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Saint-Venant's problem for inhomogeneous and anisotropic elastic solids with microstructure

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THE PRESENT paper is concerned with Saint-Venant's problem for inhomogeneous and anisotropic cylinders in the linear theory of elasticity with microstructure. The elastic coefficients are independent of the axial coordinate. The problem is solved using four generalized plane strain problems.

W pracy zajęto się zagadnieniem Saint-Venanta dla niejednorodnego walca anizotropowego z mikrostrukturą. Współczynniki sprężystości nie zależą od zmiennej osiowej. Zagadnienie rozwiązuje się za pomocą czterech uogólnionych problemów płaskiego stanu odkształcenia.

В настоящей работе рассматривается задача Сен-Венана для неоднородных и анизотропных цилиндров в теории упругости с микроструктурой. Упругие коэффициенты не зависят от аксиальной координаты. Задача разрешается при помощи четырех обобщенных плоских задач.

1. Introduction

IN THIS PAPER we consider Saint-Venant's problem in Mindlin's linear theory of elasticity with microstructure [1]. The theory of media with microstructure was developed in various papers (see e.g. [1–4]). The relation between these papers was discussed in [5]. In the linear theory of Cosserat elasticity, Saint-Venant's problem for homogeneous and isotropic solids was studied in [6, 7].

In this paper, using the results established in [7, 8], we study Saint-Venant's problem for inhomogeneous and anisotropic elastic cylinders with microstructure. We assume that the elastic coefficients are independent of the axial coordinate and are prescribed functions of the remaining coordinates. In the first part of the paper we define the generalized plane strain and give an existence theorem. In the second part we solve Saint-Venant's problem using four generalized plane strain problems.

2. Statement of the problem

Throughout this paper V denotes the interior of a right cylinder of length l with the open cross-section Σ and the lateral boundary B. We call ∂V the boundary of V and denote by L the boundary of the generic cross-section Σ. Moreover, a rectangular Cartesian coordinate system Ox_k $(k = 1, 2, 3)$ is used. The rectangular Cartesian coordinate frame is chosen such that the x_3-axis is parallel to the generators of V and the x_1Ox_2—plane contains one of the terminal sections. We call $\Sigma^{(0)}$ the cross-section located at $x_3 = 0$ and $\Sigma^{(l)}$ the cross-section which lies in the plane $x_3 = l$.

We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers (1, 2), whereas Latin subscripts to the range (1, 2, 3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

The basic equations of the static theory of elastic solids with microstructure, in absence of body-forces and body double-forces, are:

the equilibrium equations

$$\tau_{ij,i}+\sigma_{ij,i} = 0, \quad \mu_{ijk,i}+\sigma_{jk} = 0, \tag{2.1}$$

the constitutive equations

$$\begin{aligned} \tau_{ij} &= C_{ijrs}\varepsilon_{rs}+G_{rsij}\gamma_{rs}+F_{pqrij}\varkappa_{pqr}, \\ \sigma_{ij} &= G_{ijrs}\varepsilon_{rs}+B_{rsij}\gamma_{rs}+D_{ijpqr}\varkappa_{pqr}, \\ \mu_{ijk} &= F_{ijkrs}\varepsilon_{rs}+D_{rsijk}\gamma_{rs}+A_{ijkpqr}\varkappa_{pqr}, \end{aligned} \tag{2.2}$$

the geometrical equations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}), \quad \gamma_{ij} = u_{j,i}-\varphi_{ij}, \quad \varkappa_{ijk} = \varphi_{jk,i}. \tag{2.3}$$

In the above relations we have used the following notations: τ_{ij}—the classical stress tensor, σ_{ij}—the relative stress tensor, μ_{ijk}—the couple-stress tensor, ε_{ij}—the classical infinitesimal strain tensor, γ_{ij}—the relative deformation tensor, $\varkappa_{ijk}$—the microdeformation gradient tensor, u_i—the displacement vector, φ_{ij}—the microdeformation tensor, $C_{ijrs}, G_{rsij}, \ldots, A_{ijkpqr}$—the elastic coefficients. The elastic coefficients satisfy the symmetry relations

$$\begin{aligned} &C_{ijrs} = C_{rsij} = C_{jirs}, \quad B_{ijrs} = B_{rsij}, \quad A_{ijkpqr} = A_{pqrijk}, \\ &F_{ijkrs} = F_{ijksr}, \quad G_{ijrs} = G_{ijsr}. \end{aligned} \tag{2.4}$$

The surface tractions and double-tractions acting at a point $\mathbf{x}$ on the oriented surface S are given by

$$T_i = (\tau_{ji}+\sigma_{ji})n_j, \quad M_{ij} = \mu_{rij}n_r, \tag{2.5}$$

where n_j are the direction cosines of the exterior normal to S at $\mathbf{x}$.

The cylinder is supposed to be free of lateral loading so that we have the conditions

$$(\tau_{\alpha i}+\sigma_{\alpha i})n_\alpha = 0, \quad \mu_{\alpha ij}n_\alpha = 0 \quad \text{on } B, \tag{2.6}$$

where $(n_1, n_2, 0)$ are the direction cosines of the exterior normal to lateral surface.

The load of the cylinder is distributed over its ends in a way which fulfills the equilibrium conditions of a rigid body. We assume that the loading applied on $\Sigma^{(0)}$ is statically equivalent to a force $R(R_i)$ and a moment $M(M_i)$.

Saint-Venant's problem consists in determining a solution of Eqs. (2.1)–(2.3) which satisfies the conditions (2.6) and the conditions on $\Sigma^{(0)}$.

