Descriptions of disparate-mass gases

E. A. JOHNSON (SURREY)

IT IS known that one needs generalized Navier-Stokes equations to describe a gas mixture with a large disparity in species masses, in order to allow for the possibility of different temperatures for the different species. Various ways of obtaining such descriptions have been proposed. The set of thirteen-moment equations for a binary mixture of Maxwell molecules provides a set of governing equations which is completely explicit. These equations are used to demonstrate how different descriptions at the Navier-Stokes level, in terms of one temperature or two, are relevant, depending upon the magnitude of species mass-ratio and upon properties of physical situations of interest. The relation of the present treatment to previous descriptions is also discussed.

Wiadomo, że uogólnione równania Naviera-Stokesa są potrzebne do opisu mieszanin gazowych cechujących się dużą różnorodnością mas poszczególnych gatunków gazów, co pozwala na przyporządkowanie różnych temperatur różnym gatunkom. Dotąd proponowane były różne sposoby wyprowadzenia takich opisów. Układ trzynastu równań momentowych dla dwuatomowej mieszaniny cząstek Maxwella prowadzi do układu równań konstytutywnych w całkowicie jawnej postaci. Równania te posłużyły do wykazania, jak różne opisy na poziomie równań Naviera-Stokesa, zawierające jedną lub dwie temperatury, są możliwe do przyjęcia w zależności od stosunku mas gazów i własności rozważanego zjawiska fizycznego. Przedyskutowano również związek niniejszego podejścia z opisami poprzednimi.

Известно, что обобщенные уравнения Навье-Стокса нужны для описания газовых смесей, характеризующихся большой разнородностью масс отдельных сортов газов, что позволяет сопоставить разные температуры разным сортом газзов. До сих пор предлагались разные способы вывода таких описаний. Система тринадцати моментных уравнений для двухатомной смеси молекул Максвелла приводит к системе определяющих уравнений вполне явном виде. Эти уравнения послужили для показания, как разные описания на уровне уравнений Навье-Стокса, содержавшие одну или две температуры, возможно принять в зависимости от отношения масс газов и свойств рассматриваемого физического явления. Обсуждена тоже связь настоящего подхода с предыдущими описаниями.

1. Introduction

THE PROBLEM under discussion is the proper description of a gas which is composed of a mixture of very heavy particles with very light particles, in the simple case in which gradients are assumed fairly small, and in which number densities, and intermolecular forces, are assumed of similar magnitudes to each other. For simplicity, the particles are assumed to interact as Maxwell molecules, but this assumption may be easily generalized to include any inverse-power force law.

As is well known, the Navier-Stokes equations for a gas can be derived from the Boltzmann equation, for instance by the method of CHAPMAN and ENSKOG [1]. They can also be obtained from Grad's thirteen-moment approximation [2]. Mathematically, these derivations make use of the presence of a small parameter ε in the equations. Phys-

ically, in order for ε to be small, it is necessary for macroscopic lengths of interest to be large compared to the mean free path in the gas, and for macroscopic times to be long compared to the mean collision time.

For the Boltzmann equation itself, the starting point for the Chapman-Enskog solution is the nondimensional equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) f = \frac{1}{\varepsilon} J(f,f).$$

For binary mixtures, the usual generalization is

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) f_i = \frac{1}{\varepsilon} [J_{ii} + J_{ij}],$$

with an entirely similar generalization in the thirteen moment equations for the mixture. From these, one obtains the standard Navier-Stokes equations for the mixture, and in particular a description in terms of a single temperature for the gas. Physically, however, one knows that in gas mixtures in which the constituent masses are very different (for instance, in plasmas), two-temperature effects can be important. Mathematically, in a disparate-mass mixture one has a problem with the possibility of two small parameters, ε and $\delta \equiv (m_1/m_2)^{1/2}$, $m_1 \ll m_2$. Thus, it should not be surprising that the usual mixture Navier-Stokes equations should not apply [3], since their derivation involves the assumption of a single small parameter only, eg., a single characteristic relaxation time for initial transients.

