Archives of Mechanics • Archiwum Mechaniki Stosowanej • 29, 3, pp. 411–418, Warszawa 1977

On displacement functions in the discrete elasticity theory

K. H. BOJDA (GLIWICE)

THE PAPER deals with generalization of the Moisil method of resolving the system of differential equations to the case of equations of linear discrete elasticity theory. A way of reduction of the number of equations and displacement functions is presented. As the particular case of the theory the discrete Cosserat medium and monopolar discretized body are considered. The procedure was illustrated with the example of net shield with freely supported hinges constructed from the three families of rods.

W pracy uogólniono sposób Moisila rozwikłania układu równań różniczkowych na przypadek równań liniowej dyskretnej teorii sprężystości. Podano również pewien sposób redukcji liczby równań i funkcji przemieszczeń.

В работе обобщен способ Моисиля разрешения системы дифференциальных уравнений на случай уравнений линейной дискретной теории упругости. Дается тоже способ редукции числа уравнений и функций перемещений. Как частный случай рассмотрены дискретная среда Коссера и однополюсное дискретизированное тело. Целость подхода иллюстрируется на примере решетчатого диска, с шарнирными узлами, образованного из трех семейств стержней.

1. Introduction

THE MAIN role of displacements functions is to reduce the combined system of displacement equations to equations of simple structure. In the theory of elasticity a very useful way of resolving the system of partial differential equations was given by G. C. MOISIL [5]. This method was applied among others in papers ([6], pp. 279, 505), ([7], pp. 185, 189), ([2], p. 119), [1].

In this paper the Moisil procedure was generalized to the case of displacement equations of the linear discrete elasticity theory. A certain manner of reducing a number of displacement functions particularly useful in the theory of elastic discretized bodies is also discussed. This is so since in this theory the boundary conditions are prescribed in a different way than in the classic theory of elasticity. In the further part of this work the particular form of the operators appearing in the displacement equations of Cosserats discrete media [10] and unipolar discretized bodies [4] are presented. Media with regular structure and homogeneous in the sense of independence of the properties of the place are considered. The same notation as in the works [3, 4, 8, 9, 10, 11] is used.

In particular, the symbols $\Delta_\Lambda \varphi$ and $\bar{\Delta}_\Lambda \varphi$ denote the right and left derivatives of the function φ. The indices $a, b, \ldots,$ run the numbers $1, 2, \ldots, n$, the indices $\alpha, \beta, \ldots,$ — the numbers $1, 2, \ldots, r$, the indices k, l — the numbers $1, 2, 3$, the indices KL the numbers $1, 2$ and the indices Λ, Φ the series $\mathrm{I}, \mathrm{II}, \ldots, m$. The summations convention is used.

2. Displacement functions

The system of displacement equations for the discrete elasticity theory, [8], consists of the equilibrium equations

$$\bar{\Delta}_\Lambda T^\Lambda_a + t_a + f_a = 0 \tag{2.1}$$

and constitutive equations which, in the linear theory, may be presented in the following form:

$$\begin{aligned} T^\Lambda_a &= A^{\Lambda\Phi}_{ab}\Delta_\Phi q^b + B^\Lambda_{ab} q^b, \\ -t_a &= B^\Phi_{ba}\Delta_\Phi q^b + C_{ab} q^b. \end{aligned} \tag{2.2}$$

The quantities T^Λ_a and t_a in Eqs. (2.1) and (2.2) are the components of the state of stress; f_a is the exterior load, q^a are the generalized coordinates of the particle α. The quantities $A^{\Lambda\Phi}_{ab}$, B^Λ_{ab}, C_{ab} characterize the elastic properties of the considered discrete systems.

