Arch. Mech., 35, 4, pp. 541-457, Warszawa 1983

# BRIEF NOTES

## Temporal memory as a constitutive principle and its limitations

W. KOSIŃSKI (WARSZAWA) and K. C. VALANIS (CINCINNATI)

FOR VISCOPLASTIC materials instead of the relaxation property for the norm in a history space the assumption that during the infinitely long freeze of configuration a material reaches a limit state is introduced. This assumption turns out to imply the relaxation property and hence the fading memory of a material represented by any continuous response functional defined on the history space. The proof of this result is based on the theory of semi-groups of operators.

IT IS QUITE natural to apply the general Coleman–Mizel theory of materials with memory [1–3], restricted by the relaxation property (RP) for the norm of the history space, to viscoelasticity. For a viscoplastic material, however, RP cannot hold because it implies the so-called semi-elasticity in the sense of NOLL [4], which is incompatible with internal changes of the material undergoing viscoplastic deformations. In the previous paper [5] we gave a general characteristic of viscoplastic materials distinguishing them from viscoelastic ones. In the present paper we drop RP since viscoplastic materials are investigated. Moreover, in the place of a Banach function space of the Köthe–Toeplitz type introduced by COLEMAN–MIZEL in [3] we are defining a more general history space $\mathfrak{B}$. The response of a material with memory is given by a response continuous functional (operator) $\boldsymbol{r}$: $\mathfrak{B} \to \mathfrak{R}$ defining on $\mathfrak{B}$ (or a cone of $\mathfrak{B}$) with values in a finite-dimensional inner product space $\mathfrak{R}$.

### History space

We start with a nontrivial, nonnegative, sigma-finite regular Borel (i.e. Radon) measure $\mu$ on $[0, \infty)$ which has an atom at $s = 0$ and is absolutely continuous on $(0, \infty)$ with respect to the Lebesgue measure $\lambda$. Let us consider the set $\mathscr{V}$ of all $\mu$-measurable functions $\Phi$ mapping $\mathsf{R}^+ := [0, \infty)$ into the normed space $\{V_n, |\cdot|_n\}$.

DEFINITION. *By a history space we mean a Banach space* $\{\mathfrak{B}, ||\cdot||\}$ *formed of equivalent classes of elements from* $\mathscr{V}$, *such that* $\Phi$ *is equivalent to* $\Psi$ *if* $\Phi(s) = \Psi(s)$, $\mu$ — *a.e., where the norm* $||\cdot||$ *on* $\mathfrak{B}$ *has the following properties*:

*a) each Cauchy sequence in* $\mathfrak{B}$ *contains a subsequence which converges pointwise* $\mu$ — *a.e. and its limit is the same as the limit of the whole sequence,*

*b) the norm* $||\cdot||$ *is equivalent to* $||\cdot||'$ *defined by* $||\Phi||' := |\Phi(0)|_n + ||_r\Phi||_r$, *where* $||_r\Phi||_r := ||\Phi\chi_{(0,\infty)}||$ *and the space* $\mathfrak{B}_r$ *obtained from* $\mathscr{V}_r := \{_r\Phi : {}_r\Phi = \Phi|_{(0,\infty)}$ *for some* $\Phi \in \mathscr{V}\}$ *by identyfing* $\mu$ — *almost equal elements of* $\mathscr{V}_r$ *is a Banach space,* *c) if $\Omega \in V_n$ and $A$ is a $\mu$-measurable bounded subset of* $\mathrm{R}^+$ *then* $\|\Omega\chi_A\| < \infty$, *where $\chi_A$ is the characteristic function of the set $A$,*

*d) for any $\sigma \in \mathrm{R}^+$ the mappings* $T^\sigma$ and $E^\sigma$ given by

$$(T^\sigma{}_r\Phi)(s) := \begin{cases} 0 & \text{if} \quad 0 < s \leqslant \sigma, \\ {}_r\Phi(s-\sigma) & \text{if} \quad \sigma < s < \infty, \end{cases} \qquad (E^\sigma\Phi)(s) := \begin{cases} \Phi(0) & \text{if} \quad 0 \leqslant s \leqslant \sigma, \\ \Phi(s-\sigma) & \text{if} \quad \sigma \leqslant s < \infty \end{cases}$$

