}\,(t_k^i - \dot{t}_k^i)$$

or

(5.6′)
$$\underset{P}{d_k^i} = \frac{\overset{*}{t}{}_k^i}{\sqrt{\overset{**}{t}{}_s^r t_r^s}}\,\dot{\gamma}_0\left\{e^{\left[\sqrt{\frac{t_n^m t_m^n}{t_q^p t_p^q}} - 1\right]} - 1\right\}.$$

η as well as $\dot{\gamma}_0$ may depend on the state of the material. In the case of anisotropy these
approaches have to be accordingly extended.

Especially the approach Eq. (5.6′) may be modified in different ways in order to fit
the experimental results better in certain domains of strain rate and temperature. It is
not our aim to treat this matter here.

Only the work of the viscous stresses $\overset{*}{s}{}_k^i$ is correlated to non-equilibrium states. Thus,
it is only to this part of the dissipated work that we may apply the conception of thermo-
dynamic forces and fluxes taking $\overset{*}{s}{}_k^i$ as the forces and d_k^i as the fluxes or vice versa. For
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad P$
the same reason we may introduce only for that part of dissipation a dissipation function
in the usual manner, from which the fluxes can be derived as functions of the thermo-
dynamic forces. We shall renounce to introducing a dissipation function and explicitly
content ourselves with these hints.

After these preliminary general remarks we may consider a special example in order
to develop the full scheme of thermodynamics and a process description for elastic-
viscoplastic bodies. As an example we choose such a body which, for vanishing strain
rate or vanishing viscosity respectively, reduces to the elastic-plastic body considered in
Chapter 4. Therefore, we assume with respect to the specific free energy (see Eq. (4.1))

(5.7)
$$\varphi = \underset{E}{\varphi}(\underset{E}{q_k^i}, T) + f(T) + h + g\beta_k^i\beta_i^k.$$

Furthermore, concerning the entropy production we assume

(5.8)
$$\underset{D_P}{\dot{w}} = \frac{1}{\overset{\circ}{\varrho}}\,\xi\underbrace{\{\bar{s}_k^i - c\overset{\circ}{\varrho}g\beta_k^i\}}_{\bar{\mu}_k^i}\,\underset{P}{d_i^k},$$

$$\underset{D_V}{\dot{w}} = \frac{1}{\overset{\circ}{\varrho}}\,\underbrace{\{s_k^i - \bar{s}_k^i\}}_{\overset{*}{\mu}{}_k^i = \overset{*}{s}{}_k^i}\,\underset{P}{d_i^k}.$$

This leads to

$$(5.9) \qquad \underset{D}{\dot{w}} = \underset{Dp}{\dot{w}} + \underset{Dv}{\dot{w}} = \frac{1}{\overset{\circ}{\varrho}} \left\{ (\xi-1)\bar{s}_k^i - \xi c\overset{\circ}{\varrho} g\beta_k^i \right\} \underset{P}{d_i^k} + \frac{1}{\overset{\circ}{\varrho}} s_k^i \underset{P}{d_i^k} = (\xi-1)\underset{P}{\dot{w}} - \xi cg\beta_k^i \underset{P}{d_i^k} + \underset{I}{\dot{w}}.$$

Hence we obtain

$$(5.10) \qquad \underset{H}{\dot{w}} = \underset{I}{\dot{w}} - \underset{D}{\dot{w}} = (1-\xi)\underset{P}{\dot{w}} + \xi cg\beta_k^i \underset{P}{d_i^k}.$$

On the other hand the relation $(3.14)_3$ applied to the approach in Eq. (5.7) leads to

$$(5.11) \qquad \underset{H}{\dot{w}} = \dot{h} + 2g\beta_k^i \beta_i^k|_0$$

as in Chapter 4. Equations (5.10) and (5.11) are again compatible, for instance, if we put

$$(5.12) \qquad \dot{h} = (1-\xi)\underset{P}{\dot{w}} \rightarrow h - \overset{\circ}{h} = (1-\xi)\underset{P}{w},$$

$$(5.13) \qquad \beta_k^i|_0 = \frac{1}{2} c\xi \underset{P}{d_k^i} \rightarrow \beta_k^i|_0 = \zeta \underset{P}{d_k^i}.$$