In this paper we consider an inhomogeneous medium for which

$$C_{ijrs} = C_{ijrs}(x_1, x_2), \quad B_{ijrs} = B_{ijrs}(x_1, x_2),$$
$$(2.7) \qquad G_{ijrs} = G_{ijrs}(x_1, x_2), \quad F_{ijkrs} = F_{ijkrs}(x_1, x_2),$$
$$D_{ijkrs} = D_{ijkrs}(x_1, x_2), \quad A_{ijkpqr} = A_{ijkpqr}(x_1, x_2).$$

We assume that the domain Σ is C^∞-smooth [9]. The functions C_{ijrs}, B_{ijrs}, G_{ijrs}, F_{ijkrs}, D_{ijkrs}, A_{ijkpqr} are supposed to belong to C^∞. We consider only a "C^∞-theory" but it is possible to obtain a classical solution of the problem for more general assumptions of regularity. We have chosen this way so as to emphasize best our method for the solution of the underlying problem.

3. The generalized plane strain

Following [8] we define the state of generalized plane strain of the cylinder to be that state in which the functions u_i and φ_{ij} depend only on x_1 and x_2

$$(3.1) \qquad u_i = u_i(x_1, x_2), \quad \varphi_{ij} = \varphi_{ij}(x_1, x_2).$$

The above restrictions imply that ε_{ij}, γ_{ij}, $\varkappa_{ijk}$, τ_{ij}, σ_{ij}, μ_{ijk} are functions only x_1 and x_2.

The equilibrium equations with the body-forces f_i and body double-forces L_{ij} can be written in the form

$$(3.2) \qquad \tau_{\alpha i,\alpha} + \sigma_{\alpha i,\alpha} + f_i = 0,$$
$$\mu_{\alpha ij,\alpha} + \sigma_{ij} + L_{ij} = 0,$$

from which it follows that the state of generalized plane strain demands that the components of body force vector and body double-force tensor be independent of x_3.

The geometrical equations lead to

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \quad \varepsilon_{\alpha 3} = \frac{1}{2} u_{3,\alpha}, \quad \varepsilon_{33} = 0,$$
$$(3.3) \qquad \gamma_{\alpha i} = u_{i,\alpha} - \varphi_{\alpha i}, \quad \gamma_{3i} = -\varphi_{3i}, \quad \varkappa_{\alpha jk} = \varphi_{jk,\alpha}, \quad \varkappa_{3jk} = 0.$$

The constitutive equations become

$$(3.4) \qquad \tau_{\alpha i} = C_{\alpha ij\beta}\varepsilon_{j\beta} + G_{kj\alpha i}\gamma_{kj} + F_{\beta rs\alpha i}\varkappa_{\beta rs},$$
$$\sigma_{ij} = G_{ijr\beta}\varepsilon_{r\beta} + B_{krij}\gamma_{kr} + D_{ij\beta rs}\varkappa_{\beta rs},$$
$$\mu_{\alpha ij} = F_{\alpha ijr\beta}\varepsilon_{r\beta} + D_{rs\alpha ij}\gamma_{rs} + A_{\alpha ij\beta rs}\varkappa_{\beta rs},$$
$$(3.5) \qquad \tau_{3i} = C_{3ij\beta}\varepsilon_{j\beta} + G_{rj3i}\gamma_{rj} + F_{\beta rs3i}\varkappa_{\beta rs},$$
$$\mu_{3ij} = F_{3ijr\beta}\varepsilon_{r\beta} + D_{rs3ij}\gamma_{rs} + A_{3ij\beta rs}\varkappa_{\beta rs}.$$

Let us assume that on the lateral surface of the cylinder we have the conditions

$$(3.6) \qquad (\tau_{\alpha i} + \sigma_{\alpha i})n_\alpha = P_i, \quad \mu_{\alpha ij}n_\alpha = Q_{ij}.$$

Obviously the functions P_i and Q_{ij} must be independent of x_3.

The generalized plane strain problem consists in determing of the functions u_i, φ_{ij} which satisfy Eqs. (3.2)–(3.4) in Σ and the boundary conditions (3.6) on L. The functions τ_{3i}, μ_{3ij} can be calculated after the components u_i and φ_{ij} have been determined.

The conditions of equilibrium for the cylinder can be written in the form

$$\int_\Sigma f_i d\sigma + \int_L P_i ds = 0,$$
(3.7)
$$\int_\Sigma e_{3\alpha\beta}(x_\alpha f_\beta + L_{\alpha\beta}) d\sigma + \int_F e_{3\alpha\beta}(x_\alpha P_\beta + Q_{\alpha\beta}) ds = 0;$$

$$\int_\Sigma (x_2 f_3 + L_{23} - L_{32}) d\sigma + \int_L (x_2 P_3 + Q_{23} - Q_{32}) ds - \int_\Sigma (\tau_{32} + \sigma_{32}) d\sigma = 0,$$
(3.8)
$$\int_\Sigma (x_1 f_3 + L_{13} - L_{31}) d\sigma + \int_L (x_1 P_3 + Q_{13} - Q_{31}) ds - \int_\Sigma (\tau_{31} + \sigma_{31}) d\sigma = 0,$$

where e_{ijk} is the alternating symbol.

