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Existence and uniqueness of solutions of the initial boundary value problem for the flow of a barotropic viscous fluid, global in time

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IN THIS PAPER the existence and uniqueness of global in time solutions of the initial boundary value problem for the flow of a barotropic viscous fluid in a bounded domain $\Omega \subset R^n$, $n = 2, 3$, is proved. The case where the fluid enters the domain is considered only because in this case we know how to obtain an a priori estimate. Moreover, the existence of solutions is shown under the assumption that suitable norms of data functions are sufficiently small. Therefore the total inflow of mass and energy is also sufficiently small. The existence of generalized global in time solutions such that the density and velocity vector belong to $L_\infty(0, \infty; H^2(\Omega))$ and $L_\infty(0, \infty; H^2(\Omega)) \cap L_2(0, \infty; H^3(\Omega))$, respectively, is shown.

W pracy pokazano istnienie i jednoznaczność globalnych w czasie rozwiązań problemu brzegowo-początkowego dla przepływu ściśliwego barotropowego w ograniczonym obszarze $\Omega \subset R^n$ $n = 2, 3$. Problem ten rozwiązano tylko w przypadku, gdy ciecz wpływa do obszaru, gdyż w tym przypadku umiano otrzymać oszacowanie *a priori*. Ponadto, aby pokazać istnienie globalnych rozwiązań założono małość odpowiednich norm danych funkcji. Zatem całkowita masa jak i energia, które zostały dostarczone do układu, są nie tylko skończone ale muszą być dostatecznie małe. Pokazano istnienie uogólnionych rozwiązań globalnych takich, że gęstość należy do $L_\infty(0, \infty; H^2(\Omega))$, a prędkość do $L_2(0, \infty; H^2(\Omega)) \cap L_\infty(0, \infty; H^3(\Omega))$.

В работе показано существование и единственность решений, глобальных во времени, начально-краевой задачи для сжимаемого баротропного течения в ограниченной области $\Omega \subset R^n$, $n = 2, 3$. Эта задача решена только в случае, когда жидкость втекает в область, т.к. в этом случае удалось получить оценку априори. Кроме этого, чтобы показать существование глобальных решений, предполагается малость соответствующих норм заданных функций. Итак полная масса, как и энергия, которые доставляются в систему, являются не только конечными, но должны быть достаточно малыми. Показано существование обобщенных глобальных решений, таких, что плотность принадлежит к $L_\infty(0, \infty; H^2(\Omega))$, а скорость к $L_2(0, \infty; H^2(\Omega)) \cap L_\infty(0, \infty; H^3(\Omega))$.

1. Introduction

IN THIS PAPER we shall prove the existence and uniqueness of global in time solutions of the following initial boundary value problem for a compressible viscous barotropic fluid flow in a bounded domain $\Omega \subset R^n$, $n = 2, 3$, [1, 2]:

$$\varrho'_t + (\varrho' v^i)_{,x^i} = 0, \tag{1.1}$$

$$\varrho' v^i_t + \varrho' v^j v^i_{x^j} - \mu' v^i_{x^j x^j} - \nu' v^j_{x^i x^j} + p'_{x^i} = \varrho' f^i, \tag{1.2}$$

where ϱ' — density, $\varrho' > 0$; $v = (v^1, \ldots, v^n)$ — velocity; p' — pressure; $f = (f^1, \ldots, f^n)$ — external force; μ', ν' — shear and bulk viscosities, $\mu' > 0$, $\nu' > 0$,

$$\varrho'|_{t=0} = \sigma'(x), \quad v|_{t=0} = a(x), \tag{1.3}$$

$$v|_{\partial\Omega} = \eta(x', t), \quad \varrho'|_{\partial\Omega} = b'(x', t), \quad x' \in \partial\Omega. \tag{1.4}$$

Moreover, we assume the following barotropy condition:

(1.5) $$p' = R'\varrho'^{\gamma},$$

where R' is a constant. Equations (1.3) and (1.4) imply the following compatibility conditions:

(1.6) $$\eta|_{t=0} = a|_{\partial\Omega}, \quad \sigma'|_{\partial\Omega} = b'|_{t=0}.$$

Moreover, we assume

(1.7) $$-\eta\cdot\bar{n}|_{\partial\Omega} = d(x', t) \geqslant d_0 > 0, \quad x' \in \partial\Omega, \quad d_0 = \text{const}.$$

Similarly as in [1], we assume that density and its initial and boundary values have small deviation from the equilibrium condition, hence we can denote

(1.8) $$\varrho' = \varrho_0(1+\varrho), \quad b' = \varrho_0(1+b), \quad \sigma' = \varrho'(1+\sigma),$$

where ϱ_0 is a constant and denotes the equilibrium magnitude of the density. Therefore Eqs. (1.1) and (1.2), can be written in the form

(1.9) $$\varrho_t = -[(1+\varrho)v^i]_{,x^i},$$

(1.10) $$u^i_t = \frac{\mu}{1+\varrho}u^i_{x^kx^k} + \frac{\nu}{1+\varrho}u^j_{x^ix^j} - u^ju^i_{x^j} - \frac{R[(1+\varrho)^{\gamma}]_{,x^i}}{1+\varrho} - u^j\beta^i_{x^j} - \beta^j u^i_{x^j} + f'^i,$$

where

$$\mu = \frac{\mu'}{\varrho_0}, \quad \nu = \frac{\nu'}{\varrho_0}, \quad R = R'\varrho^{\gamma-1},$$

(1.11) $$f'^i = f^i + \frac{\mu}{1+\varrho}\beta^i_{x^jx^j} + \frac{\nu}{1+\varrho}\beta^j_{x^ix^j} - \beta^i_t - \beta^j\beta^i_{x^j},$$

and β is an extension of the function η such that

(1.12) $$\beta|_{\partial\Omega} = \eta(x', t), \quad x' \in \partial\Omega.$$

In the end we have the following initial and boundary conditions:

(1.13) $$u|_{t=0} = a - \beta|_{t=0}, \quad u|_{\partial\Omega} = 0,$$

(1.14) $$\varrho|_{t=0} = \sigma, \quad \varrho|_{\partial\Omega} = b.$$

This paper is a continuation of the paper [1].

Using the *a priori* estimates showed in [1], in Sect. 2 an a priori global in time estimate is obtained. Next, making use of the existence of local solutions obtained in [1], in Section 3 the existence of global solutions of the considered problem is proved.

Moreover, all notations are the same as in [1], nevertheless we define nontypical spaces used in this paper. We introduce Banach spaces $\Pi^l_{k,p}(\Omega^T) = \bigcap_{i=k}^{l} L_p^{l-i}(0, T; H^i(\Omega))$, $\Pi^{l+1/2}_{k,p}(\partial\Omega^T) = \bigcap_{i=k}^{l} L_p^{l-i}(0, T; H^{i+1/2}(\partial\Omega))$ and $\Gamma^l_k(\Omega), \Gamma^{l+1/2}_k(\partial\Omega)$, where l, k, i are natural number and $p \geqslant 1$ is real number, with the norms

$$|u|_{\Gamma_k^l(\Omega)} \equiv |u|_{l,k,\Omega} = \sum_{i=k}^{l} ||D_t^{l-i}u||_{i,2,\Omega},$$

$$|u|_{\Gamma_k^{l+1/2}(\Omega)} \equiv |u|_{l+1/2,k,\partial\Omega} = \sum_{i=k}^{l} ||D_t^{l-i}u||_{i+1/2,2,\partial\Omega},$$

where $|| \ ||_{i,2,\Omega}$ and $|| \ ||_{i+1/2,2,\partial\Omega}$ are norms in spaces $H^2(\Omega)$ and $H^{i+1/2}(\partial\Omega)$, respectively.

At last we introduce the function

$$E^0(\varrho, u) = \frac{R}{\gamma-1}[(1+\varrho)^\gamma - 1 - \gamma\varrho] + \frac{1}{2}(1+\varrho)u^2 \tag{1.15}$$

and we recall the lemma of MATSUMURA and NISHIDA [3]:

LEMMA 1.1.

There exist constants $\varrho_1 = \min\left\{1/2, \frac{3}{2|\gamma-2|2^{|\gamma-3|}}\right\}$, $C = C(\gamma)$, such that for $|\varrho| \leqslant \varrho_1$ we have

$$\frac{\gamma R}{4}\varrho^2 + \frac{1}{4}u^2 \leqslant E^0 \leqslant C(\gamma)(u^2+\varrho^2). \tag{1.16}$$

Proof. Using the expansion

$$\frac{R}{\gamma-1}[(1+\varrho)^\gamma - 1 - \gamma\varrho] = \frac{\gamma R}{2}\varrho^2 + \frac{\gamma R(\gamma-2)}{3!}\varrho^3(1+\xi\varrho)^{\gamma-3},$$

where $0 < \xi < 1$, we obtain the right-hand side inequality of Eq. (1.16) for $|\varrho| < 1/2$. The second inequality in Eq. (1.16) is obtained from

$$E^0 > \frac{R\gamma}{2}\varrho^2 - \frac{\gamma R(\gamma-2)}{6}\varrho^2|\varrho|2^{|\gamma-3|} + \frac{1}{4}u^2 = \frac{\gamma R}{2}\varrho^2\left(1 - \frac{|\gamma-2|}{3}|\varrho|2^{|\gamma-3|}\right) + \frac{1}{4}u^2$$

$$\geqslant \frac{\gamma R}{4}\varrho^2 + \frac{1}{4}u^2,$$

which is valid for $|\varrho| \leqslant \frac{3}{2|\gamma-2|2^{|\gamma-3|}}$. This completes the proof.