Formally this is the same result as in Chapter 4 (see Eqs. (4.6) and (4.7)). The only difference is that we have to introduce the athermal stresses $\bar{s}_k^i$ into the definition of $\underset{P}{\dot{w}}$ for elastic-viscoplastic bodies, whereas for elastic-plastic bodies the stresses s_k^i themselves are used in this definition.

Using the same assumptions as in the case of the elastic-plastic body (Eq. (4.8) and (4.9) etc., but now with $\bar{s}_k^i$ instead of s_k^i), namely

a) general form of the yield condition

$$(5.14) \qquad \bar{F} = (t_k^i - \alpha_k^i)(t_i^k - \alpha_i^k) - \bar{k}^2(\underset{P}{w}, T) = 0,$$

b) normality rule

$$(5.15) \qquad \underset{P}{d_k^i} = \dot{\lambda} \frac{\partial \bar{F}}{\partial \bar{s}_i^k},$$

c) $\xi, \zeta = $ const,

we may derive the constitutive relations of the process description for an elastic-viscoplastic body based on the proceeding thermodynamic considerations. However, to begin with, we still have to discuss how we can eliminate the athermal stresses $\bar{s}_k^i$ from the equations of evolution since these stresses are neither external independent process variables nor internal state variables, but only dependent (internal) process variables of the second group (see Chapter 3, Table 1). For this purpose we start with the fact that the inelastic deformations can be expressed in two different ways, one taking into account the plastic mechanism and the second considering the viscous mechanism. Choosing the simplest approach in Eq. (5.6) for the viscous mechanism, we obtain

$$(5.16) \qquad \underset{P}{d_k^i} = 2\dot{\lambda}(\bar{t}_k^i - c\overset{\circ}{\varrho} g\beta_k^i) = \frac{2(\bar{t}_s^r - c\overset{\circ}{\varrho} g\beta_s^r)\bar{t}_r^s|_0 - \dfrac{\partial k^2}{\partial T}\dot{T}}{2\zeta c\overset{\circ}{\varrho} g\bar{k}^2 + \dfrac{1}{\overset{\circ}{\varrho}} \dfrac{\partial \bar{k}^2}{\partial w} (\bar{t}_n^m - c\overset{\circ}{\varrho} g\beta_n^m)\bar{t}_m^n} (\bar{t}_k^i - c\overset{\circ}{\varrho} g\beta_k^i) \qquad \text{(plastic)},$$

$$(5.16') \quad \underset{P}{d_k^i} = \frac{1}{2\eta}\,(t_k^i - \bar{t}_k^i) = \frac{1}{2\eta}\,\{t_k^i - c\mathring{\varrho}g\beta_k^i - (\bar{t}_k^i - c\mathring{\varrho}g\beta_k^i)\} \quad \text{(viscous)}.$$

By comparing these equations for $\underset{P}{d_k^i}$, it follows

$$(5.17) \qquad\qquad t_k^i - c\mathring{\varrho}g\beta_k^i = (1 + 4\eta\,\dot{\lambda})\,(\bar{t}_k^i - c\mathring{\varrho}g\beta_k^i)$$

and hence

$$(5.18) \qquad\qquad (t_k^i - c\mathring{\varrho}g\beta_k^i)\,(t_i^k - c\mathring{\varrho}g\beta_i^k) = (1 + 4\eta\,\dot{\lambda})^2\bar{k}^2$$

or

$$(5.19) \qquad\qquad \dot{\lambda} = \frac{1}{4\eta}\left\{\sqrt{\frac{(t_k^i - c\mathring{\varrho}g\beta_k^i)\,(t_i^k - c\mathring{\varrho}g\beta_i^k)}{\bar{k}^2}} - 1\right\}.$$

Following the course of the process in each stage, the internal parameters $\dot{w}$ and β_k^i and therefore also $\bar{k}^2 = \bar{k}^2(\underset{P}{w}, \beta_k^i)$ are known. Thus we may calculate $\dot{\lambda}$ from Eq. (5.19) and then all the other quantities we need, for example $\bar{t}_k^i$ from Eq. (5.17), $\underset{P}{d_k^i}$ from Eq. (5.16) etc.