The conditions (3.8) are identically satisfied on the basis of the relations (3.2) and (3.6); thus

$$\int_\Sigma (\tau_{32} + \sigma_{32}) d\sigma = \int_\Sigma (\tau_{23} + \sigma_{23} + \sigma_{32} - \sigma_{23}) d\sigma = \int_\Sigma [\tau_{23} + \sigma_{23} + x_2(\tau_{\alpha 3,\alpha} + \sigma_{\alpha 3,\alpha} + f_3)$$
$$+ L_{23} - L_{32} + \mu_{\alpha 23,\alpha} - \mu_{\alpha 32,\alpha}] d\sigma = \int_\Sigma \{[x_2(\tau_{\alpha 3} + \sigma_{\alpha 3})]_{,\alpha} + x_2 f_3 + L_{23} - L_{32}$$
$$+ \mu_{\alpha 23,\alpha} - \mu_{\alpha 32,\alpha}\} d\sigma = \int_L (x_2 P_3 + Q_{23} - Q_{32}) ds + \int_\Sigma (x_2 f_3 + L_{23} - L_{32}) d\sigma.$$

In a similar way we can prove that the second condition from Eqs. (3.8) is satisfied.

Using the results established in [9], as in [8], we can prove

THEOREM 3.1. *The boundary value problem* (3.2)–(3.4), (3.6) *has a solution belonging to* $C^\infty(\bar{\Sigma})$ *if and only if the* C^∞ *functions* f_i, L_{ij}, P_i, Q_{ij} *satisfy the conditions* (3.7).

In what follows we will use four special problems $A^{(s)}(s = 1, 2, 3, 4)$ of generalized plane strain for the domain Σ. The problems $A^{(s)}$ correspond to the systems of loading $\{f_i^{(s)}, L_{ij}^{(s)}, P_i^{(s)}, Q_{ij}^{(s)}\}$ where

$$f_i^{(\beta)} = [(C_{\alpha i33} + G_{33\alpha i} + G_{\alpha i33} + B_{33\alpha i}) e_{\nu\beta 3} x_\nu + (D_{\alpha i3mn} + F_{3mn\alpha i}) e_{nm\beta}]_{,\alpha},$$
$$f_i^{(3)} = [(C_{\alpha i\varrho 3} + G_{3\varrho\alpha i} + G_{\alpha i\varrho 3} + B_{3\varrho\alpha i}) e_{\varrho\beta 3} x_\beta + (D_{\alpha i3mn} + F_{3mn\alpha i}) e_{nm3}]_{,\alpha},$$
$$f_i^{(4)} = (C_{\alpha i33} + G_{33\alpha i} + G_{\alpha i33} + B_{33\alpha i})_{,\alpha},$$
$$L_{ij}^{(\beta)} = [(F_{\alpha ij33} + D_{33\alpha ij}) e_{\nu\beta 3} x_\nu + A_{\alpha ij3mn} e_{nm\beta}]_{,\alpha} + (G_{ij33} + B_{33ij}) e_{\nu\beta 3} x_\nu + D_{ij3mn} e_{nm\beta},$$
$$L_{ij}^{(3)} = [(F_{\alpha ij\varrho 3} + D_{3\varrho\alpha ij}) e_{\varrho\beta 3} x_\beta + A_{\alpha ij3mn} e_{nm3}]_{,\alpha} + (B_{3\alpha ij} + G_{ij\alpha 3}) e_{\alpha\beta 3} x_\beta + D_{ij3mn} e_{nm3},$$
(3.9)
$$L_{ij}^{(4)} = (F_{\alpha ij33} + D_{33\alpha ij})_{,\alpha} + G_{ij33} + B_{33ij}, \quad \text{on } \Sigma,$$
$$P_i^{(\beta)} = -[(C_{\alpha i33} + G_{33\alpha i} + G_{\alpha i33} + B_{33\alpha i}) e_{\nu\beta 3} x_\nu + (D_{\alpha i3mn} + F_{3mn\alpha i}) e_{nm\beta}] n_\alpha,$$
$$P_i^{(3)} = -[(C_{\alpha i\varrho 3} + G_{3\varrho\alpha i} + G_{\alpha i\varrho 3} + B_{3\varrho\alpha i}) e_{\varrho\beta 3} x_\beta + (D_{\alpha i3mn} + F_{3mn\alpha i}) e_{nm3}] n_\alpha,$$

(3.9) [cont.]
$$P_i^{(4)} = -(C_{\alpha i33}+G_{33\alpha i}+G_{\alpha i33}+B_{33\alpha i})n_\alpha,$$
$$Q_{ij}^{(\beta)} = -[(F_{\alpha ij33}+D_{33\alpha ij})e_{\nu\beta 3}x_\nu+A_{\alpha ij3mn}e_{nm\beta}]n_\alpha,$$
$$Q_{ij}^{(3)} = -[(F_{\alpha ij\varrho 3}+D_{3\varrho\alpha ij})e_{\varrho\beta 3}x_\beta+A_{\alpha ij3rs}e_{sr3}]n_\alpha,$$
$$Q_{ij}^{(4)} = -(F_{\alpha ij33}+D_{33\alpha ij})n_\alpha, \text{ on } L.$$

We denote by $\{v_i^{(s)}, \varphi_{ij}^{(s)}, \varepsilon_{ij}^{(s)}, \gamma_{ij}^{(s)}, \varkappa_{ijk}^{(s)}, \tau_{ij}^{(s)}, \sigma_{ij}^{(s)}, \mu_{ijk}^{(s)}\}$ $(s = 1, 2, 3, 4)$ the elastic states corresponding to the plane strain problems $A^{(s)}$. Thus we have

(3.10)
$$\tau_{\alpha i,\alpha}^{(s)}+\sigma_{\alpha i,\alpha}^{(s)}+f_i^{(s)} = 0, \quad \mu_{\alpha ij,\alpha}^{(s)}+\sigma_{ij}^{(s)}+L_{ij}^{(s)} = 0 \quad \text{on } \Sigma,$$

and

(3.11)
$$(\tau_{\alpha i}^{(s)}+\sigma_{\alpha i}^{(s)})n_\alpha = P_i^{(s)}, \quad \mu_{\alpha ij}^{(s)}n_\alpha = Q_{ij}^{(s)} \quad \text{on } L.$$

It is easy to show that the necessary and sufficient conditions (3.7) for the existence of the solution are satisfied for each boundary value problem $A^{(s)}$. In what follows we assume that the functions $v_i^{(s)}, \varphi_{ij}^{(s)}$ $(s = 1, 2, 3, 4)$ are known.