2. A priori global in time estimate

This part is devoted to obtaining a global in time estimate of ϱ and u. Let us assume that ϱ, u are solutions of the problem (2.4)÷(2.8) [1] such, that $\varrho \in C^3(\Omega^T)$, $u \in C^3(\Omega^T)$, where T is an arbitrary. Using the function E^0 we can formulate

LEMMA 2.1.

Let us assume that $f(t) \in L_2(\Omega)$, $\beta(t) \in \Gamma_1^2(\Omega)$, $b \in H^1(\partial\Omega)$, $\beta \in L_\infty(0, T; H^1(\Omega))$, $t \in [0, T]$, $|\varrho| \leqslant 1/2$ and

$$3\alpha_4^2 \max||\beta_x||_{2,\Omega} \leqslant \mu, \tag{2.1}$$

where α_4 is the constant from Eq. (2.15) [1]. Then the following estimate

$$\text{(2.2)}\qquad \frac{d}{dt}\int_\Omega E^0 dx+\frac{\mu}{2}||u_x||^2_{2,\Omega}+\nu||\operatorname{div}u||^2_{2,\Omega} \leqslant G_1(t)+\varepsilon_1||\varrho_x||^2_{2,\Omega}$$

is valid, where E^0 is described by Eq. (1.1) and

$$\text{(2.3)}\qquad G_1(t) = C\big(||f||^2_{2,\Omega}+|\beta|^2_{2,1,\Omega}(1+||\beta||^2_{2,2,\Omega}\big)+\frac{1}{\varepsilon_1}||\beta||^2_{1,2,\Omega}+||b||^2_{1,2,\partial\Omega},$$

where C is a constant and ε_1 is an arbitrary parameter. This parameter will be assumed sufficiently small.

P r o o f. We consider the expression

$$\text{(2.4)}\qquad E^0_t = (1+\varrho)u^iu^i_t+\left[\frac{R\gamma}{\gamma-1}[(1+\varrho)^{\gamma-1}-1]+\frac{1}{2}u^2\right]\varrho_t = (1+\varrho)u^i\left[\frac{\mu}{1+\varrho}u^i_{x^jx^j}\right.$$
$$\left.+\frac{\nu}{1+\varrho}u^j_{x^ix^j}-(u^j+\beta^j)u^i_{x^j}-u^j\beta^i_{x^j}-\frac{R[(1+\varrho)^\gamma]_{,x^i}}{1+\varrho}+f'^i\right]$$
$$-\left[\frac{R\gamma}{\gamma-1}[(1+\varrho)^{\gamma-1}-1]+\frac{1}{2}u^2\right][(1+\varrho)(u^i+\beta^i)]_{,x^i}$$
$$= \mu(u^iu^i_{x^j})_{,x^j}-\mu(u^i_{x^j})^2+\nu(u^iu^j_{x^j})_{,x^i}-\nu(u^i_{x^i})^2-\frac{1}{2}[(1+\varrho)(u^i+\beta^i)u^2]_{,x^i}$$
$$-(1+\varrho)u^iu^j\beta^i_{x^j}+(1+\varrho)u^if'^i-\frac{R\gamma}{\gamma-1}[(1+\varrho)^{\gamma-1}-1][(1+\varrho)\beta^i]_{,x^i}$$
$$-\left\{R[(1+\varrho)^\gamma]_{,x^i}u^i+\frac{R\gamma}{\gamma-1}[(1+\varrho)^{\gamma-1}-1][(1+\varrho)u^i]_{,x^i}\right\}.$$

The last term in Eq. (2.4) will be expressed in the following way:

$$\text{(2.5)}\qquad -R[(1+\varrho)^\gamma]_{,x^i}u^i-\frac{R\gamma}{\gamma-1}(1+\varrho)^{\gamma-1}[(1+\varrho)u^i]_{,x^i}+\frac{R\gamma}{\gamma-1}[(1+\varrho)u^i]_{,x^i}$$
$$= -\left[R[(1+\varrho)^\gamma]_{,x^i}u^i+\frac{R}{\gamma-1}[(1+\varrho)^\gamma]_{,x^i}u^i+\frac{R\gamma}{\gamma-1}(1+\varrho)^\gamma u^i_{x^i}\right]+\frac{R\gamma}{R-1}[(1+\varrho)u^i]_{,x^i}$$
$$= -\frac{R\gamma}{\gamma-1}[(1+\varrho)^\gamma u^i]_{,x^i}+\frac{R\gamma}{\gamma-1}[(1+\varrho)u^i]_{,x^i}.$$

Using Eq. (2.5) in Eq. (2.4), integrating the result over Ω and using that $u|_{\partial\Omega} = 0$, we obtain

$$\text{(2.6)}\qquad \frac{d}{dt}\int_\Omega E^0dx+\mu||u_x||^2_{2,\Omega}+\nu||\operatorname{div}u||^2_{2,\Omega} \leqslant ||(1+\varrho)u^iu^j\beta^i_{x^j}||_{1,\Omega}+||(1+\varrho)u^if'^i||_{1,\Omega}$$
$$+\left\|\frac{R\gamma}{\gamma-1}[(1+\varrho)^{\gamma-1}-1](\varrho_{x^i}\beta^i+(1+\varrho)\beta^i_{x^i})\right\|_{1,\Omega} \leqslant \frac{3}{2}||u||^2_{4,\Omega}||\beta_x||_{2,\Omega}+C'_1||f'||^2_{2,\Omega}$$
$$+\varepsilon'||u||^2_{2,\Omega}+\frac{C'_2}{\varepsilon_1}||\beta||^2_{1,2,\Omega}+\varepsilon_1||\varrho_x||^2_{2,\Omega}+C'_3||b||^2_{1,2,\partial\Omega},$$

where we took into account $|\varrho| \leqslant 1/2$ and $||\varrho||_{2,\Omega} \leqslant ||\varrho_x||_{2,\Omega}+||\tilde{b}||_{1,2,\Omega}$, where $\tilde{b}|_{\partial\Omega} = b$. Using Eq. (4.1), the form of f' and assuming that ε' is sufficiently small, we obtain Eqs. (2.2) and (2.3). This concludes the proof.

LEMMA 4.2.

Let us assume that $f(t) \in L_2(\Omega)$, $\beta(t) \in \Gamma_1^2(\Omega) \cap L_\infty(0, T; H^3(\Omega))$, $b(t) \in \Gamma_1^2(\partial\Omega)$, $\eta(t) \in H^2(\partial\Omega)$,
and

(2.7)
$$E \leqslant \alpha_0, \quad ||\beta||_{3,2,\Omega} \leqslant \left(\frac{1}{2}\right)^{\gamma+2} \frac{R\gamma}{\mu+\nu},$$

where α_0 is a constant and

(2.8)
$$E = ||\varrho||^2_{2,2,\Omega}+|u|^2_{2,0,\Omega}.$$

Then the following estimate is valid

(2.9)
$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\varrho_x^2+\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^i}u^i\right)dx+\left(\frac{1}{2}\right)^{\gamma+1}\frac{R\gamma}{\mu+\nu}||\varrho_x||^2_{2,\Omega} \leqslant \varepsilon_2||u_{xxx}||^2_{2,\Omega}$$
$$+\varepsilon_3||u_{xx}||^2_{2,\Omega}+C[A_1\varepsilon_3^{-1}+\alpha_0^{\frac{n}{4-n}}\varepsilon_2^{-\frac{n}{4-n}}+\alpha_0+||\beta||^2_{3,2,\Omega}]||u_x||^2_{2,\Omega}+G_2(t),$$

where C is a constant, ε_1, ε_2 are arbitrary parameters, $A_1 = A_1(d_0, ||b||_{2,2,\partial\Omega})$ and

(2.10)
$$G_2(t) = ||f||^2_{2,\Omega}+||\beta^2_{2,1,\Omega}(1+||\beta||^2_{2,2,\Omega})+C(d_0)[|b|^2_{2,1,\Omega}(1+||\eta||^2_{2,2,\partial\Omega})$$
$$+(1+||b||^2_{2,2,\partial\Omega})||\beta||^2_{2,2,\Omega}].$$

P r o o f. We consider the expression

(2.11)
$$\left(\frac{1}{2}\varrho_{x^i}\varrho_{x^i}+\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^i}u^i\right)_{,t} = \left(\varrho_{x^i}+\frac{(1+\varrho)^2}{\mu+\nu}u^i\right)\varrho_{x^i t}+\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^i}u^i_t$$
$$+\frac{2(1+\varrho)}{\mu+\nu}\varrho_{x^i}u^i\varrho_t = -\left(\varrho_{x^i}+\frac{(1+\varrho)^2}{\mu+\nu}u^i\right)[(1+\varrho)(u^j+\beta^j)]_{,x^ix^j}+\frac{(1+\varrho)^2}{\mu+\nu}$$
$$\cdot\varrho_{x^i}\left[\frac{\mu}{1+\varrho}u^i_{x^jx^j}+\frac{\nu}{1+\varrho}u^j_{x^ix^j}-(u^j+\beta^j)u^i_{x^j}-u^j\beta^i_{x^j}-\frac{R[(1+\varrho)^\gamma]_{,x^i}}{1+\varrho}+f'^i\right]$$
$$-\frac{2(1+\varrho)}{\mu+\nu}\varrho_{x^i}u^i[(1+\varrho)(u^j+\beta^j)]_{,x^j},$$

where the time derivatives are replaced by Eqs. (2.4) and (2.5) [1]. At first we consider the following terms from the right-hand side of Eq. (2.11):