*are well defined operators on $\mathfrak{B}_r$ and $\mathfrak{B}$, respectively, and moreover the set $\{T^\sigma : \sigma \in \mathrm{R}^+\}$ forms a strongly continuous semi-group of linear bounded operators on $\mathfrak{B}_r$,*

*e) each element of $\mathfrak{B}$ is of absolutely continuous norm.* □

It is not difficult to show that the Banach space of histories introduced in [3] and restricted by the first three postulates listed in [3] is a particular case of a history space. In that case the third separability postulate is equivalent to condition e). It should be stressed that those three postulates do not contain RP. We recall from definition (cf. ZAANEN [6, p. 476]) that if $\Phi \in \mathfrak{B}$ is a function of an absolutely continuous norm, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any measurable set $A \subset \mathrm{R}^+$ the condition

$$\mu(A) < \delta \quad \text{implies} \quad \|\Phi\chi_A\| < \varepsilon. \tag{1}$$

In what follows we deal with the history space $\{\mathfrak{B}, \|\cdot\|\}$.

LEMMA 1. The family $\{E^\sigma : \sigma \in \mathrm{R}^+\}$ forms a semi-group of linear bounded operators on $\mathfrak{B}$ continuous in the strong operator topology.

P r o o f. It is obvious that the family $\{E^\sigma : \sigma \in \mathrm{R}^+\}$ forms a semi-group of linear operators. To prove their boundedness let us notice that the condition b) implies that there exist two positive constants $c_1$ and $c_2$ such that, for any $\Phi \in \mathfrak{B}$ and $\sigma \geqslant 0$,

$$\begin{aligned} \|E^\sigma\Phi\| \leqslant c_1\{|(E^\sigma\Phi)(0)|_n + \|{}_r(E^\sigma\Phi)\|_r\} &= c_1\{|\Phi(0)|_n + \|T^\sigma_r\Phi + \Phi(0)\chi_{(0,\sigma]}\|\} \\ &\leqslant c_1\{|\Phi(0)|_n + \|T^\sigma\|\,\|{}_r\Phi\|_r + \|\Phi(0)\chi_{(0,\sigma]}\|\} \\ &\leqslant c_1\{(1 + \|1\chi_{(0,\sigma]}\|)|\Phi(0)|_n + \|T^\sigma\|\,\|{}_r\Phi\|_r\} \\ &\leqslant c_1c_2\{1 + \|1\chi_{(0,\sigma]}\| + \|T^\sigma\|\}\|\Phi\|. \end{aligned}$$

Since $\{T^\sigma : \sigma \in \mathrm{R}^+\}$ forms a semi-group of class $\mathscr{C}_0$ (cf. condition d)) the norm $\|T^\sigma\|$ is finite for any $\sigma$ and by condition c) for any $\sigma$ the operator $E^\sigma$ is bounded. To prove the strong continuity of $\{E^\sigma\}$ let us take two arbitrary non-negative numbers $\sigma_0$ and $\sigma_1$ to get the estimation

$$\|E^{\sigma_1}\Phi - E^{\sigma_0}\Phi\| \leqslant \|T^{\sigma_1}{}_r\Phi - T^{\sigma_0}{}_r\Phi\|_r + \|\Phi(0)\chi_{(\sigma_0',\sigma_1']}\|,$$

where $\sigma_0' := \min\{\sigma_0, \sigma_1\}$, $\sigma_1' := \max\{\sigma_0, \sigma_1\}$. From condition e) and (1) as well as from the absolute continuity of $\mu$ with respect to $\lambda$ it follows that given $\sigma_0 \geqslant 0$ for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\Phi(0)\chi_{(\sigma_0',\sigma_1']}\| < \varepsilon/2$ if $\sigma_1$ is such that $\sigma_1' - \sigma_0' < \delta$. The strong continuity of $\{T^\sigma : \sigma \in \mathrm{R}^+\}$ implies that $\|T^\sigma{}_r\Phi - T^{\sigma_0}{}_r\Phi\| < \varepsilon/2$ whenever $\sigma \geqslant 0$ and $|\sigma - \sigma_0| < \delta_1$ with some positive $\delta_1$. Applying these to the above estimation we can see that