4. Extension, bending and torsion

Let the loading applied on $\Sigma^{(0)}$ be statically equivalent to a force $R(0, 0, R_3)$ and a moment $M(M_i)$. Thus, for $x_3 = 0$ we have the following conditions:

(4.1)
$$\int_\Sigma (\tau_{3\alpha}+\sigma_{3\alpha})d\sigma = 0,$$

(4.2)
$$\int_\Sigma (\tau_{33}+\sigma_{33})d\sigma = -R_3,$$

(4.3)
$$\int_\Sigma [x_\alpha(\tau_{33}+\sigma_{33})+\mu_{3\alpha 3}-\mu_{33\alpha}]d\sigma = e_{\alpha\beta 3}M_\beta,$$

(4.4)
$$\int_\Sigma e_{3\alpha\beta}[x_\alpha(\tau_{3\beta}+\sigma_{3\beta})+\mu_{3\alpha\beta}]d\sigma = -M_3.$$

The problem consists in solving Eqs. (2.1)–(2.3) with the conditions (2.6), (4.1)–(4.4). We try to solve this problem assuming that

(4.5)
$$u_\alpha = e_{\alpha\beta 3}\left(-\frac{1}{2}b_\beta x_3+b_3x_\beta\right)x_3+\sum_{s=1}^4 b_s v_\alpha^{(s)},$$
$$u_3 = (e_{3\alpha\beta}x_\alpha b_\beta+b_4)x_3+\sum_{s=1}^4 b_s v_3^{(s)},$$
$$\varphi_{ij} = e_{jik}b_kx_3+\sum_{s=1}^4 b_s\varphi_{ij}^{(s)},$$

where $v_i^{(s)}, \varphi_{ij}^{(s)}$ are the solutions of the problems $A^{(s)}$, and b_r are unknown constants.

From Eqs. (2.3) and (4.5) we get

$$\varepsilon_{\alpha\beta} = \sum_{s=1}^{4} b_s \varepsilon_{\alpha\beta}^{(s)}, \quad 2\varepsilon_{\alpha 3} = e_{\alpha\beta 3} b_3 x_\beta + 2\sum_{s=1}^{4} b_s \varepsilon_{\alpha 3}^{(s)},$$

(4.6)
$$\varepsilon_{33} = e_{3\alpha\beta} x_\alpha b_\beta + b_4, \quad \gamma_{\alpha i} = \sum_{s=1}^{4} b_s \gamma_{\alpha i}^{(s)}, \quad \gamma_{3\alpha} = e_{\alpha\beta 3} b_3 x_\beta + \sum_{s=1}^{4} b_s \gamma_{3\alpha}^{(s)},$$

$$\gamma_{33} = e_{3\alpha\beta} x_\alpha b_\beta + b_4 + \sum_{s=1}^{4} b_s \gamma_{33}^{(s)}, \quad \varkappa_{\alpha jk} = \sum_{s=1}^{4} b_s \varkappa_{\alpha jk}^{(s)}, \quad \varkappa_{3jk} = e_{kjr} b_r.$$

Taking into account Eqs. (4.6), from Eqs. (2.2) we obtain

$$\tau_{ij} = (C_{ij33} + G_{33ij})(e_{3\alpha\beta} x_\alpha b_\beta + b_4) + (C_{ij\alpha 3} + G_{3\alpha ij}) e_{\alpha\beta 3} b_3 x_\beta + F_{3mnij} e_{nmr} b_r + \sum_{s=1}^{4} b_s \tau_{ij}^{(s)},$$

(4.7)
$$\sigma_{ij} = (G_{ij33} + B_{33ij})(e_{3\alpha\beta} x_\alpha b_\beta + b_4) + (B_{3\alpha ij} + G_{ij\alpha 3}) e_{\alpha\beta 3} b_3 x_\beta + D_{ij3mn} e_{nmr} b_r + \sum_{s=1}^{3} b_s \sigma_{ij}^{(s)},$$

$$\mu_{ijk} = (F_{ijk33} + D_{33ijk})(e_{3\alpha\beta} x_\alpha b_\beta + b_4) + (F_{ijk\alpha 3} + D_{3\alpha ijk}) e_{\alpha\beta 3} b_3 x_\beta + A_{ijk3mn} e_{nmr} b_r + \sum_{s=1}^{4} b_s \mu_{ijk}^{(s)}.$$

The equilibrium (2.1) and the boundary conditions (2.6) are satisfied on the basis of the relations (3.10), (3.11), (3.9). The conditions (4.1) are identically satisfied on the basis of the equilibrium equations and the boundary conditions (2.6). Thus for the first condition of (4.1) we have

$$\int_\Sigma (\tau_{31} + \sigma_{31}) d\sigma = \int_\Sigma (\tau_{13} + \sigma_{13} + \sigma_{31} - \sigma_{13}) d\sigma = \int_\Sigma [\tau_{13} + \sigma_{13} + x_1(\tau_{\alpha 3,\alpha} + \sigma_{\alpha 3,\alpha}) + \mu_{\alpha 13,\alpha} - \mu_{\alpha 31,\alpha}] d\sigma = \int_\Sigma \{[x_1(\tau_{\alpha 3} + \sigma_{\alpha 3})]_{,\alpha} + \mu_{\alpha 13,\alpha} - \mu_{\alpha 31,\alpha}\} d\sigma$$
$$= \int_L [x_1(\tau_{\alpha 3} + \sigma_{\alpha 3}) n_\alpha + \mu_{\alpha 13} n_\alpha - \mu_{\alpha 31} n_\alpha] ds = 0.$$

In a similar way we can prove that the second condition of Eqs. (4.1) is satisfied.