(2.12)
$$-(1+\varrho)\varrho_{x^i}u^j_{x^ix^j}+\frac{(1+\varrho)}{\mu+\nu}\varrho_{x^i}[\mu u^i_{x^jx^j}+\nu u^j_{x^ix^j}] = \frac{1+\varrho}{\mu+\nu}\mu\varrho_{x^i}(u^i_{x^jx^j}-u^j_{x^ix^j})$$
$$= \frac{\mu}{\mu+\nu}(1+\varrho)\varrho_{x^i}u^i_{x^jx^j}-\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^i}u^j_{x^i}]_{,x^j}+\frac{\mu}{\mu+\nu}\varrho_{x^i}\varrho_{x^j}u^j_{x^i}+\frac{\mu}{\mu+\nu}(1+\varrho)\varrho_{x^ix^j}u^j_{x^i}$$
$$= \frac{\mu}{\mu+\nu}(1+\varrho)\varrho_{x^i}u^i_{x^jx^j}-\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^i}u^j_{x^i}]_{,x^j}+\frac{\mu}{\mu+\nu}\varrho_{x^i}\varrho_{x^j}u^j_{x^i}+\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^j}u^j_{x^i}]_{,x^i}$$

$$-\frac{\mu}{\mu+\nu}\varrho_{x^i}\varrho_{x^j}u^j_{x^i}-\frac{\mu}{\mu+\nu}(1+\varrho)\varrho_{x^j}u^j_{x^ix^i}=-\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^i}u^j_{x^i}]_{,x^j}$$
$$+\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^j}u^j_{x^i}]_{,x^i}.$$

Using Eq. (2.12) in Eq. (2.11), we obtain

$$(2.13)\quad \left(\frac{1}{2}\varrho_{x^i}\varrho_{x^i}+\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^i}u^i\right)_{,t}=-\varrho_{x^i}[\varrho_{x^ix^j}(u^j+\beta^j)+\varrho_{x^i}(u^j+\beta^j)_{,x^j}+\varrho_{x^j}(u^j+\beta^j)_{,x^i}]$$
$$-(1+\varrho)\varrho_{x^i}\beta^j_{x^ix^j}-\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^i}[(u^j+\beta^j)u^i_{x^j}+u^j\beta^i_{x^j}]-\frac{1+\varrho}{\mu+\nu}R[(1+\varrho)^\gamma]_{,x^i}\varrho_{x^i}$$
$$+\frac{(1+\varrho)^2}{\mu+\nu}f^{i\prime}\varrho_{x^i}-\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^i}u^j_{x^i}]_{,x^i}+\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^j}u^j_{x^i}]_{,x^i}$$
$$-\frac{2(1+\varrho)}{\mu+\nu}\varrho_{x^i}u^i[(1+\varrho)(u^j+\beta^j)]_{,x^j}-\frac{(1+\varrho)^2}{\mu+\nu}u^i[(1+\varrho)(u^j+\beta^j)]_{,x^ix^j},$$

where the last term in Eq. (2.13) is replaced by

$$(2.14)\quad -\frac{1}{\mu+\nu}[(1+\varrho)^2u^i[(1+\varrho)(u^j+\beta^j)]_{,x^j}]_{,x^i}+\frac{1}{\mu+\nu}[(1+\varrho)^2u^i]_{,x^i}\cdot[(1+\varrho)(u^j+\beta^j)]_{,x^j}.$$

Using Eq. (2.14) in Eq. (2.13), we have

$$(2.15)\quad \left(\frac{1}{2}\varrho_x^2+\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^i}u^i\right)_{,t}+\frac{R\gamma}{\mu+\nu}(1+\varrho)^\gamma\varrho_x^2=-\frac{1}{2}(u+\beta)\cdot\nabla\varrho_x^2-\varrho_x^2\operatorname{div}(u+\beta)$$
$$-\varrho_{x^i}\varrho_{x^j}(u^j+\beta^j)_{,x^i}-(1+\varrho)\varrho_{x^i}\beta^j_{x^ix^j}-\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^i}[(u^j+\beta^j)u^i_{x^j}+u^j\beta^i_{x^j}]$$
$$+\frac{(1+\varrho)^2}{\mu+\nu}f^{\prime i}\varrho_{x^i}-\frac{2(1+\varrho)}{\mu+\nu}u\cdot\nabla\varrho(u+\beta)\cdot\nabla\varrho-\frac{2(1+\varrho)^2}{\mu+\nu}u\cdot\nabla\varrho\operatorname{div}(u+\beta)$$
$$+\frac{1}{\mu+\nu}[2(1+\varrho)u\cdot\nabla\varrho(u+\beta)\cdot\nabla\varrho+(1+\varrho)^2\operatorname{div}u(u+\beta)\cdot\nabla\varrho+2(1+\varrho)^2u\cdot\nabla\varrho\operatorname{div}(u+\beta)$$
$$+(1+\varrho)^3\operatorname{div}u\operatorname{div}(u+\beta)]-\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^i}u^j_{x^i}]_{,x^j}+\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^j}u^j_{x^i}]_{,x^i}$$
$$-\frac{1}{\mu+\nu}[(1+\varrho)^2u^i[(1+\varrho)(u^j+\beta^j)]_{,x^j}]_{,x^i}$$
$$=-\frac{1}{2}\operatorname{div}[(u+\beta)\varrho_x^2]-\frac{1}{2}\varrho_x^2\operatorname{div}(u+\beta)-\varrho_{x^i}\varrho_{x^j}(u^i+\beta^i)_{,x^j}-(1+\varrho)\varrho_{x^i}\beta^j_{x^ix^j}$$
$$-\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^i}[(u^j+\beta^j)u^i_{x^j}+u^j\beta^i_{x^j}]+\frac{(1+\varrho)^2}{\mu+\nu}f^{\prime i}\varrho_{x^i}+\frac{1}{\mu+\nu}[(1+\varrho)^2\operatorname{div}u(u+\beta)\cdot\nabla\varrho$$
$$+(1+\varrho)^3\operatorname{div}u\operatorname{div}(u+\beta)]-\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^i}u^j_{x^i}]_{,x^j}+\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^j}u^j_{x^i}]_{,x^i}$$
$$-\frac{1}{\mu+\nu}[(1+\varrho)^2u^i[(1+\varrho)(u^j+\beta^j)]_{,x^j}]_{,x^i}.$$

Integrating Eq. (2.15) over Ω, using $u|_{\partial\Omega} = 0$ and $|\varrho| \leqslant \frac{1}{2}$, we get

$$(2.16)\quad \frac{d}{dt}\int_{\Omega}\left(\varrho_x^2+\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^i}u^i\right)dx+\frac{R\gamma}{\mu+\nu}\left(\frac{1}{2}\right)^{\gamma}||\varrho_x||_2^2 \leqslant -\frac{1}{2}\int_{\partial\Omega}\beta\cdot\bar{n}\varrho_x^2$$
$$+\frac{27}{9(\mu+\nu)}||\mathrm{div}\,u||_{2,\Omega}^2+C_1'[||\varrho_x^2u_x||_{1,\Omega}+||\varrho_x^2\beta_x||_{1,\Omega}+||\varrho_x\beta_{xx}||_{1,\Omega}+||\varrho_x uu_x||_{1,\Omega}$$
$$+||\varrho_x u_x\beta||_{1,\Omega}+||\varrho_x u\beta_x||_{1,\Omega}+||f'\varrho_x||_{1,\Omega}+||u_x\beta_x||_{1,\Omega}]+C_2'||\varrho_x u_x||_{1,\partial\Omega}.$$

Now we estimate particular terms from the right-hand side of Eq. (2.16). The first term from the right-hand side of Eq. (2.16) has the following estimate:

$$-\int_{\partial\Omega}\beta_n\varrho_x^2\,ds \leqslant \int_{\partial\Omega}d\varrho_x^2\,ds \leqslant C||\eta||_{2,2,\partial\Omega}||b||_{1,2,\partial\Omega}^2+\frac{C}{d_0}(||b_t||_{2,\partial\Omega}^2+||\eta||_{1,2,\partial\Omega}^2||b||_{2,2,\partial\Omega}^2$$
$$+(1+||b||_{2,2,\partial\Omega}^2)||v_x||_{2,\partial\Omega}^2,$$

where we made use of the fact that $||\varrho_x||_{2,\partial\Omega} \leqslant C(||\varrho_{,\tau}||_{2,\partial\Omega}+||\varrho_{,n}||_{2,\partial\Omega})$, where $\varrho_{,n} = \frac{1}{d}[\varrho_t+v_\mu\varrho_{,\tau_\mu}+(1+\varrho)\mathrm{div}\,v]$ and $d \geqslant d_0 > 0$. At last, using Eq. (2.17) [1] we obtain

$$(2.17)\quad \int_{\partial\Omega}d\varrho_x^2\,ds \leqslant C(d_0)|b|_{2,1,\partial\Omega}^2(1+||\eta||_{2,2,\partial\Omega}^2)+\varepsilon_1||u_{xx}||_{2,\Omega}^2+A_1(||b||_{2,2,\partial\Omega},d_0)$$
$$\cdot\varepsilon_1^{-1}||u_x||_{2,\Omega}^2+C(d_0)(1+||b||_{2,2,\partial\Omega}^2)||\beta||_{2,2,\Omega}^2,$$

where the function A_1 is determined by the above considerations and ε_1 is an arbitrary parameter. Now we shall estimate terms from Eq. (2.16) which are multiplied by C_1':

$$||\varrho_x^2u_x||_{1,\Omega} \leqslant \sup_{\Omega}|u_x|\,||\varrho_x||_{2,\Omega}^2 \leqslant \frac{\varepsilon_2}{2}||\varrho_x||_{2,\Omega}^2+\frac{||\varrho_x||_{2,\Omega}^2}{2\varepsilon_2}(\sup_{\Omega}|u_x|)^2$$
$$\leqslant \frac{\varepsilon_2}{2}||\varrho_x||_{2,\Omega}^2+\varepsilon_3||u_{xxx}||_{2,\Omega}^2+C\left(\frac{||\varrho_x||_{2,\Omega}^2}{2\varepsilon_2}\right)^{\frac{4}{4-n}}\varepsilon_3^{-\frac{n}{4-n}}||u_x||_{2,\Omega}^2,$$