$$\|E^{\sigma_1}\Phi - E^{\sigma_0}\Phi\| < \varepsilon,$$

whenever $\sigma_1 \geqslant 0$ and $|\sigma_1 - \sigma_0| < \min\{\delta, \delta_1\}$. This ends the proof. □

**Infinitesimal generator**

To find the form of the infinitesimal generator $A$ of $\{E^\sigma: \sigma \in \mathsf{R}^+\}$ and its domain $D(A)$, i.e. the set of those $\Psi \in \mathfrak{B}$ for which the limit

$$\lim_{\sigma\to 0^+} \sigma^{-1}\{E^\sigma\Psi - \Psi\} = :A\Psi \tag{2}$$

exists, it should be pointed out first, that any $\Psi_\eta$ given by

$$\Psi_\eta := \int_0^\eta E^\tau \Phi d\tau \tag{3}$$

with some $\Phi \in \mathfrak{B}$ and $\eta > 0$ belongs to $D(A)$ (cf. HILLE and PHILIPS [7]). Furthermore, in view of $\lim\limits_{\sigma\to 0^+} E^\sigma\Phi = \Phi$ the relation $\lim\limits_{\eta\to 0} \frac{1}{\eta}\Psi_\eta = \Phi$ holds. Integrating Eq. (3) gives

$$\Psi_\eta(s) = \begin{cases} \Phi(0)(\eta - s) + \int\limits_0^s \Phi(s-\sigma)d\sigma & \text{if} \quad 0 \leqslant s \leqslant \eta, \\ \int\limits_0^\eta \Phi(s-\sigma)d\sigma & \text{if} \quad s > \eta. \end{cases}$$

We can see that $\Psi_\eta$ is absolutely continuous and hence possesses, $\mu$ — a.e., the derivative $\dot{\Psi}_\eta$, defined pointwise by $\dot{\Psi}_\eta(s) := \frac{d}{ds}\Psi_\eta(s)$. This derivative satisfies the equation

$$\dot{\Psi}_\eta = \Phi - E^\eta \Phi, \tag{4}$$

and at $s = 0$ vanishes. To check the properties of elements of $D(A)$ let us take $\Psi \in D(A)$, then the pointwise calculation gives

$$\lim_{\sigma\to 0^+}\left(\frac{E^\sigma\Psi - \Psi}{\sigma}\right)(s) = \begin{cases} \lim\limits_{\sigma\to 0^+}\frac{1}{\sigma}\{\Psi(s-\sigma) - \Psi(s)\} = -\dot{\Psi}(s) & \text{if} \quad s > 0, \\ \lim\limits_{\sigma\to 0^+}\frac{1}{\sigma}\{\Psi(0) - \Psi(0)\} = 0 & \text{if} \quad s = 0. \end{cases}$$

If in Eq. (2) the limit in the norm $\|\cdot\|$ exists, then $\mu$ — almost everywhere we have

$$-(A\Psi)(s) = \begin{cases} \dot{\Psi}(s) & \text{if} \quad s > 0, \\ 0 & \text{if} \quad s = 0 \end{cases} \tag{5}$$

in view of condition a) of Definition.

LEMMA 2. Let $A_0$ be a linear transformation defined by

$$A_0\Psi := -\dot{\Psi} \tag{6}$$

for any $\Psi \in D(A_0) := \{\Psi \in \mathfrak{B}: \dot{\Psi}$ exists, $\mu$ — a.e., belongs to $\mathfrak{B}$ and $\dot{\Psi}(0) = 0\}$. Then the infinitesimal generator $A$ of $\{E^\sigma: \sigma \in \mathsf{R}^+\}$ is an extension of $A_0$ to $D(A)$.