The relations (4.7) can be written in the form

(4.8)
$$\tau_{ij} = \sum_{s=1}^{4} b_s t_{ij}^{(s)}, \quad \sigma_{ij} = \sum_{s=1}^{4} b_s \pi_{ij}^{(s)}, \quad \mu_{ijk} = \sum_{s=1}^{4} b_s m_{ijk}^{(s)},$$

where

(4.9)
$$\begin{aligned} t_{ij}^{(\beta)} &= (C_{ij33} + G_{33ij}) e_{\nu\beta 3} x_\nu + F_{3mnij} e_{nm\beta} + \tau_{ij}^{(\beta)}, \\ t_{ij}^{(3)} &= (C_{ij\alpha 3} + G_{3\alpha ij}) e_{\alpha\beta 3} x_\beta + F_{3mnij} e_{nm3} + \tau_{ij}^{(3)}, \\ t_{ij}^{(4)} &= C_{ij33} + G_{33ij} + \tau_{ij}^{(4)}, \\ \pi_{ij}^{(\beta)} &= (G_{ij33} + B_{33ij}) e_{\nu\beta 3} x_\nu + D_{ij3mn} e_{nm\beta} + \sigma_{ij}^{(\beta)}, \end{aligned}$$

(4.9) [cont.]

$$\begin{aligned} \pi_{ij}^{(3)} &= (B_{3\alpha ij}+G_{ij\alpha 3})e_{\alpha\beta 3}x_\beta+D_{ij3mn}e_{nm3}+\sigma_{ij}^{(3)},\\ \pi_{ij}^{(4)} &= G_{ij33}+B_{33ij}+\sigma_{ij}^{(4)},\\ m_{ijk}^{(\beta)} &= (F_{ijk33}+D_{33ijk})e_{\nu\beta 3}x_\nu+A_{ijk3mn}e_{nm\beta}+\mu_{ijk}^{(\beta)},\\ m_{ijk}^{(3)} &= (F_{ijk\alpha 3}+D_{3\alpha ijk})e_{\alpha\beta 3}x_\beta+A_{ijk3mn}e_{nm3}+\mu_{ijk}^{(3)},\\ m_{ijk}^{(4)} &= F_{ijk33}+D_{33ijk}+\mu_{ijk}^{(4)}. \end{aligned}$$

From Eqs. (4.2)–(4.4), (4.9) we obtain the following system for the unknown constants

(4.10)
$$\sum_{s=1}^{4} D_{\alpha s}b_s = e_{\alpha\beta 3}M_\beta,$$
$$\sum_{s=1}^{4} D_{3s}b_s = -R_3, \qquad \sum_{s=1}^{4} D_{4s}b_s = -M_3,$$

where we have used the notations

(4.11)
$$D_{\alpha s} = \int_\Sigma [x_\alpha(t_{33}^{(s)}+\pi_{33}^{(s)})+m_{3\alpha 3}^{(s)}-m_{33\alpha}^{(s)}]d\sigma,$$
$$D_{3s} = \int_\Sigma (t_{33}^{(s)}+\pi_{33}^{(s)})d\sigma,$$
$$D_{4s} = \int_\Sigma e_{3\alpha\beta}[x_\alpha(t_{3\beta}^{(s)}+\pi_{3\beta}^{(s)})+m_{3\alpha\beta}^{(s)}]d\sigma, \qquad s = 1, 2, 3, 4.$$

Let us prove that the system (4.10) determines the constants b_s ($s = 1, 2, 3, 4$). We assume that the internal energy density

(4.12)
$$U(u) = \frac{1}{2}C_{ijkr}\varepsilon_{ij}\varepsilon_{kr}+\frac{1}{2}B_{ijkr}\gamma_{ij}\gamma_{kr}+\frac{1}{2}A_{ijkrmn}\varkappa_{ijk}\varkappa_{rmn}$$
$$+D_{ijkrm}\gamma_{ij}\varkappa_{krm}+F_{ijkrm}\varkappa_{ijk}\varepsilon_{rm}+G_{ijkr}\gamma_{ij}\varepsilon_{kr},$$

is a positive definite quadratic form. In Eq. (4.12) we used the notation $u = \{u_i, \varphi_{jk}\}$. Let us consider two elastic states $\{u_i', \varphi_{ij}', \ldots, \mu_{ijk}'\}$ and $\{u_i'', \varphi_{ij}'', \ldots, \mu_{ijk}''\}$. If we denote

(4.13)
$$2U(u', u'') = \tau_{ij}'\varepsilon_{ij}''+\sigma_{ij}'\gamma_{ij}''+\mu_{ijk}'\varkappa_{ijk}'',$$

it follows that

(4.14)
$$U(u', u'') = U(u'', u'), \qquad U(u, u) = U(u).$$

It is easy to obtain the reciprocity relation

(4.15)
$$2\int_V U(u', u'')dv = \int_{\partial V}(T_i'u_i''+M_{ij}'\varphi_{ij}'')d\sigma = \int_{\partial V}(T_i''u_i'+M_{ij}''\varphi_{ij}')d\sigma.$$