$$||\varrho_x^2\beta_x||_{1,\Omega} \leqslant \sup_{\Omega}|\beta_x|\,||\varrho_x||_{2,\Omega}^2 \leqslant ||\beta||_{3,2,\Omega}||\varrho_x||_{2,\Omega}^2,$$

$$||\varrho_x\beta_{xx}||_{1,\Omega} \leqslant \frac{\varepsilon_2}{2}||\varrho_x||_{2,\Omega}^2+\frac{1}{2\varepsilon_2}||\beta_{xx}||_{2,\Omega}^2,$$

$$(2.18)\quad ||\varrho_x uu_x||_{1,\Omega} \leqslant \sup_{\Omega}|u|\,||\varrho_x||_{2,\Omega}||u_x||_{2,\Omega} \leqslant \frac{\varepsilon_2}{2}||\varrho_x||_{2,\Omega}^2+\frac{(\sup_{\Omega}|u|)^2}{2\varepsilon_2}||u_x||_{2,\Omega}^2,$$

$$||\varrho_x u_x\beta||_{1,\Omega} \leqslant \frac{\varepsilon_2}{2}||\varrho_x||_{2,\Omega}^2+\frac{(\sup_{\Omega}|\beta|)^2}{2\varepsilon_2}||u_x||_{2,\Omega}^2,$$

$$||\varrho_x u\beta_x||_{1,\Omega} \leqslant \frac{\varepsilon_2}{2}||\varrho_x||_{2,\Omega}^2+\frac{(\sup_{\Omega}|\beta_x|^2)}{2\varepsilon_2}||u||_{2,\Omega}^2,$$

$$||f'\varrho_x||_{1,\Omega} \leqslant \frac{\varepsilon_2}{2}||\varrho_x||_{2,\Omega}^2+\frac{1}{2\varepsilon_2}||f'||_{2,\Omega}^2,$$

$$||u_x\beta_x||_{1,\Omega} \leqslant \frac{\varepsilon_4}{2}||u_x||^2_{2,\Omega}+\frac{1}{2\varepsilon_4}||\beta_x||^2_{2,\Omega},$$

where ε_2, ε_3, ε_4 are arbitrary parameters. In the end we shall estimate the term

$$(2.19)\quad ||\varrho_x u_x||_{1,\partial\Omega} \leqslant C||(\varrho_{,\tau}+\varrho_{,n})v_x||_{1,\partial\Omega} \leqslant C||b_\tau v_x||_{1,\partial\Omega}+C||\frac{1}{d}(b_t+\eta_\mu b_{,\tau_\mu}$$
$$+(1+b)\operatorname{div}v)v_x||_{1,\partial\Omega} \leqslant C(d_0)|b|^2_{2,1,\partial\Omega}(1+||\eta||^2_{2,2,\partial\Omega})+C(d_0)(1+||b||_{2,2,\Omega})||v_x||^2_{2,\partial\Omega}$$
$$\leqslant C(d_0)|b|^2_{2,1,\partial\Omega}(1+||\eta||^2_{2,2,\partial\Omega})+C(d_0)(1+||b||^2_{2,2,\Omega})||\beta||^2_{2,2,\Omega}+\varepsilon_1||u_{xx}||^2_{2,\Omega}$$
$$+A_1(||b||_{2,2,\partial\Omega},d_0)\,\varepsilon_1^{-1}||u_x||^2_{2,\Omega}.$$

Using Eqs. (2.17), (2.18) and (2.19) in Eq. (2.16) and assuming that $\varepsilon_2 = \varepsilon_3$, $\varepsilon_3 = 2\varepsilon_1$, $3\varepsilon_2 \leqslant \left(\frac{1}{2}\right)^{\gamma+2}\frac{R\gamma}{\mu+\nu}$, $||\beta||_{3,2,\Omega} \leqslant \left(\frac{1}{2}\right)^{\gamma+2}\frac{R\gamma}{\mu+\nu}$, we have

$$(2.20)\quad \frac{d}{dt}\int_\Omega\left(\varrho_x^2+\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^i}u^i\right)dx+\left(\frac{1}{2}\right)^{\gamma+1}\frac{R\gamma}{\mu+\nu}||\varrho_x||^2_{2,\Omega} \leqslant \varepsilon_2||u_{xxx}||^2_{2,\Omega}+\varepsilon_3||u_{xx}||^2_{2,\Omega}$$
$$+C\left[A_1\varepsilon_3^{-1}+||\varrho_x||^{\frac{8}{4-n}}_{2,\Omega}\,\varepsilon_2^{-\frac{n}{4-n}}+(\sup_\Omega|u|)^2+(\sup_\Omega|\beta|)^2+(\sup_\Omega|\beta_x|)^2\right]||u_x||^2_{2,\Omega}$$
$$+C[||f'||^2_{2,\Omega}+||\beta_{xx}||^2_{2,\Omega}+C(d_0)[|b|^2_{2,1,\partial\Omega}(1+||\eta||^2_{2,2,\partial\Omega})+(1+||b||^2_{2,2,\partial\Omega})||\beta||^2_{2,2,\Omega}],$$

where ε_2, ε_3 are arbitrary parameters and C is a constant. From Eq. (2.20) we obtain Eq. (2.9). This completes the proof.

LEMMA 4.3.

Let us assume that $f(t)\in L_2(\Omega)$, $\eta(t)\in H^2(\partial\Omega)$, $\beta\in L_\infty(0,T;H^3(\Omega))$, $\beta(t)\in\Gamma_1^2(\Omega)$, $b(t)\in\Gamma_1^2(\partial\Omega)$ for $t\in[0,T]$ and Eqs. (2.1) and (2.7) are satisfied. Then the following estimate is valid:

$$(2.21)\quad D_1\frac{d}{dt}\int_\Omega E^0dx+\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\varrho_x^2+\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^i}u^i\right)dx+\frac{d}{dt}\int_\Omega(1+\varrho)(u_x^2+u_t^2)dx$$
$$+\frac{d}{dt}\int_\Omega((\mu u_x^2+\nu(\operatorname{div}u)^2)dx+||u_x||^2_{2,\Omega}+||u_t||^2_{2,\Omega}+||\varrho_x||^2_{2,\Omega}+\frac{\mu}{2}(||u_{xx}||^2_{2,\Omega}$$
$$+||u_{xt}||^2_{2,\Omega}) \leqslant G_3(t)+\varepsilon_4||u_{xxx}||^2_{2,\Omega},$$

where ε_4 can be assumed sufficiently small, D_1 is determined by Eq. (2.23) and

$$(2.22)\quad G_3(t) = C(\alpha_0,||\beta||_{2,2,\Omega})[||f||^2_{2,\Omega}+|\beta|^2_{2,1,\Omega}(1+|\beta|^2_{2,1,\Omega}+|b|^2_{2,1,\Omega})$$
$$+(1+||\eta||^2_{2,2,\partial\Omega})|b|^2_{2,1,\partial\Omega}].$$

P r o o f. Multiplying Eq. (2.2) by a constant D_1 such that

$$(2.23)\quad D_1\varepsilon_1 \leqslant \left(\frac{1}{2}\right)^{\gamma+2}\frac{R\gamma}{\mu+\nu},\qquad \frac{1}{4}D_1\mu \geqslant C\left[A_1\varepsilon_3^{-1}+\alpha_0^{\frac{4}{4-n}}\varepsilon_2^{-\frac{n}{4-n}}+\alpha_0+||\beta||^2_{3,2,\Omega}\right],$$

and adding the result to Eq. (2.9), we obtain

$$(2.24)\quad D_1\frac{d}{dt}\int_\Omega\left(E^0+\frac{1}{2}\varrho_x^2+\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^i}u^i\right)dx+\frac{1}{4}D_1\mu||u_x||^2_{2,\Omega}+\left(\frac{1}{2}\right)^{\gamma+2}\frac{R\gamma}{\mu+\nu}||\varrho_x||^2_{2,\Omega}$$
$$\leqslant \varepsilon_2||u_{xxx}||^2_{2,\Omega}+\varepsilon_3||u_{xx}||^2_{2,\Omega}+G_1'(t),$$

where

$$G_1'(t) = G_2(t)+D_1 G_1(t). \tag{2.25}$$

From Eq. (4.22) [1] we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (1+\varrho)(u_t^2+u_x^2)dx+\frac{\mu}{2}\int_\Omega (u_{xx}^2+u_{xt}^2)dx \leqslant C_1'(||u_x||^2_{2,\Omega}+||u_t||^2_{2,\Omega})$$
$$+C_2'||\varrho_x||^2_{2,\Omega}+C_3'[||f||^2_{2,\Omega}+|\beta|^2_{2,1,\Omega}(1+|\beta|^2_{2,1,\Omega})], \tag{2.26}$$

where

$$C_1' = C[1+0(\alpha_0, |\beta|_{2,1,\Omega})] \geqslant C\Big[1+||1+\varrho||^2_{2,2,\Omega}(1+||u+\beta||^2_{2,2,\Omega})$$
$$+|1+\varrho|^{\frac{4}{4-n}}_{2,0,\Omega}\Big(1+|u+\beta|^{\frac{4}{4-n}}_{2,0,\Omega}\Big)\Big],$$

$$C_2' = C[1+0(\alpha_0, |\beta|_{2,1,\Omega})] \geqslant C[1+||u+\beta||^2_{2,2,\Omega}],$$

$$C_3' = C[1+0(\alpha_0, |\beta|_{2,1,\Omega})] \geqslant C[1+|\varrho|^2_{2,0,\Omega}].$$