Substituting Eq (2.2) into Eq. (2.1), we obtain the following displacement equations:

$$A^{\Lambda\Phi}_{ab}\bar{\Delta}_\Lambda\Delta_\Phi q^b + B^\Lambda_{ab}\bar{\Delta}_\Lambda q^b - B^\Phi_{ba}\Delta_\Phi q^b - C_{ab} q^b + f_a = 0. \tag{2.3}$$

Introducing the operators

$$L_{ab} = A^{\Lambda\Phi}_{ab}\bar{\Delta}_\Lambda\Delta_\Phi + B^\Lambda_{ab}\bar{\Delta}_\Lambda - B^\Phi_{ba}\Delta_\Phi - C_{ab}. \tag{2.4}$$

Equation (2.3) may be written in the form

$$L_{ab} q^b + f_a = 0. \tag{2.5}$$

From Eq. (2.4) it follows that the operators L_{ab} are commutative and linear.

The linearity of operation is understood here in its common sense, that is, if

$$\varphi_1, \varphi_2 \in \{\varphi\colon (\varphi\colon D \to R)\},$$

$$\alpha_1, \alpha_2 \in R,$$

then

$$\bigwedge_{a,b}[L_{ab}(\alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1 L_{ab}\varphi_1 + \alpha_2 L_{ab}\varphi_2],$$

where D is the set of particles α of a discrete medium, and R is the set of real numbers.

Then, expressing the generalized coordinates q^a of the particle a in terms of the displacement functions

$$F_a\colon D \to R$$

in the following manner,

$$q^a = \det \mathbf{L}^a, \tag{2.6}$$

we receive the equations

$$\det \mathbf{L} F_a + f_a = 0, \tag{2.7}$$

where $\mathbf{L} = [\mathbf{L}_{ab}]$ and the matrix $\mathbf{L}^a$ is obtained from the matrix $\mathbf{L}$ by replacing a-th column by the vector $\mathbf{F}$.

The system of Eqs. (2.7) consists of n equations, each of which contains only one unknown function F_a. Thus, each equation may be solved independently.

Let us now discuss a certain way reducing the system (2.7) to the smaller number of equations.

Denote by L_a^α the commutative endomorphisms with the operators L_{ab}

$$L_a^\alpha : C(\tilde{D}) \to C(\tilde{D}),$$
$$C(\tilde{D}) = \{\varphi : (\varphi : \tilde{D} \to R)\}, \quad \tilde{D} \subset D,$$

such that

$$\bigwedge_{\alpha, a} (\{\varphi \equiv 0\} \subset \operatorname{Ker} L_a^\alpha), \tag{2.8}$$

where $\operatorname{Ker} L_a^\alpha$ is a kernel of L_a^α.

As it results from Eq. (2.8), the operators L_a^α cannot be the injections. If one may determine such functions

$$\varphi_\alpha : D \to R,$$

that

$$f_a = L_a^\alpha \varphi_\alpha, \tag{2.9}$$

then expressing F_a in terms of the new displacement functions Φ_α in the following way,

$$F_a = L_a^\alpha \Phi_\alpha,$$

we obtain the equations

$$\det \mathbf{L} \Phi_a + \varphi_a = 0. \tag{2.10}$$

The quantities q^a are calculated in this case from the formulae

$$q^a = L_b^\alpha \mathbf{D}^{ba} \Phi_\alpha = \det \mathbf{L}^{a\alpha} \Phi_\alpha, \tag{2.11}$$

where $\mathbf{D}^{ba}$ is a co-factor of the element L_{ba} of the matrix $\mathbf{L}$; the matrix $\mathbf{L}^{a\alpha}$ being obtained from the matrix $\mathbf{L}$ by replacing the a-th column by the vector $\mathbf{L}^\alpha \equiv (L_a^\alpha)$.

When $r < n$, the system of Eqs. (2.10) contains fewer equations than the system (2.7). For example, if for a certain fixed value

$$f_1 \equiv f_2 \equiv \ldots \equiv f_{k-1} \equiv f_{k+1} \equiv \ldots \equiv f_n \equiv 0, \quad f_k \neq 0,$$

occurs, then

$$f_a = \delta_a^{(k)} \varphi$$

and

$$F_a = \delta_a^{(k)} \Phi.$$

Hence, the multiplications through $\delta_a^{(k)}$ are here the endomorphisms L_a^α ($\alpha = (k)$). The system (2.10) reduces in this case to the single equation

$$\det \mathbf{L} \Phi + \varphi = 0,$$

and q^a is evaluated from the formula

$$q^a = \mathbf{D}^{(k)a} \Phi.$$

The procedure analogous to that described by the relation (2.9) is broadly used in the classical theory of elasticity; for example, in deriving the stress equation for shields it is frequently assumed that the mass forces X_i have the potential V

$$X_i = -\partial_i V, \quad i = 1, 2.$$

In all particular cases the equations for the displacement functions are determined from Eq. (2.7) or Eq. (2.10), and only the shape of the operators is different.