Obviously,

(4.16)
$$2\int_V U(u)dv = \int_{\partial V}(T_iu_i+M_{ij}\varphi_{ij})d\sigma.$$

The relations (4.5) can be written in the form

(4.17) $$u_i = \sum_{s=1}^{4} b_s u_i^{(s)}, \quad \varphi_{ij} = \sum_{s=1}^{4} b_s \psi_{ij}^{(s)},$$

where

$$u_\alpha^{(\beta)} = -\frac{1}{2} e_{\alpha\beta 3} x_3^2 + v_\alpha^{(\beta)}, \quad u_3^{(\beta)} = e_{\nu\beta 3} x_\nu x_3 + v_3^{(\beta)},$$

(4.18) $$u_\alpha^{(3)} = e_{\alpha\beta 3} x_\beta x_3 + v_\alpha^{(3)}, \quad u_3^{(3)} = v_3^{(3)}, \quad u_\alpha^{(4)} = v_\alpha^{(4)},$$

$$u_3^{(4)} = x_3 + v_3^{(4)}, \quad \psi_{ij}^{(k)} = e_{jik} x_3 + \varphi_{ij}^{(k)}, \quad \varphi_{ij}^{(4)} = \psi_{ij}^{(4)}.$$

It is easy to see that

(4.19) $$U(u) = \sum_{r,s=1}^{4} U_{rs} b_r b_s,$$

where

(4.20) $$U_{rs} = U(u^{(r)}, u^{(s)}), \quad u^{(r)} = \{u_i^{(r)}, \psi_{jk}^{(r)}\}, \quad r, s = 1, 2, 3, 4.$$

The total elastic energy is

(4.21) $$E = \int_V U(u) dv = \sum_{r,s=1}^{4} E_{rs} b_r b_s,$$

where

(4.22) $$E_{rs} = \int_V U_{rs} dv.$$

We note that

(4.23) $$(t_{\alpha i}^{(s)} + \pi_{\alpha i}^{(s)})_{,\alpha} = 0, \quad m_{\alpha ij,\alpha}^{(s)} + \pi_{ij}^{(s)} = 0 \quad \text{on } \Sigma,$$

and

(4.24) $$(t_{\alpha i}^{(s)} + \pi_{\alpha i}^{(s)}) n_\alpha = 0, \quad m_{\alpha ij}^{(s)} n_\alpha = 0 \quad \text{on } L.$$

Taking into account Eqs. (4.23) and (4.24) we get

(4.25) $$\int_\Sigma (t_{3\alpha}^{(s)} + \pi_{3\alpha}^{(s)}) d\sigma = 0.$$

Let us apply the relations (4.15), (4.16) to the elastic states $\{u_i^{(s)}, \psi_{ij}^{(s)}, \ldots, m_{ijk}^{(s)}\}$, $(s = 1, 2, 3, 4)$. Using the relations (4.18), (4.25), we obtain

(4.26) $$2E_{\alpha s} = l e_{3\beta\alpha} D_{\beta s}, \quad 2E_{3s} = -l D_{4s}, \quad 2E_{4s} = l D_{3s}, \quad s = 1, 2, 3, 4.$$

From Eqs. (4.26) and (4.21) it follows

(4.27) $$\det(D_{rs}) \neq 0,$$

so that the system (4.10) uniquely determines the constants b_s. The problem is therefore solved.

5. Flexure

Let us assume that the loading applied on $\Sigma^{(0)}$ is statically equivalent to a force $R(R_1, R_2, 0)$ and a moment $M(0, 0, 0)$. Thus, for $x_3 = 0$ we have the following conditions

$$\int_{\Sigma} (\tau_{3\alpha}+\sigma_{3\alpha})d\sigma = -R_\alpha, \tag{5.1}$$

$$\int_{\Sigma} (\tau_{33}+\sigma_{33})d\sigma = 0, \tag{5.2}$$

$$\int_{\Sigma} [x_\alpha(\tau_{33}+\sigma_{33})+\mu_{3\alpha 3}-\mu_{33\alpha}]d\sigma = 0, \tag{5.3}$$

$$\int_{\Sigma} e_{3\alpha\beta}[x_\alpha(\tau_{3\beta}+\sigma_{3\beta})+\mu_{3\alpha\beta}]d\sigma = 0. \tag{5.4}$$

The problem consists in solving Eqs. (2.1)–(2.3) with the conditions (2.6), (5.1)–(5.4).

We call the solution (4.5) the primary solution and denote by $\hat{u}[b_r]$ the vector $\{u_i, \varphi_{rs}\}$ with the components (4.5), indicating its dependence on the constants b_r. Let $v = \{v_i, \psi_{rs}\}$ be a vector with the components $v_i = v_i(x_1, x_2)$, $\psi_{rs} = \psi_{rs}(x_1, x_2)$. In what follows we assume that the functions v_i, ψ_{rs} and the constants b_r, c_r $(r = 1, 2, 3, 4)$ are unknown and we seek the solution $u = \{u_i, \varphi_{rs}\}$ of the flexure problem in the form

$$u = \hat{u}[b_r]+\int_0^{x_3} \hat{u}[c_r]dx_3+v, \tag{5.5}$$

i.e.,

$$u_\alpha = e_{\alpha\beta 3}\left(-\frac{1}{2}b_\beta x_3^2+b_3x_\beta x_3-\frac{1}{6}c_\beta x_3^3+\frac{1}{2}c_3x_\beta x_3^2\right)+\sum_{s=1}^{4}(b_s+c_sx_3)v_\alpha^{(s)}+v_\alpha,$$

$$u_3 = (e_{3\alpha\beta}x_\alpha b_\beta+b_4)x_3+\frac{1}{2}(e_{3\alpha\beta}x_\alpha c_\beta+c_4)x_3^2+\sum_{s=1}^{4}(b_s+x_3c_s)v_3^{(s)}+v_3, \tag{5.6}$$

$$\varphi_{ij} = e_{jik}\left(b_kx_3+\frac{1}{2}c_kx_3^2\right)+\sum_{s=1}^{4}(b_s+x_3c_s)\varphi_{ij}^{(s)}+\psi_{ij}.$$