In C_i', $i = 1, 2, 3$, we used the following estimates implied by Eq. (2.4) [1]

$$||\varrho_t||^2_{2,\Omega} \leqslant ||u+\beta||^2_{2,2,\Omega}(1+||1+\varrho||^2_{2,2,\Omega}) \leqslant C(||\beta||_{2,2,\Omega})(1+\alpha_0^2), \tag{2.27}$$

$$||\varrho_{tt}||^2_{2,\Omega} \leqslant |u+\beta|^2_{2,1,\Omega}(1+||1+\varrho||^2_{2,2,\Omega}+||1+\varrho||^2_{2,2,\Omega}||u+\beta||^2_{2,2,\Omega})$$
$$\leqslant C(|\beta|_{2,1,\Omega})(1+\alpha_0^3). \tag{2.28}$$

Moreover, from Eq. (4.9) [1] we have

$$\frac{1}{2}||u_t||^2_{2,\Omega}+\frac{d}{dt}\int_\Omega [\mu u_x^2+\nu(\operatorname{div} u)^2]dx \leqslant C_4'||u_x||^2_{2,\Omega}+C||\varrho_x||^2_{2,\Omega}$$
$$+C[||f||^2_{2,\Omega}+|\beta|^2_{2,1,\Omega}(1+|\beta|^2_{2,1,\Omega})], \tag{2.29}$$

where $C_4' = 0(\alpha_0, ||\beta||_{2,2,\Omega}) \geqslant C||u+\beta||^2_{2,2,\Omega}$. Multiplying Eq. (2.29) by a constant D_2 such that

$$\frac{1}{4}D_2 \geqslant C_1' \tag{2.30}$$

and adding to Eq. (2.26), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (1+\varrho)(u_x^2+u_t^2)dx+\frac{D_2}{4}||u_t||^2_{2,\Omega}+\frac{\mu}{2}(||u_{xx}||^2_{2,\Omega}+||u_{xt}||^2_{2,\Omega})+D_2\frac{d}{dt}\int_\Omega [\mu u_x^2$$
$$+\nu(\operatorname{div} u)^2]dx \leqslant (C_1'+D_2C_4')||u_x||^2_{2,\Omega}+(C_2'+D_2C)||\varrho_x||^2_{2,\Omega}+C(D_2+1)[||f||^2_{2,\Omega}$$
$$+|\beta|^2_{2,1,\Omega}(1+|\beta|^2_{2,1,\Omega})]. \tag{2.31}$$

Multiplying Eq. (2.24) by D_3 such that

$$\frac{D_1D_3\mu}{8} \geqslant C_1'+D_2C_4', \qquad \frac{R\gamma}{\mu+\nu}\left(\frac{1}{2}\right)^{\gamma+3}D_3 \geqslant C_2'+D_2C, \qquad D_3\varepsilon_3 \leqslant \frac{1}{4}\mu \tag{2.32}$$

and adding the result to Eq. (2.31), we obtain

$$D_1D_3\frac{d}{dt}\int_\Omega \left(E^0+\frac{1}{2}\varrho_x^2+\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^i}u^i\right)dx+\frac{1}{2}\frac{d}{dt}\int_\Omega (1+\varrho)(u_t^2+u_x^2)dx \tag{2.33}$$

$$+D_2 \frac{d}{dt} \int_\Omega (\mu u_x^2 + \nu |\operatorname{div} u|^2) dx + \frac{D_1 D_3}{8} ||u_x||^2_{2,\Omega} + \frac{D_2}{4} ||u_t||^2_{2,\Omega} + \frac{R\gamma D_3}{\mu+\nu} \left(\frac{1}{2}\right)^{\gamma+3} ||\varrho_x||^2_{2,\Omega}$$

$$+ \frac{\mu}{4} (||u_{xx}||^2_{2,\Omega} + ||u_{xt}||^2_{2,\Omega}) \leqslant G_2'(t) + \varepsilon_2 D_3 ||u_{xxx}||^2_{2,\Omega},$$

where

$$G_2'(t) = D_3 G_1'(t) + C(1+\alpha_0^3) [||f||^2_{2\Omega} + |\beta|^2_{2,1,\Omega}(1+|\beta|^2_{2,1,\Omega})]. \tag{2.34}$$

From Eqs. (2.33) and (2.34) we have Eqs. (2.21) and (2.22). This concludes the proof.

LEMMA 2.4.

Let us assume that $f(t) \in H^1(\Omega)$, $\beta(t) \in \Gamma_2^3(\Omega)$, $\beta \in L_\infty(0, T; H^3(\Omega))$, $\eta(t) \in \Gamma_0^2(\partial\Omega)$, $b(t) \in \Gamma_0^2(\partial\Omega)$ for $t \in [0, T]$ and Eqs. (2.1) and (2.7) are satisfied. Then the following estimate is valid:

$$\frac{d}{dt} \int_\Omega \left(\frac{1}{2} \varrho_{xx}^2 + \frac{(1+\varrho)^2}{\mu+\nu} u^i_{x^k} \varrho_{x^i x^k} \right) dx + \frac{R\gamma}{\mu+\nu} \left(\frac{1}{2}\right)^{\gamma+1} ||\varrho_{xx}||^2_{2,\Omega} \tag{2.35}$$

$$\leqslant \varepsilon_5 (||u_{xxx}||^2_{2,\Omega} + ||u_{xxt}||^2_{2,\Omega}) + C_1 ||\varrho_x||^2_{2,\Omega} + C_2 (||u_{xx}||^2_{2,\Omega}$$

$$+ ||u_{xt}||^2_{2,\Omega} + ||u_x||^2_{2,\Omega} + ||u_t||^2_{2,\Omega} + 1) + G_4(t),$$

where

$$C_1 = C\big(1 + 0(\alpha_0, ||\beta||_{3,2,\Omega})\big)\left(1 + \frac{1}{\varepsilon_6}\right), \quad C_2 = \Gamma(|b|_{2,0,\partial\Omega}, |\eta|_{2,0,\partial\Omega}, |\beta|_{3,2,\Omega}),$$

has a polynomial form with respect to its arguments and

$$G_4(t) = C[||f||^2_{1,2,\Omega} + (1+|\beta|^2_{2,1,\Omega}) |\beta|^2_{3,1,\Omega} + G(|b|_{2,0,\partial\Omega}, |\eta|_{2,0,\partial\Omega}) |b|^2_{2,1,\partial\Omega}]. \tag{2.36}$$

P r o o f. We consider the expression

$$\left(\frac{1}{2} \varrho_{x^i x^k} \varrho_{x^i x^k} + \frac{(1+\varrho)^2}{\mu+\nu} u^i_{x^k} \varrho_{x^i x^k} \right)_{,t} = \varrho_{x^i x^k} \varrho_{x^i x^k t} + \frac{(1+\varrho)^2}{\mu+\nu} (u^i_{x^k} \varrho_{x^i x^k t} \tag{2.37}$$

$$+ u^i_{x^k t} \varrho_{x^i x^k}) + \frac{2(1+\varrho)}{\mu+\nu} \varrho_t u^i_{x^k} \varrho_{x^i x^k} = -\left(\varrho_{x^i x^k} + \frac{(1+\varrho)^2}{\mu+\nu} u^i_{x^k} \right) [(1+\varrho)(u^j + \beta^j)]_{,x^j x^i x^k}$$

$$+ \frac{(1+\varrho)^2}{\mu+\nu} \varrho_{x^i x^k} \left[\frac{\mu}{1+\varrho} u^i_{x^j x^j} + \frac{\nu}{1+\varrho} u^j_{x^i x^j} - (u^j + \beta^j) u^i_{x^j} - u^j \beta^i_{x^j} - \frac{R[(1+\varrho)^\gamma]_{,x^i}}{1+\varrho} + f'^i \right]_{,x^k}$$

$$- \frac{2(1+\varrho)}{\mu+\nu} u^i_{x^k} \varrho_{x^i x^k} [(1+\varrho)(u^j + \beta^j)]_{,x^j}.$$

In the expression (2.37) we consider the following terms:

$$-(1+\varrho) \varrho_{x^i x^k} u^j_{x^i x^k x^j} + \frac{(1+\varrho)}{\mu+\nu} \varrho_{x^i x^k} [\mu u^i_{x^j x^j x^k} + \nu u^j_{x^i x^j x^k}] \tag{2.38}$$

$$= \frac{\mu}{\mu+\nu} (1+\varrho) \varrho_{x^i x^k} [u^i_{x^j x^j x^k} - u^j_{x^i x^j x^k}] = \frac{\mu}{\nu+\mu} (1+\varrho) \varrho_{x^i x^k} u^i_{x^j x^j x^k}$$

(2.38) [cont.]
$$-\frac{\mu}{\mu+\nu}(1+\varrho)\varrho_{x^lx^k}u^j_{x^lx^k}]_{,x^j}+\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^lx^k}]_{,x^j}u^j_{x^lx^k}=\frac{\mu}{\mu+\nu}(1+\varrho)\varrho_{x^lx^k}u^i_{x^jx^jx^k}$$
$$-\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^lx^k}u^j_{x^lx^k}]_{,x^j}+\frac{\mu}{\mu+\nu}\varrho_{x^j}\varrho_{x^lx^k}u^j_{x^lx^k}+\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^jx^k}u^j_{x^lx^k}]_{,x^l}$$
$$-\frac{\mu}{\mu+\nu}\varrho_{x^l}\varrho_{x^jx^k}u^j_{x^lx^k}-\frac{\mu}{\mu+\nu}(1+\varrho)\varrho_{x^jx^k}u^j_{x^lx^lx^k}=$$
$$=-\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^lx^k}u^j_{x^lx^k}]_{,x^j}+\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^jx^k}u^j_{x^lx^k}]_{,x^l}$$
$$+\frac{\mu}{\mu+\nu}\varrho_{x^j}\varrho_{x^lx^k}u^j_{x^lx^k}-\frac{\mu}{\mu+\nu}\varrho_{x^l}\varrho_{x^jx^k}u^j_{x^lx^k}.$$