2. Review of previous treatments

To get a description suitable for the particular case of a disparate-mass gas, people have therefore tried reordering the relevant coupled Boltzmann equations so as to obtain solutions of a Navier-Stokes type which would nevertheless apply to a disparate-mass gas, and contain the possibility of separate species temperatures.

The first such suggestion was that of GRAD [4]:

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) f_1 = \frac{1}{\varepsilon^2} J_{11} + J_{12},$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) f_2 = J_{21} + \frac{1}{\varepsilon} J_{22}.$$

This does indeed give zero-order distribution functions which are locally Maxwellian about independent species temperatures T_i and species flow velocities $\mathbf{U}_i$. It has been shown, however [5], that other consequences of these equations give unphysical predictions, at least if applied to disparate-mass gases.

A different reordering suggestion was made by CHMIELESKI and FERZIGER [6], and independently by GALKIN [7]:

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)f_1 = \frac{1}{\varepsilon}[J_{11}+J_{12}^{\infty}]+[J_{12}-J_{12}^{\infty}],$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)f_2 = J_{21}+\frac{1}{\varepsilon}J_{22},$$

where J_{12}^{∞} is the collision term one would obtain if $m_2^{-\infty}$, in a straightforward generalization of the early treatment of LORENTZ [8]. This reordering gives zero-order distribution functions which are local Maxwellians about separate temperatures (but a common flow velocity). It seems to be a correct reordering for the collision terms, and to contain the potentiality for giving good results for transport coefficients.

Recently, PETIT and DARROZES [3] have considered the problem again. These authors suggest that the correct equations for a disparate-mass mixture are best found, first by explicitly non-dimensionalizing the relevant Boltzmann equations, and then by making a "composite expansion" [9] in ε and δ so as to make explicit all consequences following from the possibility that δ can be of the same order as ε. When one explicitly non-dimensionalizes the Boltzmann equations, it becomes obvious that the left-hand-sides of these equations have different dependences on δ. Thus, Petit and Darrozes have been the first to point out what should, perhaps, have been obvious: that for a proper treatment of disparate-mass gases it is necessary to reorder the left-hand sides, as well as the right-hand sides, of the Boltzmann equations.

A different approach, more closely related to the thirteen-moment approach to be presented here, is the two-fluid treatment of GOLDMAN and SIROVICH [10]. The justification for this approach is not straightforward. The results of the present work, however, suggest that the treatment of Goldman and Sirovich gives transport coefficients correctly (that is, correct in an approximate sense, as the correct first-order approximations to the exact results obtainable from a correct reordering procedure). Our results also suggest that the governing equations, and equations of change, derived by Goldman and Sirovich are potentially correct, but that some specific conclusions, obtained by specialization to disparate-mass gases or to the Chapman-Enskog regime, are not correct.

3. Thirteen moment approach

The present work has been done in association with GOEBEL and HARRIS [11]. General thirteen-moment equations have been obtained for a binary mixture of Maxwell molecules, for arbitrary masses, number densities, and intermolecular force constants. This has been done by choosing approximate distribution functions of the form

$$f_i = f_i^{(0)}\left[1-\frac{2}{5}\frac{\varrho_i\mathbf{q}_i}{p_i^2}\cdot\mathbf{V}_i\left(\frac{5}{2}-\alpha_i V_i^2\right)+\frac{\varrho_i}{2p_i^2}\{\mathbf{P}_i\}:\{\mathbf{V}_i\mathbf{V}_i\}\right],$$

with $f_i^{(0)}$ taken for generality to be locally Maxwellian about independent species temperatures T_i and flow velocities $\mathbf{U}_i$. Here $\mathbf{V}_i$ is the peculiar velocity with respect to $\mathbf{U}_i$,

and the species pressure deviators and heat fluxes (calculated with respect to U_i) are $\{P_i\}$ and q_i respectively. This choice was expected to provide the most flexibility for various uses.