From Eqs. (2.2), (2.3) and (5.6) we obtain

$$\tau_{ij} = (C_{ij33}+G_{33ij})[(e_{3\alpha\beta}x_\alpha b_\beta+b_4)+(e_{3\alpha\beta}x_\alpha c_\beta+c_4)x_3]+(C_{ij\alpha 3}$$
$$+G_{3\alpha ij})e_{\alpha\beta 3}(b_3+x_3c_3)x_\beta+\sum_{s=1}^{4}(b_s+x_3c_s)\tau_{ij}^{(s)}+F_{3mnij}e_{nmr}(b_r+x_3c_r)+t_{ij}+K_{ij},$$

$$\sigma_{ij} = (G_{ij33}+B_{33ij})[(e_{3\alpha\beta}x_\alpha b_\beta+b_4)+(e_{3\alpha\beta}x_\alpha c_\beta+c_4)x_3]$$
$$+(B_{3\alpha ij}+G_{ij\alpha 3})e_{\alpha\beta 3}(b_3+x_3c_3)x_\beta+D_{ij3mn}e_{nmr}(b_r+x_3c_r)+\pi_{ij}+H_{ij}, \tag{5.7}$$

$$\mu_{ijk} = (F_{ijk33}+D_{33ijk})[(e_{3\alpha\beta}x_\alpha b_\beta+b_4)+(e_{3\alpha\beta}x_\alpha c_\beta+c_4)x_3]+(F_{ijk\alpha 3}$$
$$+D_{3\alpha ijk})e_{\alpha\beta 3}(b_3+x_3c_3)x_\beta+A_{ijk3mn}e_{nmr}(b_r+x_3c_r)+\sum_{s=1}^{4}(b_s+x_3c_s)\mu_{ijk}^{(s)}+m_{ijk}+R_{ijk},$$

where

$$t_{ij} = C_{ijr\beta}e_{r\beta}+G_{rsij}\eta_{rs}+F_{\beta rsij}\nu_{\beta rs}, \tag{5.8}$$
$$\pi_{ij} = G_{ijr\beta}e_{r\beta}+B_{rsij}\eta_{rs}+D_{ij\beta rs}\nu_{\beta rs},$$
$$m_{ijk} = F_{ijkr\beta}e_{r\beta}+D_{rsijk}\eta_{rs}+A_{ijk\beta rs}\nu_{\beta rs},$$

$$e_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta}+v_{\beta,\alpha}), \quad e_{\alpha 3} = e_{3\alpha} = \frac{1}{2}v_{3,\alpha}, \tag{5.9}$$
$$\eta_{\alpha i} = v_{i,\alpha}-\psi_{\alpha i}, \quad \eta_{3i} = -\psi_{3i}, \quad \nu_{\alpha jk} = \psi_{jk,\alpha},$$

and

$$K_{ij} = \sum_{s=1}^{4}[c_s(C_{ijk3}+G_{3kij})v_k^{(s)}+F_{3qrij}\varphi_{qr}^{(s)}],$$
$$H_{ij} = \sum_{s=1}^{4}[c_s(G_{ijk3}+B_{3kij})v_k^{(s)}+D_{ij3qr}\varphi_{qr}^{(s)}], \tag{5.10}$$
$$R_{ijk} = \sum_{s=1}^{4}[c_s(F_{ijkr3}+D_{3rijk})v_r^{(s)}+A_{ijk3qr}\varphi_{qr}^{(s)}].$$

Using the notations (4.9) we can write

$$\tau_{ij} = \sum_{s=1}^{4}(b_s+x_3c_s)t_{ij}^{(s)}+t_{ij}+K_{ij},$$
$$\sigma_{ij} = \sum_{s=1}^{4}(b_s+x_3c_s)\pi_{ij}^{(s)}+\pi_{ij}+H_{ij}, \tag{5.11}$$
$$\mu_{ijk} = \sum_{s=1}^{4}(b_s+x_3c_s)m_{ijk}^{(s)}+m_{ijk}+R_{ijk}.$$

On the basis of the relations (4.23) the equilibrium equations reduce to

$$t_{\alpha i,\alpha}+\pi_{\alpha i,\alpha}+F_i = 0, \quad m_{\alpha ij,\alpha}+\pi_{ij}+G_{ij} = 0, \tag{5.12}$$

where

$$F_i = K_{\alpha i,\alpha}+H_{\alpha i,\alpha}+\sum_{s=1}^{4}c_s(t_{3i}^{(s)}+\pi_{3i}^{(s)}), \tag{5.13}$$
$$G_{ij} = R_{\alpha ij,\alpha}+H_{ij}+\sum_{s=1}^{4}c_s m_{3ij}^{(s)}.$$

In view of the relations (4.24) the conditions on the lateral surface become

$$(t_{\alpha i}+\pi_{\alpha i})n_\alpha = p_i, \quad m_{\alpha ij}n_\alpha = q_{ij} \quad \text{on } L, \tag{5.14}$$

where

$$p_i = -(K_{\alpha i}+H_{\alpha i})n_\alpha, \quad q_{ij} = -R_{\alpha ij}n_\alpha. \tag{5.15}$$