P r o o f. Let us for $\Psi \in D(A_0)$ and $z \in \mathsf{R}^+$ calculate $\Gamma := \int\limits_0^z E^\sigma A_0 \Psi d\sigma$. We get, pointwise,

$$\Gamma(s) = -\left(\int\limits_0^z E^\sigma \dot{\Psi} d\sigma\right)(s) = \begin{Bmatrix} -\int\limits_0^s \dot{\Psi}(\tau)\,d\tau & \text{if} \quad 0 \leqslant s \leqslant z \\ -\int\limits_{s-z}^s \dot{\Psi}(\tau)\,d\tau & \text{if} \quad s > z \end{Bmatrix} = (E^z\Psi)(s) - \Psi(s).$$

The means that $A_0$ has the property $E^z\Psi - \Psi = \int\limits_0^z E^\sigma A_0 \Psi d\sigma$ for each $\Psi \in D(A_0)$ and $z \in \mathrm{R}^+$. Thus in view of Theorem 10.5.2 in [7] $A_0 = A|_{D(A_0)}$ which ends the proof. □

Corollary. Each $\Psi_\eta$ defined by (3) belongs to $D(A_0)$. □

## Constitutive asymptotic stability

The general Coleman–Mizel theory involves additionally the relaxation property for the norm formulated as a postulate. In the present paper we reject this postulate, introducing

*Constitutive asymptotic stability property*. For any continuous functional $\mathfrak{r}$ defined on $\mathfrak{B}$ and any $\Phi \in \mathfrak{B}$

(CASP) $$\lim_{\sigma\to\infty} \mathfrak{r}(E^\sigma\Phi) \text{ exists}.$$

By such a formulation, which exactly follows that of Coleman and Mizel, a property to the history space $\mathfrak{B}$ is introduced. In fact the following result holds (for its proof see Appendix):

Lemma 3. The constitutive asymptotic property holds if and only if in the norm of $\mathfrak{B}$

(ARP) $$\lim_{\sigma\to\infty} E^\sigma\Phi \text{ exists}$$

for any $\Phi \in \mathfrak{B}$. □

Note that ARP (the asymptotic rest property) is the mathematical expression of the Noll relaxation axiom $\bar{\mathrm{V}}$ of the material element [4] if the state space of the material with memory is identified with the history space $\mathfrak{B}$. The relaxation axiom, as well as ARP, postulate that if a material element is frozen in a definite configuration, i.e. in $E^\sigma\Phi$, its state will approach a relaxed state, i.e. a limit value $\Phi^f := \lim\limits_{\sigma\to\infty} E^\sigma\Phi$.

Now we are ready to formulate the main

Theorem 2. *If the Banach space $\mathfrak{B}$ has CASP, then the norm $\|\cdot\|$ has the Coleman–Mizel relaxation property, i.e. the only limit of $E^\sigma\Phi$ for $\sigma$ tending to infinity is the constant function* $\Phi(0)^\dagger(s) \equiv \Phi(0)$, $s \in \mathrm{R}^+$.

Hence we can obtain the following

Corollary. If the Banach space $\mathfrak{B}$ possesses ARP, then the constitutive model represented by a continuous response function $\mathfrak{r}$ defined on $\mathfrak{B}$ has fading memory. □

P r o o f of Theorem 2. Let $\Psi$ be an element of $\mathfrak{B}$ and take any $\tau > 0$. Since $\{E^\sigma\}$ forms a semi-group of class $\mathscr{C}_0$, ASP results in

$$E^\tau(\lim_{\sigma\to\infty} E^\sigma\Psi) = \lim_{\sigma\to\infty} E^\tau(E^\sigma\Psi) = \lim_{\sigma\to\infty} E^{\tau+\sigma}\Psi = \lim_{\sigma\to\infty} E^\sigma\Psi = \Psi^f.$$