This procedure fails at the point of transition from the elastic domain to elastic-viscoplastic deformations since in this instant $t_k^i = \bar{t}_k^i$ and therefore $\dot{\lambda}$, as well as $\underset{P}{d_k^i}$, becomes zero. But in this case we may calculate $\underset{P}{d_k^i}|_0$ from the following considerations. From Eq. (4.25) we obtain on account of $\underset{P}{d_k^i} = 0$ and $t_k^i = \bar{t}_k^i$

$$(5.20) \qquad\qquad 2(t_k^i - c\mathring{\varrho}g\beta_k^i)\,\bar{t}_i^k|_0 - \frac{\partial\bar{k}^2}{\partial T}\,\dot{T} = 0.$$

On the other hand, we derive from Eqs. (5.16) and (5.16') (observing $\dot{\lambda} = 0$ and $t_k^i = \bar{t}_k^i$)

$$(5.21) \qquad\qquad \underset{P}{d_k^i}|_0 = 2\ddot{\lambda}(t_k^i - c\mathring{\varrho}g\beta_k^i) = \frac{1}{2\eta}\,(\dot{t}_k^i|_0 - \dot{\bar{t}}_k^i|_0).$$

Equating Eqs. (5.21) and multiplying by $(t_i^k - c\mathring{\varrho}g\beta_i^k)$ we get

$$(5.22) \qquad\qquad 2\ddot{\lambda}\bar{k}^2 = \frac{1}{2\eta}\,(\dot{t}_k^i|_0 - \dot{\bar{t}}_k^i|_0)\,(t_i^k - c\mathring{\varrho}g\beta_i^k).$$

Together with Eq. (5.20) this leads to

$$(5.23) \qquad\qquad \ddot{\lambda} = \frac{1}{8\eta\bar{k}^2}\left\{2(t_k^i - c\mathring{\varrho}g\beta_k^i)\,\dot{t}_i^k|_0 - \frac{\partial\bar{k}^2}{\partial T}\,\dot{T}\right\}.$$

With this value of $\ddot{\lambda}$ we may calculate $\underset{P}{d_k^i}|_0$ from Eq. (5.21)$_1$, $\dot{\bar{t}}_k^i|_0$ from Eq. (5.21)$_2$ etc.

Thus we finally obtain the following system of constitutive equations in the description of processes of elastic-viscoplastic bodies (see Table 4).

Table 4.

independent process variables: $\quad s_k^i, T$

dependent process variables:

    internal parameters: $\quad \underset{P}{w}, \beta_k^i$

    others: $\quad \bar{s}_k^i, q \ldots$

constitutive law:

    a) if

$$F = (t_k^i - c_\varrho^\circ g \beta_k^i)(t_i^k - c_\varrho^\circ g \beta_i^k) - \underset{P}{\bar{k}^2}(w, T) > 0$$

        then

$$d_k^i = \underbrace{\frac{1}{2G_v}\left\{ s_k^i|_0 - \frac{v}{1+v}s_r^r|_0 \delta_k^i \right\} + \alpha \dot{T}\delta_k^i}_{\underset{E}{d_k^i}} + \underbrace{2\dot{\lambda}(\bar{t}_k^i - c_\varrho^\circ g \beta_k^i)}_{\underset{P}{d_k^i}}$$