Thus, the functions v_i, ψ_{rs} are the components of the displacement vector and microdeformation tensor in the generalized plane strain problem (5.8), (5.9), (5.12), (5.14). The necessary and sufficient conditions to solve this problem are

$$(5.16)\qquad \int_{\Sigma} F_i d\sigma + \int_{L} p_i ds = 0, \qquad \int_{\Sigma} e_{3\alpha\beta}(x_\alpha F_\beta + G_{\alpha\beta}) d\sigma + \int_{L} e_{3\alpha\beta}(x_\alpha p_\beta + q_{\alpha\beta}) ds = 0.$$

The first two conditions (5.16) are satisfied in view of the relations (5.13), (5.15), (4.25). From the remaining conditions we get

$$(5.17)\qquad \sum_{s=1}^{4} D_{rs} c_s = 0, \quad r = 3, 4,$$

where D_{rs} are given by Eqs. (4.11).

Taking into account the equilibrium equations and the boundary conditions (2.6) we can write

$$(5.18)\qquad \int_{\Sigma} (\tau_{31} + \sigma_{31}) d\sigma = \int_{\Sigma} (\tau_{13} + \sigma_{13} + \sigma_{31} - \sigma_{13}) d\sigma = \int_{\Sigma} [\tau_{13} + \sigma_{13} + x_1(\tau_{\alpha 3,\alpha} + \sigma_{\alpha 3,\alpha}$$
$$+ \tau_{33,3} + \sigma_{33,3}) + \mu_{i13,i} - \mu_{i31,i}] d\sigma = \int_{L} [x_1(\tau_{\alpha 3} + \sigma_{\alpha 3}) n_\alpha + (\mu_{\alpha 13} - \mu_{\alpha 31}) n_\alpha] ds$$
$$+ \int_{\Sigma} [x_1(\tau_{33} + \sigma_{33})_{,3} + \mu_{313,3} - \mu_{331,3}] d\sigma = \int_{\Sigma} [x_1(\tau_{33} + \sigma_{33})_{,3} + \mu_{313,3} - \mu_{331,3}] d\sigma.$$

In a similar way we have

$$(5.19)\qquad \int_{\Sigma} (\tau_{32} + \sigma_{32}) d\sigma = \int_{\Sigma} [x_1(\tau_{33} + \sigma_{33})_{,3} + \mu_{323,3} - \mu_{332,3}] d\sigma.$$

Using Eqs. (5.11), (5.18), (5.19), (4.11) the conditions (5.1) reduce to

$$(5.20)\qquad \sum_{s=1}^{4} D_{\alpha s} c_s = -R_\alpha.$$

The system (5.17), (5.20) uniquely determines the constants c_s. Thus the conditions (5.16) are satisfied and in what follows we assume that the functions v_i, ψ_{rs} are known.

Let us consider now the conditions (5.2)–(5.4). From Eqs. (5.11) and (5.2)–(5.4) we obtain the following system for the unknown constants b_s:

$$(5.21)\qquad \sum_{s=1}^{4} D_{rs} b_s = d_r, \quad r = 1, 2, 3, 4,$$

where

$$d_\alpha = -\int_{\Sigma} [(t_{33} + \pi_{33} + K_{33} + H_{33}) x_\alpha + m_{3\alpha 3} - m_{33\alpha} + R_{3\alpha 3} - R_{33\alpha}] d\sigma,$$

$$(5.22)\qquad d_3 = -\int_{\Sigma} (t_{33} + \pi_{33} + K_{33} + H_{33}) d\sigma,$$

$$d_4 = -\int_{\Sigma} e_{3\alpha\beta} [x_\alpha (t_{3\beta} + \pi_{3\beta} + K_{3\beta} + H_{3\beta}) + m_{3\alpha\beta} + R_{3\alpha\beta}] d\sigma.$$

The system (5.21) uniquely determines the constants b_s. Thus the flexure problem is solved.

6. Conclusions

In this paper we established the procedure for determining the solution of Saint-Venant's problem for elastic solids with microstructure.

As in classical elasticity the problem is reduced to solving plane problems. The components of the displacement vector have the same form as in the classical theory. The effect of microstructure is present by means of the auxiliary plane strain problems $A^{(s)}$.

The solutions of auxiliary plane strain problems are independent of the loading of the beam. They can be determined when the elastic coefficients and the domain of cross-section are prescribed. The solutions of these problems in the classical theory, for homogeneous and isotropic solids, are

$$v_1^{(1)} = -\frac{\lambda}{4(\lambda+\mu)}(x_1^2-x_2^2), \quad v_2^{(1)} = -\frac{\lambda}{2(\lambda+\mu)}x_1x_2, \quad v_3^{(1)} = 0,$$

$$v_1^{(2)} = -\frac{\lambda}{2(\lambda+\mu)}x_1x_2, \quad v_2^{(2)} = \frac{\lambda}{4(\lambda+\mu)}(x_1^2-x_2^2), \quad v_3^{(2)} = 0,$$

$$v^{(3)} = -\frac{\lambda}{2(\lambda+\mu)}x_\alpha, \quad v_3^{(3)} = 0, \quad v_\alpha^{(4)} = 0, \quad v_3^{(4)} = \varphi(x_1, x_2),$$

where λ, μ are the Lamé moduli and φ is the solution of the boundary value problem

$$\varphi_{,\alpha\alpha} = 0 \quad \text{on } \Sigma, \quad \varphi_{,\alpha}n_\alpha = e_{\alpha\beta 3}n_\alpha x_\beta \quad \text{on } L.$$

The case of micropolar elastic solids was studied in [7].

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