This means that

$$E^{\tau}\Psi^{f} = \Psi^{f} \tag{7}$$

for any $\Psi^{f} := \lim\limits_{\sigma\to\infty} E^{\sigma}\Psi$, and $\tau \in \mathrm{R}^{+}$ with arbitrary $\Psi \in \mathfrak{B}$. Now we notice that $\Psi^{f}$ belongs to $D(A)$. In view of (7) and (3), for any

$$\Psi \in \mathfrak{B}, \quad D(A) \ni \frac{1}{\eta}\int\limits_{0}^{\eta} E^{\tau}\Psi^{f} d\tau = \Psi^{f}.$$

By Corollary to Lemma 2 $\Psi^{f}$ belongs to $D(A_0)$, too, and satisfies the equation $A_0\Psi^{f} = 0$ with $A_0 = -\dfrac{d}{ds}$.

This means that $\Psi^{f} = \text{const}$, $\mu$ — a.e. and is equal to $\Psi(0)$. In fact, since Eq. (2) has to hold in the norm $\|\cdot\|$, the point b) of Definition implies that

$$\lim_{\sigma\to\infty} (E^{\sigma}\Psi)(0) = \Psi^{f}(0).$$

But $(E^{\sigma}\Psi)(0) = \Psi(0)$ for any $\sigma \geqslant 0$, hence $\Psi^{f} \equiv \Psi(0)^{\dagger}$, which ends the proof. □

## Concluding remarks

It is not difficult to check that the material with memory represented by $r$ defined on the Banach function space $\mathfrak{B}$ with ARP for its norm, can be treated as a material element in the sense of NOLL [4]. We shall call it a *material element* with *temporal memory*. The mapping $\hat{a}:\mathfrak{B} \to \mathfrak{B}$ defined by $\hat{a}(\Psi) := \Psi^{f}$ (above) plays the role of the state relaxation mapping of Noll's theory; the members of the range of $\hat{a}$, i.e. $\mathfrak{B}_{\mathrm{rel}} := a(\mathfrak{B})$ will be called relaxed states.

According to Noll's classification a *semi-elastic* material element is defined by the condition that the relaxed states be in one-to-one correspondence with the deformations, i.e. elements of $V_n$.

Now in view of Theorem 2 and the above remarks we have the following

THEOREM 3. *A material element with temporal memory is semi-elastic.*

P r o o f. The function $\hat{\lambda} := \delta_0 \circ \hat{a}$ with $\delta_0(\Phi) := \Phi(0)$ defined by $\hat{\lambda}(\Psi) = \delta_0(\hat{a}(\Psi)) = \hat{a}(\Psi)(0)$ is the desired correspondence. □

A visco-plastic material and plastic one are materials with structure (cf. [5]) and are examples of a *non-semi elastic* material element in the sense of NOLL [4], since for them to one value of deformation may correspond more than one different relaxed states. Hence Theorem 3 implies:

COROLLARY. If the history space $\mathfrak{B}$ possesses ARP, then the constitutive model represented by any continuous constitutive response function $r$ defined on $\mathfrak{B}$ is improper for the description of visco-plastic as well as plastic materials. □

**Acknowledgement**

This paper is based on the unpublished report prepared in 1976 for the National Science Foundation in the course of a study supported by Grant No. ENG74–11125 to the University of Iowa.

## Appendix (Proof of Lemma 3)

It is necessary to demonstrate is the implication "⇒". To this end let us assume that there exists an element $\underset{0}{\Phi} \in \mathfrak{B}$ such that $\lim_{\sigma\to\infty} E^{\sigma}\underset{0}{\Phi}$ does not exist. Then one of the possibilities is that no subsequence of $\{E^{\sigma_n}\underset{0}{\Phi}\}_0^\infty$ converges. Take any subsequence $\{E^{\sigma_n}\underset{0}{\Phi}\}_0^\infty$. Then the set

$$B := \{E^{\sigma_n}\underset{0}{\Phi} : n \text{ — natural and } \sigma_n \to \infty\}$$

is closed, since $\mathfrak{B}\setminus B$ is open in $\mathfrak{B}$. Let us define a real-valued function $f$ on $B$ by the formula, for any $\Psi \in B$,

$$f(\Psi) := \min\{k \text{ — natural: } E^{\sigma_k}\underset{0}{\Phi} = \Psi\}. \tag{A.1}$$