3. Some particular cases

The discrete Cosserats medium is a particular case of the discrete media considered in the previous sections. In the theory of discrete Cosserats media one assumes [10]

$$\begin{aligned}
q^a &= \delta^a_k u^k + \delta^{a-3}_k v^k,\\
T^\Lambda_a &= \delta^k_a T^\Lambda_k + \delta^k_{a-3} M^\Lambda_k,\\
t_a &= \delta^k_{a-3} \varepsilon_{kp}{}^r T^\Lambda_r l^p_\Lambda,\\
f_a &= \delta^k_a f_k + \delta^k_{a-3} n_k,
\end{aligned}$$

where the quantities u^k and v^k are the components of the displacement state and l^k_Λ are the components of the vector connecting the centre of the mass body d with the centre of the mass body $f_\Lambda d$ in a reference configuration.

The linear constitutive equations have the form

$$\begin{aligned}
T^\Lambda_k &= A^{\Lambda\Phi}_{kl} \gamma^l_\Phi + B^{\Lambda\Phi}_{kl} \varkappa^l_\Phi,\\
M^\Lambda_k &= B^{\Phi\Lambda}_{lk} \gamma^l_\Phi + F^{\Lambda\Phi}_{kl} \varkappa^l_\Phi,
\end{aligned}$$

where

$$\begin{aligned}
\gamma^k_\Lambda &= \Delta_\Lambda u^k + \varepsilon^k_{pr} v^r l^p_\Lambda,\\
\varkappa^k_\Lambda &= \Delta_\Lambda v^k,
\end{aligned}$$

are the components of the displacement state.

Hence, the following relations hold:

$$\begin{aligned}
A^{\Lambda\Phi}_{ab} &= \delta^k_a \delta^l_b A^{\Lambda\Phi}_{kl} + \delta^k_a \delta^l_{b-3} B^{\Lambda\Phi}_{kl} + \delta^k_{a-3} \delta^l_b B^{\Phi\Lambda}_{lk} + \delta^k_{a-3} \delta^l_{b-3} F^{\Lambda\Phi}_{kl},\\
B^\Lambda_{ab} &= \delta^k_a \delta^r_{b-3} \varepsilon^l_{mr} l^m_\Phi A^{\Lambda\Phi}_{kl} + \delta^k_{a-3} \delta^r_{b-3} \varepsilon^l_{mr} l^m_\Phi B^{\Phi\Lambda}_{lk},\\
C_{ab} &= \delta^r_{a-3} \delta^p_{b-3} \varepsilon^l{}_{kr} \varepsilon^m{}_{np} l^k_\Lambda l^n_\Phi A^{\Lambda\Phi}_{lm}.
\end{aligned} \tag{3.1}$$

Substituting Eq. (3.1) into Eq. (2.4), we obtain

$$L_{ab} = \delta^k_a \delta^l_b A_{kl} + \delta^k_a \delta^l_{b-3} B_{kl} + \delta^k_{a-3} \delta^l_b C_{kl} + \delta^k_{a-3} \delta^l_{b-3} G_{kl}. \tag{3.2}$$