Using Eq. (2.38) in Eq. (2.37), we obtain

(2.39)
$$\left(\frac{1}{2}\varrho_{x^lx^k}\varrho_{x^lx^k}+\frac{(1+\varrho)^2}{\mu+\nu}u^i_{x^k}\varrho_{x^lx^k}\right)_{,t}=-\varrho_{x^lx^k}[(1+\varrho)\beta^j]_{,x^lx^kx^j}-\varrho_{x^lx^k}\varrho_x\{{}_{l}{}_{x^k}u^j_{x^j}\}$$
$$-\varrho_{x^lx^k}\varrho_x\{{}_{l}u^j_{x^kx^j}\}+\frac{\mu}{\mu+\nu}\varrho_{x^j}\varrho_{x^lx^k}u^j_{x^lx^k}-\frac{\mu}{\mu+\nu}\varrho_{x^l}\varrho_{x^jx^k}u^j_{x^lx^k}-\varrho_{x^lx^k}\varrho_{x^lx^kx^j}u^j$$
$$-\frac{1}{\mu+\nu}\lfloor(1+\varrho)^2u^i_{x^k}[(1+\varrho)(u^j+\beta^j)]_{,x^jx^k}\rfloor_{,x^l}+\frac{1}{\mu+\nu}[(1+\varrho)^2u^i_{x^k}]_{,x^l}[(1+\varrho)(u^j+\beta^j)]_{,x^jx^k}$$
$$-\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^lx^k}u^j_{x^lx^k}]_{,x^j}+\frac{\mu}{\mu+\nu}[(1+\varrho)\varrho_{x^jx^k}u^j_{x^lx^k}]_{,x^l}$$
$$-\frac{1}{\mu+\nu}\varrho_{x^lx^k}[\mu\varrho_{x^k}u^i_{x^jx^j}+\nu\varrho_{x^k}u^j_{x^lx^j}]-\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^lx^k}[(u^j+\beta^j)u^i_{x^j}+u^j\beta^i_{x^j}]_{,x^k}$$
$$-\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^lx^k}\left[\frac{R[(1+\varrho)^\gamma]_{,x^l}}{1+\varrho}\right]_{,x^k}+\frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^lx^k}f'^i_{x^k}$$
$$-\frac{2(1+\varrho)}{\mu+\nu}u^i_{x^k}\varrho_{x^lx^k}[(1+\varrho)(u^j+\beta^j)]_{,x^j},$$

where $\{i,j,k\}$ denotes all permutations of i, j, k. Integrating Eq. (2.39) over Ω, we get

(2.40)
$$\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\varrho^2_{xx}+\frac{(1+\varrho)^2}{\mu+\nu}u^i_{x^k}\varrho_{x^lx^k}\right)dx+\left(\frac{1}{2}\right)^\gamma\frac{R\gamma}{\mu+\nu}||\varrho_{xx}||^2_{2,\Omega}\leqslant-\frac{1}{2}\int_\Omega\operatorname{div}(\beta\varrho^2_{xx})dx$$
$$+C'_1[||\varrho^2_{xx}u_x||_{1,\Omega}+||\varrho_{xx}\varrho_xu_{xx}||_{1,\Omega}+||(\varrho_xu_x+u_{xx})[\varrho_{xx}(u+\beta)+\varrho_x(u_x+\beta_x)+u_{xx}+\beta_{xx}]||_{1,\Omega}$$
$$+||\varrho_{xx}[(u+\beta)_xu_x+(u+\beta)u_{xx}]||_{1,\Omega}+||\varrho_{xx}(u_x\beta_x+u\beta_{xx})||_{1,\Omega}+||\varrho_{xx}\varrho^2_x||_{1,\Omega}+||\varrho_{xx}f'_x||_{1,\Omega}$$
$$+||u_x\varrho_{xx}(u_x+\beta_x)||_{1,\Omega}+||u_x\varrho_{xx}\varrho_x(u+\beta)||_{1,\Omega}+||\varrho^2_{xx}\beta_x||_{1,\Omega}+||\varrho_{xx}\varrho_x\beta_{xx}||_{1,\Omega}+||\varrho_{xx}\beta_{xxx}||_{1,\Omega}]$$
$$+C'_2(||u_x[(1+\varrho)(u+\beta)]_{xx}||_{1,\partial\Omega}+||\varrho_{xx}u_{xx}||_{1,\partial\Omega})$$
$$\leqslant\frac{1}{2}||d\varrho^2_{xx}||_{1,\partial\Omega}+C'_1(||\varrho_{xx}u_{xx}||_{1,\partial\Omega}+||u_x[(1+\varrho)(u+\beta)]_{xx}||_{1,\partial\Omega})$$
$$+C'_2(||\varrho^2_{xx}\beta_x||_{1,\Omega}+||\varrho_{xx}\varrho_x\beta_{xx}||_{1,\Omega}+||\varrho_{xx}\beta_{xxx}||_{1,\Omega}+||\varrho^2_{xx}u_x||_{1,\Omega}+||\varrho_{xx}\varrho_xu_{xx}||_{1,\Omega}$$

$$+\|\varrho_{xx}\varrho_x u_x(u+\beta)\|_{1,\Omega}+\|\varrho_{xx}u_{xx}(u+\beta)\|_{1,\Omega}+\|\varrho_x^2 u_x(u_x+\beta_x)\|_{1,\Omega}+\|u_{xx}\varrho_x(u_x+\beta_x)\|_{1,\Omega}$$
$$+\|\varrho_x u_x\beta_{xx}\|_{1,\Omega}+\|u_{xx}(u_{xx}+\beta_{xx})\|_{1,\Omega}+\|\varrho_{xx}u_x^2\|_{1,\Omega}+\|\varrho_{xx}(u_x\beta_x+u\beta_{xx})\|_{1,\Omega}$$
$$+\|\varrho_{xx}\varrho_x^2\|_{1,\Omega}+\|\varrho_{xx}f_x'\|_{1\,\Omega},$$

where C_1', C_2' are constants. Now we shall estimate the terms from the right-hand side of the inequality (2.40). At first we shall consider the coefficients of C_2'. Using the interpolation inequality (2.16) [1], we have

$$\sup|u_x|\,\|\varrho_{xx}\|_{2,\Omega}^2 \leqslant \frac{\varepsilon}{2}\|\varrho_{xx}\|_{2,\Omega}^2+\varepsilon_1\|u_{xxx}\|_{2,\Omega}^2+C\left(\frac{\|\varrho_x\|_{2,\Omega}^2}{2\varepsilon}\right)^{\frac{4-n}{4}}\varepsilon_1^{-\frac{n}{4-n}}\|u_x\|_{2,\Omega}^2,$$

$$\|\varrho_x\varrho_{xx}u_{xx}\|_{1,\Omega} \leqslant \frac{\varepsilon}{2}\|\varrho_{xx}\|_{2,\Omega}^2+\varepsilon_2\|u_{xxx}\|_{2,\Omega}^2+C\left(\frac{\|\varrho_x\|_{1,2,\Omega}^2}{2\varepsilon}\right)^{\frac{4}{4-n}}\varepsilon_2^{-\frac{n}{4-n}}\|u_{xx}\|_{2,\Omega}^2,$$

$$\|\varrho_x u_x u_{xx}\|_{1,\Omega} \leqslant \frac{1}{2}\|\varrho_x\|_{2,\Omega}^2\|u_x\|_{4,\Omega}^2+\varepsilon_3\|u_{xxx}\|_{2,\Omega}^2+C\varepsilon_3^{-\frac{4}{4-n}}\|u_{xx}\|_{2,\Omega}^2,$$

$$\|\varrho_x^2u_x^2\|_{1,\Omega} \leqslant C(\|\varrho_{xx}\|_{2,\Omega}^2+\|\varrho_x\|_{2,\Omega}^2)\|u_x\|_{1,2,\Omega}^2,$$

$$\|\varrho_x u_x u\varrho_{xx}\|_{1,\Omega} \leqslant \varepsilon\|\varrho_{xx}\|_{2,\Omega}^2+C\|u\|_{2,2,\Omega}^2\|\varrho\|_{2,2,\Omega}^2\|u\|_{2,2,\Omega}^2,$$

(2.41) $$\|u\varrho_{xx}u_{xx}\|_{1,\Omega} \leqslant \frac{\varepsilon}{2}\|\varrho_{xx}\|_{2,\Omega}^2+\frac{1}{2\varepsilon}(\sup_\Omega|u|)^2\|u_{xx}\|_{2,\Omega}^2,$$

$$\|\varrho_{xx}\varrho_x^2\|_{1,\Omega} \leqslant \varepsilon\|\varrho_{xx}\|_{2,\Omega}^2+C\|\varrho\|_{2,2,\Omega}^{\frac{4}{4-n}}\|\varrho_x\|_{2,\Omega}^2,$$

$$\|u_x^2\varrho_{xx}\|_{1,\Omega} \leqslant \frac{\varepsilon}{2}\|\varrho_{xx}\|_{2,\Omega}^2+\frac{1}{2\varepsilon}\|u_x\|_{1,2,\Omega}^2,$$

$$\|\varrho_{xx}(u_x\beta_x+u\beta_{xx})\|_{1,\Omega} \leqslant \varepsilon\|\varrho_{xx}\|_{2,\Omega}^2+\frac{1}{2\varepsilon}(\sup_\Omega|\beta_x|)^2\|u_{xx}\|_{2,\Omega}^2+\frac{1}{2\varepsilon}(\sup_\Omega|u|)^2\|\beta_x\|_{2,\Omega}^2.$$