3.1. Ordinary mixtures

For ordinary gas mixtures, with a single small parameter ε, the thirteen moment equations so obtained describe the relaxation of initial transients on a time scale of order ε of a macroscopic time. Thereafter, they give back the familiar Navier-Stokes equations for a mixture. The diffusion velocity $W \equiv U_2 - U_1$, $\{P_i\}$, and q_i are small effects in this regime, of order ε of suitable reference quantities, as one must expect. There is also an additional equation for the temperature difference between the species, $\varDelta T \equiv T_2 - T_1$. $\varDelta T$ appears as a second order effect, $\sim \varepsilon^2 T_0$ (T_0 a reference temperature). Thus, as expected, only a single temperature appears to first order, just as in the Chapman-Enskog development. The regime is a normal one describable in terms of species number densities n_i, overall flow velocity U, and common temeprature T.

3.2. Disparate-mass mixtures

One may next consider disparate-mass mixtures, for which $\delta \ll 1$ (but force constants and number densities are comparable). A small value of δ implies the presence of multiple relaxation times: initial transients in W, $\{P_1\}$, and q_1 relax in times of the order of the light species self-collision time τ_1; those in $\{P_2\}$ and q_2 relax in times of the order of the heavy species self-collision time $\tau_2 \sim (\tau_1/\delta)$; and those in $\varDelta T$ relax over a time $\tau_{\varDelta T}$ which is longer still, $\tau_{\varDelta T} \sim (\tau_2/\delta)$. This is the epochal relaxation predicted by GRAD [4]. A second consequence of small δ is some simplification of the full equations, for instance some decoupling of the equations for $\{P_i\}$ and for q_i, and simpler expressions for transport coefficients. A third consequence of small δ is the different descriptions found necessary depending upon the regime under consideration.

i) Normal regime

In those situations in which all characteristic times of the system are much longer than $\tau_{\varDelta T}$ (the regime $\delta \gg \varepsilon$), one is dealing with an asymptotic solution in a single small parameter ε. The description obtained is a normal one, of Chapman-Enskog type, with the simplifications noted above. A typical simplification is the relation obtained for the diffusion velocity,

$$W = -D\nabla \ln p_1,$$

where $p_1 = n_1 kT$, the light species pressure. (GOLDMAN and SIROVICH [10], in contrast, have asserted that when $\tau_{\varDelta T} \gg \tau_2$, no normal solution is possible).

ii) "Near-normal" regime

In situations in which a characteristic time in the system can be of the order of $\tau_{\varDelta T}$, one is in a regime in which $\delta \sim \varepsilon$. In this case, the thirteen moment equations can be reordered in the same way as that used by PETIT and DARROZES [3] on the Boltzmann equations themselves. When this is done, one finds that the nondimensional diffusion

velocity $\mathbf{W}$ and light species heat flux $\mathbf{q}_1$ are terms of order $(\varepsilon/\delta) \sim 1$, and that the (non-dimensional) temperature difference is a term of order $(\varepsilon/\delta)^2 \sim 1$. Thus these effects, which are small effects, of first and second order, respectively, in usual mixtures, must appear as zero-order effects in a disparate-mass gas in this regime. One also finds that the nondimensional parameters $\{\mathbf{P}_2\}$ and $\mathbf{q}_2$ are of order ε, while $\{\mathbf{P}_1\}$ is in effect smaller still, of order $(\delta\varepsilon)$.

The second point to be made about the present regime is that (ΔT) has not yet relaxed to normal behavior. Thus T_1 and T_2 (or, equivalently, T and ΔT) are still independent parameters, still dependent upon their initial conditions. All other parameters $\mathbf{W}$, $\{\mathbf{P}_i\}$, and $\mathbf{q}_i$ will depend upon initial conditions, and thus be non-normal, by virtue of their dependence on T_1 and T_2.

Finally, in this regime the governing equations for the system will be different from the usual mixture equations. From the conservation equations for mass, momentum, and energy, one has respectively

1. [zero order = first order]:

$$\frac{D\varrho_1}{Dt} + \varrho_1 \nabla \cdot \mathbf{U} - \nabla \cdot \varrho_1 \mathbf{W} = 0,$$

$$\frac{D\varrho_2}{Dt} + \varrho_2 \nabla \cdot \mathbf{U} = 0.$$

2. [zero order]:

[first order]:

$$\left. \begin{array}{c} \varrho_2 \dfrac{D\mathbf{U}}{Dt} + \nabla(n_1 kT_1 + n_2 kT_2) \\[2mm] + \nabla \cdot \{\mathbf{P}_2\} \end{array} \right\} = 0.$$

3. [zero order]:

[first order]:

$$\left. \begin{array}{c} \dfrac{D}{Dt}\left(\dfrac{3}{2}nkT\right) + \dfrac{5}{2}nkT\nabla \cdot \mathbf{U} + \nabla \cdot \mathbf{q}_{10} \\[2mm] + \nabla \cdot \mathbf{q}_{20} + \{\mathbf{P}_2\} : \nabla\mathbf{U} \end{array} \right\} = 0.$$

Here $D/Dt \equiv (\partial/\partial t + \mathbf{U} \cdot \nabla)$, and $\mathbf{q}_{i0}$ are the heat fluxes taken with respect to the overall flow velocity $\mathbf{U}$:

$$\mathbf{q}_{10} = \mathbf{q}_1 - \frac{5}{2}p_1\mathbf{W}, \qquad \mathbf{q}_{20} = \mathbf{q}_2$$

in this regime.

For the equations of change, one finds

$$\mathbf{W} = -D\nabla\ln p_1 \qquad (p_1 = n_1 kT_1)$$

for the diffusion velocity. This is the usual classical equation for $\mathbf{W}$, when $\delta \ll 1$, plus an additional term in ΔT. This relation has already been derived by GOLDMAN and SIROVICH [10]. For $\{\mathbf{P}_2\}$ one obtains

$$\{\mathbf{P}_2\} = -2\mu_2(T_2)\{\nabla\mathbf{U}\},$$

and for the heat fluxes,

$$\mathbf{q}_i = -\lambda_i(T_i)\nabla T_i.$$

The equation governing the temperature separation ΔT seems to have been derived previously only by GOLDMAN and SIROVICH [10], who have neglected terms which must necessarily be included. When these inclusions are made, the expression for ΔT simplifies considerably, and one obtains

$$\frac{D}{Dt}(\Delta T) + \frac{2}{3}(\Delta T)\nabla\cdot\mathbf{U} - \frac{2}{3kn_1}\nabla\cdot\mathbf{q}_{10} - \frac{T_1}{n_1}\nabla\cdot(n_1\mathbf{W}) = -R(\Delta T).$$

In these equations, all transport coefficients are known explicitly [11]. The results obtained here are found to agree with those derivable from the work of PETIT and DARROZES [3], to lowest order, once account is taken of the difference between the Maxwell gas treated here and the Coulomb case considered by those authors.

References

1. S. CHAPMAN and T. G. COWLING, *Mathematical theory of non-uniform gases*, Cambridge University Press, London, 1970.
2. H. GRAD, Commun. Pure Appl. Math., **11**, 33 1949; in Handbuch der Physik, edited by S. Flügge, Springer-Verlag, Berlin, **12**, Sec. 28, 1958.
3. This is a singular perturbation problem, as discussed in detail by J. P. PETIT and J. S. DARROZES, Journal de Mécanique (to be published).
4. H. GRAD, in Rarefied Gas Dynamics, edited by F. Devienne, Pergamon, New York 1960, p. 100, Sec. 22.
5. E. A. JOHNSON, Transp. Theory Stat. Phys., **4**, 37, 1975.
6. R. M. CHMIELESKI and J. H. FERZIGER, Phys. Fluids, **10**, 364, 1967.
7. V. S. GALKIN, Mekh. Zhid. Gaza, **2**, 58, 1967 [Fluid Dyn., **2**, 38 1967].
8. H. A. LORENTZ, Proc. Amst. Acad., **7**, 438, 585, 684, 1905.
9. J. S. DARROZES, Fl. Dyn. Trans., **6**, 119, 1972.
10. E. GOLDMAN and L. SIROVICH, Phys. Fluids, **10**, 1928, 1967; **12**, 245, 1969.
11. C. J. GOEBEL, S. M. HARRIS and E. A. JOHNSON, Phys. Fluids, **19**, 627, 1976.

DEPARTMENT OF PHYSICS
UNIVERSITY OF SURREY, ENGLAND.