The sequence $\{E^{\sigma_n}\underset{0}{\Phi}\}_0^\infty$ contains no convergent subsequence, and hence for each $\Psi$ the set on the right-hand side of (A.1) has finite number of elements and the function $f$ is well-defined and continuous, as its range is discrete. Now, let us note that $\mathfrak{B}$ is a normal topological space and hence any continuous function defined on a closed set has a continuous extension to the whole space $\mathfrak{B}$. The extended function $f$ may be used to construct a continuous response functional $r_f$ on $\mathfrak{B}$ by the identity

$$r_f := r_0 f,$$

where $r_0$ is a constant non-zero element in the space of responses $\mathfrak{R}$. It is seen from (A.1) that $\lim_{\sigma\to\infty} r_f(E^{\sigma}\underset{0}{\Phi})$ does not exist, for $\lim_{n\to\infty} f(E^{\sigma_n}\underset{0}{\Phi}) = \infty$. This proves that the sequence $\{E^{\sigma_n}\underset{0}{\Phi}\}_0^\infty$ must contain at least one convergent subsequence. If $\{E^{\sigma_p}\underset{0}{\Phi}\}_0^\infty$ is that subsequence and no other convergent subsequence exists, then the set

$$\{E^{\sigma}\underset{0}{\Phi} : \sigma \in \mathrm{R}^+\}\setminus\{E^{\sigma_p}\underset{0}{\Phi} : p \text{ — natural and } \sigma_p \to \infty\}$$

does not contain any convergent subsequence. But in view of the previous part of the proof this is impossible.

Hence the only possibility is that the sequence $\{E^{\sigma}\underset{0}{\Phi}\}_0^\infty$ contains two subsequences convergent to different limits. If $\underset{0}{\Psi}$ and $\underset{1}{\Psi}$ are those limits and $\underset{0}{\Psi} \neq \underset{1}{\Psi}$ then we may define by the normality of $\mathfrak{B}$ a continuous real-valued function $g$ on $\mathfrak{B}$ such that

$$g(\underset{0}{\Psi}) = 0 \quad \text{and} \quad g(\underset{1}{\Psi}) = 1. \tag{A.2}$$

Note that in this case we may construct the continuous respose functional $r_g := r_0 g$ such that the limit of the sequence $\{\underset{g}{r}(E^{\sigma}\underset{0}{\Phi})\}_0^{\infty}$ does not exist, because its two subsequences, by (A.2), converge to two different limits 0 and $r_0$, respectively. This ends the whole proof, the idea of which comes from K. FRISCHMUTH and S. SPAHN.

## References


1. B. D. COLEMAN and V. J. MIZEL, *Norms and semi-groups in the theory of fading memory*, Arch. Rational Mech. Anal., **23**, 87–123, 1966.
2. B. D. COLEMAN and V. J. MIZEL, *A general theory of dissipation in materials with memory*, Arch. Rational Mech. Anal., **27**, 255–274, 1967.
3. B. D. COLEMAN and V. J. MIZEL, *On the general theory of fading memory*, Arch. Rational Mech. Anal., **29**, 18–31, 1968.
4. W. NOLL, *A new mathematical theory of simple materials*, Arch. Rational Mech. Anal., **48**, 1–50, 1972.
5. W. KOSIŃSKI and K. C. VALANIS, *Intrinsic physical limits to the theory of materials with memory*, Arch. Mech., **30**, 6, 699–715, 1978.
6. A. C. ZAANEN, *Integration*, North-Holland, Amsterdam 1967.
7. E. HILLE and R. PHILLIPS, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ. Providence, **31**, 1957.



POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH
and
UNIVERSITY OF CINCINNATI
COLLEGE OF ENGINEERING, CINCINNATI, USA.


*Received November* 25, 1982.