Using Eq. (2.41), the term in Eq. (2.40) multiplied by C_2' can be estimated by

(2.42) $$(\varepsilon_4+\alpha_0)\|\varrho_{xx}\|_{2,\Omega}^2+\varepsilon_5\|u_{xxx}\|_{2,\Omega}^2+C_3'(\|\varrho_x\|_{2,\Omega}^2+\|u_x\|_{2,\Omega}^2+\|u_{xx}\|_{2,\Omega}^2)+G_1'(t),$$

where ε_4, ε_5 can be assumed sufficiently small:

(2.43)
$$C_3' = C\left[1+0(\alpha_0,\|\beta\|_{3,2,\Omega})\left(1+0\left(\frac{1}{\varepsilon_5}\right)\right)\right],$$
$$G_1'(t) = C(\|f_x\|_{2,\Omega}^2+(1+\|\beta\|_{2,2,\Omega}^2)\|\beta\|_{3,2,\Omega}^2).$$

Using the curvilinear coordinates, the boundary conditions (1.4), the expressions (3.5)÷(3.9) [1], the interpolation inequality (2.17) [1] the boundary terms in Eq. (2.40), we estimate in the following way:

(2.44) $$\int_{\partial\Omega}(1+\varrho)|\varrho_{xx}u_{xx}|\,ds+\int_{\partial\Omega}|u_x[(1+\varrho)(u+\beta)]_{,xx}|\,ds-\int_{\partial\Omega}\beta n\varrho_{xx}^2\,ds \leqslant \Gamma_1(b',\eta',\beta',d')$$
$$+\Gamma_2(b',\eta',\beta',d')(\|u_{xx}\|_{2,\partial\Omega}^2+\|u_{xt}\|_{2,\partial\Omega}^2+\|u_x\|_{2,\partial\Omega}^2+\|u_t\|_{2,\partial\Omega}^2) \leqslant \varepsilon_6(\|u_{xxx}\|_{2,\Omega}^2$$
$$+\|u_{xxt}\|_{2,\Omega}^2)+\Gamma_3(b',\eta',\beta',d')[\|u_{xx}\|_{2,\Omega}^2+\|u_{xt}\|_{2,\Omega}^2+\|u_x\|_{2,\Omega}^2+\|u_t\|_{2,\Omega}^2+1],$$

where

$$b' = |b|_{2,0,\partial\Omega}, \quad \eta' = |\eta|_{2,0,\partial\Omega}, \quad \beta' = |\beta|_{2,1,\partial\Omega}, \quad d' = ||d||_{2,2,\partial\Omega}\,.$$

and Γ_i, $i = 1, 2, 3$, are polynomials of their arguments. Using Eqs. (2.42), (2.43), (2.44) in Eq. (2.40), we conclude the proof.

LEMMA 2.5.

Let us assume that $f(t) \in \Gamma_0^1(\Omega)$, $f_{tt}(t) \in L_2(\Omega)$, $\beta(t) \in \Gamma_0^3(\Omega)$, $\beta_{xxtt}(t) \in L_2(\Omega)$, $\beta \in L_\infty(0, T; H^3(\Omega))$, $\eta(t) \in \Gamma_0^2(\partial\Omega)$, $b(t) \in \Gamma_0^2(\partial\Omega)$, $t \in [0, T]$, Eqs. (2.1), (2.7) and (2.37) are satisfied and

$$\alpha_2^2 0(\alpha_0) \leqslant 1/2\mu, \tag{2.45}$$

where α_2 is the constant from Eq. (2.15) [1]. Then the following estimate is valid:

$$\begin{aligned}(2.46)\quad & D_1 D_4 \frac{d}{dt}\int_\Omega E^0 dx + D_4 \frac{d}{dt}\int_\Omega \left(\frac{1}{2}\varrho_x^2 + \frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^i}u^i\right)dx \\ & + D_4\frac{d}{dt}\int_\Omega (1+\varrho)(u_x^2+u_t^2)dx + D_4\frac{d}{dt}\int_\Omega [\mu u_x^2 + \nu(\operatorname{div} u)^2]dx + \frac{1}{4}D_4(||u_x||_{2,\Omega}^2 \\ & + ||u_t||_{2,\Omega}^2 + ||\varrho_x||_{2,\Omega}^2) + \frac{\mu}{8}D_4(||u_{xx}||_{2,\Omega}^2 + ||u_{xt}||_{2,\Omega}^2) + \frac{d}{dt}\int_\Omega\left(\frac{1}{2}\varrho_{xx}^2\right. \\ & \left. + \frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^ix^k}u^i_{x^k}\right)dx + \frac{R\gamma}{\mu+\nu}\left(\frac{1}{2}\right)^{\gamma+1}||\varrho_{xx}||_{2,\Omega}^2 + \frac{d}{dt}\int_\Omega (1+\varrho)(u_{xx}^2+u_{xt}^2+u_{tt}^2)dx \\ & + \frac{\mu}{2}(||u_{xxx}||_{2,\Omega}^2 + ||u_{xxt}||_{2,\Omega}^2 + ||u_{xtt}||_{2,\Omega}^2) \leqslant G_5(t),\end{aligned}$$

where D_4 is determined by Eq. (2.48) and

$$\begin{aligned}(2.47)\quad G_5(t) = {} & C(\alpha_0, ||\beta||_{2,2,\Omega})[|f|_{1,0,\Omega}^2 + ||f_{tt}||_{2,\Omega}^2 + (1+|\beta|_{2,0,\Omega}^2 + |b|_{2,1,\partial\Omega}^2)|\beta|_{3,0,\Omega}^2 \\ & + ||\beta_{xxtt}||_{2,\Omega}^2 + G(|b|_{2,0,\partial\Omega}, |\eta|_{2,0,\partial\Omega})|b|_{2,1,\partial\Omega}^2].\end{aligned}$$

Proof. Multiplying Eq. (2.21) by D_4 such that

$$\frac{1}{4}D_4 \geqslant C_1, \quad \frac{1}{4}\mu D_4 \geqslant C_2, \tag{2.48}$$

and adding the result to Eq. (2.35), we obtain

$$\begin{aligned}(2.49)\quad & D_1 D_4 \frac{d}{dt}\int_\Omega E^0 dx + D_4 \frac{d}{dt}\int_\Omega \left(\frac{1}{2}\varrho_x^2 + \frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^i}u^i\right)dx \\ & + D_4\frac{d}{dt}\int_\Omega (1+\varrho)(u_x^2+u_t^2)dx + D_4\frac{d}{dt}\int_\Omega [\mu u_x^2 + \nu(\operatorname{div} u)^2]dx + \frac{1}{2}D_4(||u_x||_{2,\Omega}^2 \\ & + ||u_t||_{2,\Omega}^2 + ||\varrho_x||_{2,\Omega}^2) + \frac{\mu}{4}D_4(||u_{xx}||_{2,\Omega}^2 + ||u_{xt}||_{2,\Omega}^2 + \frac{d}{dt}\int_\Omega\left(\frac{1}{2}\varrho_{xx}^2\right. \\ & \left. + \frac{(1+\varrho)^2}{\mu+\nu}\varrho_{x^ix^k}u^i_{x^k}\right)dx + \frac{R\gamma}{\mu+\nu}\left(\frac{1}{2}\right)^{\gamma+1}||\varrho_{xx}||_{2,\Omega}^2 \leqslant D_4 G_3 + G_4 \\ & + \varepsilon_4 D_4||u_{xxx}||_{2,\Omega}^2 + \varepsilon_5(||u_{xxx}||_{2,\Omega}^2 + ||u_{xxt}||_{2,\Omega}^2).\end{aligned}$$

From Eq. (4.37) [1] we have

$$(2.50)\quad \frac{d}{dt}\int_\Omega (1+\varrho)(u_{xx}^2+u_{xt}^2+u_{tt}^2)dx+\mu(||u_{xxx}||_{2,\Omega}^2+||u_{xxt}||_{2,\Omega}^2+||u_{xtt}||_{2,\Omega}^2)$$
$$\leqslant C\big(0(\alpha_0)+1\big)(1+|\beta|_{3,1,\Omega}^2)|u|_{2,1,\Omega}^2+0(\alpha_0)||u_{tt}||_{2,\Omega}^2+0(\alpha_0)(||\varrho_{xx}||_{2,\Omega}^2+||\varrho_x||_{2,\Omega}^2$$
$$+||u||_{2,2,\Omega}^2)+(1+\alpha_0)|f|_{1,0,\Omega}^2+||f_{tt}||_{2,\Omega}^2+|\beta|_{3,0,\Omega}^2+||\beta_{xxtt}||_{2,\Omega}^2$$
$$+0(\alpha_0)(1+|\beta|_{2,0,\Omega}^2)|\beta|_{3,0,\Omega}^2.$$

Adding Eq. (2.50) to Eq. (2.49), using $||u_{tt}||_{2,\Omega}\leqslant \alpha_2||u_{xtt}||_{2,\Omega}$, Eq. (2.45), assuming that ε_4, ε_5 are sufficiently small and

$$\frac{1}{4}D_4+\frac{\mu}{8}D_4\geqslant 0(\alpha_0)+C\big(0(\alpha_0)+1\big)(1+|\beta|_{3,1,\Omega}^2),$$

we obtain Eq. (2.46). This completes the proof.

Using the above lemmas we shall formulate the following main result of this section.