The operators appearing at the right hand side of Eq. (3.2) have the following form:

$$\begin{aligned}
A_{kl} &= \bar{A}^{\Lambda\Phi}_{kl} \Delta_\Lambda \Delta_\Phi,\\
B_{kl} &= B^{\Lambda\Phi}_{kl} \bar{\Delta}_\Lambda \Delta_\Phi + \varepsilon^n{}_{ml} l^m_\Phi A^{\Lambda\Phi}_{kn} \bar{\Delta}_\Lambda,\\
C_{kl} &= B^{\Phi\Lambda}_{lk} \bar{\Delta}_\Lambda \Delta_\Phi + \varepsilon_{km}{}^n l^m_\Lambda A^{\Lambda\Phi}_{nl} \Delta_\Phi,\\
G_{kl} &= F^{\Lambda\Phi}_{kl} \bar{\Delta}_\Lambda \Delta_\Phi + \varepsilon^n{}_{ml} l^m_\Phi B^{\Phi\Lambda}_{nk} \bar{\Delta}_\Lambda + \varepsilon_{km}{}^n l^m_\Lambda B^{\Lambda\Phi}_{nl} \Delta_\Phi + \varepsilon_{km}{}^n \varepsilon^r{}_{pl} l^m_\Lambda l^p_\Phi A^{\Lambda\Phi}_{nr}.
\end{aligned}$$

The components of the displacement state are evaluated from the formulae

$$u^k = q^k,$$
$$v^k = q^{k+3}.$$

For the net shields we have [3]

$$q^a = \delta^a_K u^K + \delta^a_3 v.$$

Hence,

$$u^K = q^K,$$
$$v = q^3,$$

and the operators L_{ab} have the following form

$$L_{ab} = \delta^K_a \delta^l_b A_{KL} + \delta^K_a \delta^3_b B_K + \delta^3_a \delta^L_b C_L + \delta^3_a \delta^3_b G,$$

where

$$A_{KL} = A^{\Lambda\Phi}_{KL} \bar{\Delta}_\Lambda \Delta_\Phi,$$
$$B_K = B^{\Lambda\Phi}_K \bar{\Delta}_\Lambda \Delta_\Phi + \varepsilon^L{}_M l^M_\Phi A^{\Lambda\Phi}_{KL} \bar{\Delta}_\Lambda,$$
$$C_L = B^{\Phi\Lambda}_L \bar{\Delta}_\Lambda \Delta_\Phi + \varepsilon_M{}^N l^M_\Lambda A^{\Lambda\Phi}_{NL} \Delta_\Phi,$$
$$G = F^{\Lambda\Omega} \bar{\Delta}_\Lambda \Delta_\Phi + \varepsilon^N{}_M l^M_\Phi B^{\Phi\Lambda}_N \bar{\Delta}_\Lambda + \varepsilon_M{}^N l^M_\Lambda B^{\Lambda\Phi}_N \Delta_\Phi + \varepsilon_M{}^N \varepsilon^R{}_P l^M_\Lambda l^P_\Phi A^{\Lambda\Phi}_{NR}.$$

But for the net plates [3]

$$q^a = \delta^a_3 u + \delta^a_K v^K.$$

Thus

$$u = q^3,$$
$$v^K = q^K$$

and

$$L_{ab} = \delta^3_a \delta^3_b \tilde{A} + \delta^3_a \delta^L_b \tilde{B}_L + \delta^K_a \delta^3_b \tilde{C}_K + \delta^K_a \delta^L_b \tilde{G}_{KL},$$

where

$$\tilde{A} = A^{\Lambda\Phi} \bar{\Delta}_\Lambda \Delta_\Phi,$$
$$\tilde{B}_L = B^{\Lambda\Phi}{}_L \bar{\Delta}_\Lambda \Delta_\Phi + \varepsilon_{ML} l^M_\Phi A^{\Lambda\Phi} \bar{\Delta}_\Lambda,$$
$$\tilde{C} = B^{\Phi\Lambda}_K \bar{\Delta}_\Lambda \Delta_\Phi + \varepsilon_{KM} l^M_\Lambda A^{\Lambda\Phi} \Delta_\Phi,$$
$$\tilde{D}_{KL} = F^{\Lambda\Phi}_{KL} \bar{\Delta}_\Lambda \Delta_\Phi + \varepsilon_{ML} l^M_\Phi B^{\Phi\Lambda}_K \bar{\Delta}_\Lambda + \varepsilon_{KM} l^M_\Lambda B^{\Lambda\Phi}_L \Delta_\Phi + \varepsilon_{KM} \varepsilon_{RL} l^M_\Lambda l^R_\Phi A^{\Lambda\Phi}.$$