        with

$$\dot{\lambda} = \frac{1}{4\eta}\left\{ \sqrt{\frac{(t_k^i - c_\varrho^\circ g \beta_k^i)(t_i^k - c_\varrho^\circ g \beta_i^k)}{\bar{k}^2}} - 1 \right\}$$

$$\bar{t}_k^i = \frac{1}{1 + 4\eta\dot{\lambda}}(t_k^i - c_\varrho^\circ g \beta_k^i) + c_\varrho^\circ g \beta_k^i$$

$$\underset{P}{\dot{w}} = \frac{1}{\overset{\circ}{\varrho}}\,\bar{t}_k^i \underset{P}{d_i^k}$$

$$\beta_k^i|_0 = \zeta \underset{P}{d_k^i}$$

. . . .

    b) if $\qquad\qquad F = 0$

        and $\qquad \dfrac{\partial F}{\partial s_k^i}\,s_k^i|_0 + \dfrac{\partial F}{\partial T}\,\dot{T} > 0$

        then $\qquad d_k^i = \underset{E}{d_k^i}$

$$\underset{P}{d_k^i} = 0, \quad \underset{P}{d_k^i}|_0 = 2\ddot{\lambda}(t_k^i - c_\varrho^\circ g \beta_k^i)$$

        with $\qquad \ddot{\lambda} = \dfrac{1}{8\eta\bar{k}^2}\left\{ 2(t_k^i - c_\varrho^\circ g \beta_k^i)t_i^k|_0 - \dfrac{\partial \bar{k}^2}{\partial T}\dot{T} \right\}$

. . . .

    c) if $\qquad\qquad F = 0 \quad$ and $\quad \dfrac{\partial F}{\partial s_k^i}\,s_k^i|_0 + \dfrac{\partial F}{\partial T}\dot{T} \leqslant 0$

        or $\qquad\qquad F < 0$

        then

$$d_k^i = \underset{E}{d_k^i}$$

$$\underset{P}{\dot{w}} = 0$$

$$\beta_k^i|_0 = 0$$

. . . .

6. Some complementary remarks

We may extend our considerations to more complex constitutive equations by introducing more internal parameters or state variables, respectively. In some cases it may be more realistic to base the constitutive relations on another material model. Such a possibility is sketched in Fig. 2. This model shows two different mechanisms of plastic
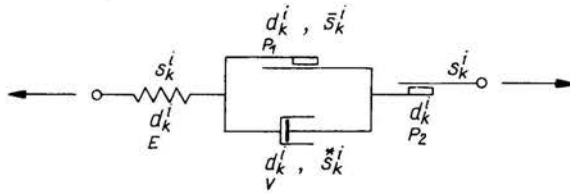

FIG. 2.

deformations. One corresponding approach for the stress-strain relations which is easy to treat is

$$d_k^i = \underset{E}{d_k^i} + \underset{P_1}{d_k^i} + \underset{P_2}{d_k^i} = \underset{E}{d_k^i} + \underbrace{\lambda \frac{\partial \bar{F}}{\partial \bar{s}_i^k}}_{\underset{P_1}{d_k^i}} + \underbrace{\varkappa_{ks}^{ir} s_r^s|_0}_{\underset{P_2}{d_k^i}},$$

(6.1)

$$\underset{P_1}{d_k^i} = \underset{V}{d_k^i}.$$

This model seems to be more suitable to cover some experimental results concerning second-order effects and some observed deviations of the normality rule. Sometimes the normality rule is considered as a fundamental law based on some principles concerning entropy production. But we should keep in mind that not the whole plastic work is dissipated since one part is used for the change of the internal structure. Thus we cannot expect that the total plastic strain rate obeys the theory of plastic potential (normality rule), even when the mentioned principles of entropy production are correct.

On the other hand, the scheme of the material behaviour based on Fig. 2 and the approach in Eq. (6.1) should only be considered as one possibility of an extended theory. Many other possibilities are still open.

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LEHRSTUHL FÜR MECHANIK I
RUHR-UNIVERSITÄT BOCHUM.