THEOREM 2.1. *Let us assume that*

1) $a\in H^4(\Omega)$, $\sigma\in H^3(\Omega)$,
2) $f(0)\in H^1(\Omega)$, $f_t(0)\in L_2(\Omega)$,
3) $f\in \Pi_{0,2}^1(\Omega^\infty)$, $f_{tt}\in L_2(\Omega^\infty)$,
4) $\beta(0)\in H^2(\Omega)$, $\beta_t(0)\in H^1(\Omega)$, $\beta_{tt}(0)\in L_2(\Omega)$,
5) $\beta\in \Pi_{0,\infty}^2(\Omega^\infty)$, $b\in \Pi_{0,\infty}^2(\partial\Omega^\infty)$, $\eta\in \Pi_{0,\infty}^2(\partial\Omega^\infty)$, $b\in \Pi_{0,2}^2(\partial\Omega^\infty)$,
6) $\beta\in \Pi_{0,2}^3(\Omega^\infty)$, $\beta_{xxtt}\in L_2(\Omega^\infty)$,

and

$$(2.51)\quad \sup_t||\varrho||_{2,2,\Omega}^2+|u|_{2,0,\infty,\Omega^\infty}^2\leqslant \alpha_0,\qquad \sup_t||\beta||_{3,2,\Omega}\leqslant\left(\frac{1}{2}\right)^{\gamma+2}\frac{R\gamma}{\mu+\nu},$$

where

$$(2.52)\quad \alpha_2^2 0(\alpha_0)\leqslant \frac{1}{2}\mu,$$

then the following global in time estimate

$$(2.52)\quad |\varrho|_{2,0,\infty,\Omega^t}^2+|u|_{2,0,\infty,\Omega^t}^2+\frac{\mu}{2}(|\varrho|_{2,0,2,\Omega^t}^2+|u|_{3,1,2,\Omega^t}^2)\leqslant K_1(R_1)N_1+K_2(R_2)N_2,$$

is valid, where $t\in[0,\infty]$ *and*

$$R_1=\alpha_0+|\beta|_{2,0,\infty\Omega^\infty}^2+|b|_{2,0,\infty,\partial\Omega^\infty}^2+|\eta|_{2,0,\infty,\partial\Omega^\infty}^2,$$
$$R_2=||a||_{2,2,\Omega}^2+||\sigma||_{2,2,\Omega}^2,$$
$$N_1=|f|_{1,0,2,\Omega^\infty}^2+||f_{tt}||_{2,\Omega^\infty}^2+|\beta|_{3,0,2,\Omega^\infty}^2+||\beta_{xxtt}||_{2,\Omega^\infty}^2+|b|_{2,0,2,\partial\Omega^\infty}^2,$$
$$N_2=||a||_{4,2,\Omega}^2+||\sigma||_{3,2,\Omega}^2+|f|_{1,0,\Omega}^2\big|_{t=0}+|\beta|_{2,0,\Omega}^2\big|_{t=0}.$$

At last, from the inequalities (2.51) *and* (2.53) *the following restriction on the given data functions must be assumed:*

$$(2.54)\quad K_1(R_1)N_1+K_2(R_2)N_2\leqslant \alpha_0.$$

P r o o f. Integrating Eq. (2.46) with respect to time, using the inequalities: $E^0 \geqslant \geqslant \frac{\gamma R}{4}\varrho^2+\frac{1}{4}u^2$, $\frac{(1+\varrho)}{\mu+\nu}u^i\varrho_{x^i} \geqslant -\frac{1}{4}\varrho_x^2-\frac{(1+\varrho)^4}{(\mu+\nu)^2}u^2$, $\frac{(1+\varrho)^2}{\mu+\nu}u^i_{x^k}\varrho_{x^ix^k} \geqslant -\frac{1}{4}\varrho_{xx}^2-\frac{(1+\varrho)^4}{(\mu+\nu)^2}u_x^2$ and assuming that $D_1 \geqslant 8\frac{(1+\varrho)^4}{\mu+\nu}$, $D_4 \geqslant 4\frac{(1+\varrho)^3}{\mu+\nu}$, we obtain Eq. (2.53). This concludes the proof.

3. Global in time solutions

The global estimates of Sect. 2 are crucial for the extension of the existence and uniqueness of solution results obtained in Sect. 5 of [1] which are local in time. Using Eq. (1.12), similarly as in [3] the following theorem can be stated.

THEOREM 3.1. *Let us assume that*

$a \in H^4(\Omega)$, $\sigma \in H^3(\Omega)$,

$f(0) \in \Gamma_0^1(\Omega)$, $f \in \Pi^1_{0,2}(\Omega^\infty)$, $f_{tt} \in L_2(\Omega^\infty)$,

$\eta(0) \in \Gamma_0^{2+1/2}(\partial\Omega)$, $\eta \in \Pi^{3+1/2}_{0,2}(\partial\Omega^\infty)\cap\Pi^{2+1/2}_{0,\infty}(\partial\Omega^\infty)$, $\eta_{tt} \in \Pi^{2+1/2}_{2,2}(\partial\Omega^\infty)$,

$b \in \Pi^2_{0,2}(\partial\Omega^\infty)\cap\Pi^2_{0,\infty}(\partial\Omega^\infty)$,

Eqs. (2.51) *and* (2.52) *are satisfied and* $\sup_t \|\eta\|_{3+1/2,2,\partial\Omega} \leqslant \left(\frac{1}{2}\right)^{\gamma+2}\frac{R\gamma}{\mu+\nu}$, *then there exists a global solution of the problem* (1.9)÷(1.14) *such that* $\varrho \in \Pi^2_{0,\infty}(\Omega^\infty)\cap\Pi^2_{0,2}(\Omega^\infty)$, $u \in \mathfrak{M}(\Omega^\infty) \equiv \Pi^3_{1,2}(\Omega^\infty)\cap\Pi^2_{0,\infty}(\Omega^\infty)$ *and*

$$(3.1)\qquad |\varrho|^2_{2,0,\infty,\Omega^t}+|u|^2_{2,0,\infty,\Omega^t}+\frac{\mu}{2}(|\varrho|^2_{2,0,2,\Omega^t}+|u|^2_{3,1,2,\Omega^t}) \leqslant \bar{K}_1(\bar{R}_1)\bar{N}_1+\bar{K}_2(\bar{R}_2)\bar{N}_2.$$

is valid, where $t \in [0,\infty]$ *and*

$$\bar{R}_1 = \alpha_0+|\eta|^2_{5/2,0,\infty,\partial\Omega^\infty}+|b|^2_{2,0,\infty,\partial\Omega^\infty},$$

$$\bar{R}_2 = \|a\|^2_{2,2,\Omega}+\|\sigma\|^2_{2,2,\Omega},$$

$$\bar{N}_1 = |f|^2_{1,0,2,\Omega^\infty}+\|f_{tt}\|^2_{2,\Omega^\infty}+|\eta|^2_{3+1/2,0,2,\Omega\partial^\infty}+|\eta_{tt}|^2_{2+1/2,2,2,\partial\Omega^\infty}+|b|^2_{2,0,2,\partial\Omega^\infty},$$

$$\bar{N}_2 = \|a\|^2_{4,2,\Omega}+\|\sigma\|^2_{3,2,\Omega}+|f|^2_{1,0,\Omega}|_{t=0}+|\eta|^2_{2+1/2,0,\partial\Omega}|_{t=0}.$$

Moreover, a similar restriction to Eq. (2.54) *is satisfied.*

P r o o f. From the global estimate (2.53) it follows that the function $G(T, \mathsf{R}_t, \mathsf{N}_t, \mathsf{M}_t, \chi_T)$ described by Eq. (5.2) of [1], where $t \in [kT, (k+1)T]$, $k \geqslant 0$ is an integer, has the following estimate:

$$(3.2)\qquad G(T, \mathsf{R}_t, \mathsf{N}_t, \mathsf{M}, \chi_T) \leqslant G(T, \mathsf{R}_\infty, \mathsf{N}_\infty, \mathsf{M}, \chi_T),$$

where $\mathsf{R} = \mathsf{R}_t$, $\mathsf{N} = \mathsf{N}_s$, $\chi = \chi_T$ are described in the explanation of Eq. (5.2) of [1]. Therefore there exist R_∞, N_∞, M (we must add some additional restriction on the data functions which can be expressed by the assumption that α_0 is sufficiently small) and T, such that Eq. (5.3) of [1] is valid for $\chi = \chi_T$. Hence Eq. (5.4) of [1] is valid also. Thus for $t \in [0, T]$ Theorem 5.1 of [1] implies that there exists a solution $\varrho(t)$, $u(t)$. Then $\varrho(T)$, $u(T)$ are known and we can use Theorem (2.1) of [1] for $t \in [T, 2T]$ because of Eq. (3.2) with the initial values $\varrho(T)$, $u(T)$ and so on. Knowing that the global estimate (2.53) is valid, we have the existence of global solutions. The uniqueness can be proved in the same way as in [4] on each interval $[kT, (k+1)T]$ where $k \geqslant 0$ is an integer. This concludes the proof.

4. Closing remarks

The existence and uniqueness of solutions of the initial-boundary value problem of a barotropic viscous flow global in time is proved for the case of continuous density and first derivatives of velocity. The same method can be used to deduce similarly arbitrary smooth solutions if the initial and boundary conditions as well as the external forces are sufficiently smooth. The dimensionality of the physical space considered has no effect on the proofs. It should be noted that to complete the proofs it is necessary to have a strictly negative normal component of velocity on the whole boundary and the density must be given there. However, it should be emphasized the proof is valid for the case when the total inflow of mass and energy are sufficiently small as shown in the assumptions of Theorems 2.1, 3.1. We would like to remark also that the barotropic condition used can be replaced by the more general heat conducting fluid model.

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