In the formulae for the net shields and plates the following notations are introduced

$$A^{\Lambda\Phi}_{33} \equiv A^{\Lambda\Phi}, \quad B^{\Lambda\Phi}_{3L} \equiv B^{\Lambda\Phi}{}_L. \quad B^{\Lambda\Phi}_{K3} \equiv B^{\Lambda\Phi}_K, \quad F^{\Lambda\Phi}_{33} = F^{\Lambda\Phi}.$$

The monopolar discretized body is a particular case of the discrete Cosserat medium [4]. For this case

$$q^a = \delta^a_k u^k, \quad A^{\Lambda\Phi}_{ab} = \delta^k_a \delta^l_b A^{\Lambda\Phi}_{kl}, \quad B^\Lambda_{ab} \equiv C_{ab} \equiv 0.$$

Hence,

$$u^k = q^k, \qquad (3.3)$$
$$L_{ab} = \delta_a^k \delta_b^l A_{kl},$$

where

$$A_{kl} = A_{kl}^{\Lambda\Phi} \bar{\Delta}_\Lambda \Delta_\Phi.$$

In plane problems the indices $k, l, \ldots$, should be replaced by $K, L, \ldots$.

4. Example

Consider a net shield with free supported hinges and formed from three families of rods. For the purpose of computation the two-dimensional monopolar discretized body was chosen. The range of the difference structure m for this body is three. For the case considered according to Eq. (3.3) we have

$$L_{ab} = \delta_a^K \delta_b^L A_{KL}^{\Lambda\Phi} \bar{\Delta}_\Lambda \Delta_\Phi.$$

Hence, Eqs. (2.7) now assume the form

$$C^{\Lambda\Omega\Phi\Gamma} \bar{\Delta}_\Lambda \bar{\Delta}_\Omega \Delta_\Phi \Delta_\Gamma F_L + f_K = 0, \qquad (4.1)$$

where

$$C^{\Lambda\Omega\Phi\Gamma} = A_{11}^{\Lambda\Phi} A_{22}^{\Omega\Gamma} - A_{12}^{\Lambda\Phi} A_{21}^{\Omega\Gamma}.$$

Since the expression for $A_{KL}^{\Lambda\Phi}$ is

$$A_{KL}^{\Lambda\Phi} = \begin{cases} 0, & \text{where} \quad \Lambda \neq \Phi, \\ \dfrac{EA_\Lambda}{l_\Lambda} t_K^\Lambda t_L^\Lambda, & \text{where} \quad \Lambda = \Phi, \end{cases} \qquad (4.2)$$

then

$$C^{\Lambda\Omega\Phi\Gamma} = \begin{cases} 0, \quad \text{where} \quad \Lambda \neq \Phi \quad \text{or} \quad \Omega \neq T, \\ \dfrac{E^2 A_\Lambda A_\Omega}{l_\Lambda l_\Omega} (t_1^\Lambda t_1^\Lambda t_2^\Omega t_2^\Omega - t_1^\Lambda t_2^\Lambda t_1^\Omega t_2^\Omega), \quad \text{where} \quad \Lambda = \Phi, \Omega = \Gamma. \end{cases} \qquad (4.3)$$

In formulae (4.2) and (4.3), E is the modulus of longitudinal elasticity of the materia of the rods, A_Λ is the cross-section area of the rods belonging to the family Λ, t_k^Λ are the components of the unit vector parallel to the direction of the axis of the rod of the family Λ. The components of the displacement vector u^K are evaluated from the following formulae

$$u^1 = A_{22}^{\Lambda\Omega} \bar{\Delta}_\Lambda \Delta_\Phi F_1 - A_{12}^{\Lambda\Phi} \bar{\Delta}_\Lambda \Delta_\Phi F_2,$$
$$u^2 = A_{11}^{\Lambda\Phi} \bar{\Delta}_\Lambda \Delta_\Phi F_2 - A_{21}^{\Lambda\Phi} \bar{\Delta}_\Lambda \Delta_\Phi F_1.$$

If

$$f_2 \equiv 0,$$

then,

$$f_K = \delta_K^1 \varphi,$$
$$F_K = \delta_K^1 \Phi.$$

In this case the system (4.1) reduces to the single equation

$$C^{\Lambda\Omega\Phi\Gamma}\bar{\Delta}_\Lambda \bar{\Delta}_\Omega \Delta_\Phi \Delta_\Gamma \Phi + \varphi = 0, \tag{4.4}$$

and the components of the displacement state are evaluated from the following formulae:

$$\begin{aligned} u^1 &= A_{22}^{\Lambda\Phi}\bar{\Delta}_\Lambda \Delta_\Phi \Phi, \\ u^2 &= -A_{21}^{\Lambda\Phi}\bar{\Delta}_\Lambda \Delta_\Phi \Phi. \end{aligned} \tag{4.5}$$

If, moreover,

$$\varphi \equiv 0,$$

then, Eq. (4.4) is homogeneous

$$C^{\Lambda\Omega\Phi\Gamma}\bar{\Delta}_\Lambda \bar{\Delta}_\Omega \Delta_\Phi \Delta_\Gamma \Phi = 0. \tag{4.6}$$

The displacement function satisfying Eq. (4.6) determines two possible displacement states: one given by Eq. (4.5) and the second given by the following formulae

$$\begin{aligned} u^1 &= -A_{12}^{\Lambda\Phi}\bar{\Delta}_\Lambda \Delta_\Phi \Phi, \\ u^2 &= A_{11}^{\Lambda\Phi}\bar{\Delta}_\Lambda \Delta_\Phi \Phi. \end{aligned} \tag{4.7}$$

The formulae (4.7) result from the relation

$$f_K = \delta_K^2 \varphi,$$
$$F_K = \delta_K^2 \Phi,$$

where

$$\varphi \equiv 0$$

should be assumed.

The application of the displacement functions discussed above will be specially suitable when the displacements are prescribed on the boundary.

Acknowledgment

The author is indebted to Professor CZ. WOŹNIAK for his valuable suggestions during the preparation of this article.

References

1. A. GAWĘCKI, *Płyta Reissnera o niejednorodności wykładniczej*, Rozpr. Inż., **21**, 3, 1973.
2. Z. KĄCZKOWSKI, *Płyty. Obliczenia statyczne*, Arkady, Warszawa 1968.
3. S. KONIECZNY, F. PIETRAS, CZ. WOŹNIAK, *O liniowych zagadnieniach dyskretnej teorii sprężystości*. I. Rozpr. Inż., **20**, 2, 1972.
4. W. KUFEL, *O liniowych zagadnieniach teorii sprężystości ciał dyskretyzowanych*, Mech. Teoret. Stos., **11**, 1, 1973.

5. G. C. Moisil, *Aspura sistemelor de ecuatii cu derivate partiale lineare si cu coeficienti constanti*, Biull. Acad. RPR, ser. A, 1, 1949.
6. W. Nowacki, *Teoria sprężystości*, PWN, Warszawa 1970.
7. W. Nowacki, *Teoria niesymetrycznej sprężystości*, PWN, Warszawa 1971.
8. Cz. Woźniak, *Discrete elastisity*, Arch. Mech. Stos., **23**, 6, 1971.
9. Cz. Woźniak, *Podstawy mechaniki ciał dyskretyzowanych*, Mech. Teoret. Stos., **11**, 1, 1973.
10. Cz. Woźniak, *Discrete elastic Cosserat media*, Arch. Mech. Stos., **25**, 2, 1973.
11. K. H. Bojda, *Analogia tarczowo-płytowa w teorii dźwigarów siatkowych*, Mech. Teoret. Stos., **12**,, 2, 1974.

TECHNICAL UNIVERSITY OF GLIWICE.

Received September 12, 1976.