Archives of Mechanics • Archiwum Mechaniki Stosowanej • 28, 5-6, pp. 923-932, Warszawa 1976

Approximate analytical solution of the steady axisymmetric supersonic free jet of a reacting gas

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THIS PAPER presents a theoretical investigation of a steady axisymmetric free jet of a gas in which the thermodynamic state depends not only on pressure and density but also on an additional state variable which indicates the degree of deviation from local equilibrium (e.g., the degree of dissociation). An integral method is used and by linearizing the obtained approximate equations, an analytical solution can be found without specifying the equations of caloric-state and rate. The interaction of the flow and the reaction is demonstrated by some examples.

Niniejsza praca zawiera teoretyczną analizę ustalonego, osiowo-symetrycznego swobodnego, przepływu gazu, w którym stan termodynamiczny zależy nie tylko od ciśnienia i temperatury, lecz również od dodatkowej zmiennej stanu określającej stopień odchylenia od lokalnego stanu równowagi (np. rząd dysocjacji). Rozwiązując równania wyjściowe zastosowano metodę całkowitą. Po zlinearyzowaniu otrzymanych przybliżonych równań znaleziono rozwiązanie analityczne bez specyfikacji równania kalorycznego i równania opisującego proces relaksacji. Wzajemne oddziaływanie przepływu i procesu relaksacji pokazano na kilku przykładach.

В работе представлен теоретический анализ стационарного осесимметричного истечения газа, термодинамическое состояние которого зависит не только от давления и температуры, но также от дополнительного параметра, определяющего отклонение от локального равновесия (примером такого параметра может быть порядок диссоциации). Решение фундаментальных уравнений найдено методом интегралов. Путем линеаризации полученных приближенных уравнений можно найти аналитическое решение без определения калорического урабнения и релаксационного уравнения. На нескольких примерах показано взаимное воздействие течения и процесса релаксации.

1. Introduction

THE WAVE structure of a supersonic free jet causes a non-monotonic variation of the flow variables along the axis of such a jet. Therefore, the supersonic jet is well-suited for the study of non-equilibrium effects under the conditions of both expansion and compression.

This paper presents a theoretical investigation of steady axisymmetric free jet of a gas in which the thermodynamic state depends not only on pressure p and density ϱ but also on an additional variable α (e.g., the degree of dissociation) which may deviate from its local equilibrium value. Approximate formulas will be given which possess the same (rough) accuracy as the well-known Prandtl-formula in the case of a perfect gas [1].

2. Approximate differential equation

The basic equations in integral form are applied to finite parts of the jet. The equations of mass and momentum of a sectional part of the jet as shown in Fig. 1 are given as follows:

$$\int_0^R \varrho w \cos\vartheta \, r \, dr = c_1, \tag{2.1}$$

(2.1) [cont.]

$$\int_0^R (\varrho w^2 \cos^2\vartheta + p) r\,dr - \frac{p_R R^2}{2} = c_2 - \frac{p_R R_N^2}{2},$$

$$\int_0^R \varrho w^2 \sin\vartheta \cos\vartheta\, r\,dr - \int_0^X \int_0^R p\,dr\,dX + \int_0^X p_R R\,dX = c_3,$$

X and r are the axisymmetric coordinates (Fig. 1); w is the velocity and ϑ the angle between the axis and the vector of velocity. R is the distance between the axis and the boundary of the jet. The subscripts N and R refer to the values at the exit of the nozzle and at the boundary of the jet. c_1, c_2, c_3 are constants.

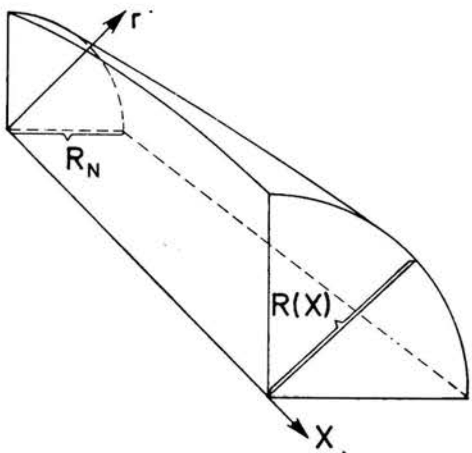

FIG. 1. Control surface.

The energy equation is used in the form

$$\frac{w^2}{2} + h = \text{const},$$

h is the enthalpy per unit mass. We assume that the state equation is of the form

$$h = h(p, \varrho, \alpha).$$

The rate equation

$$\mathbf{w}\,\text{grad}\,\alpha = L(p, \varrho, \alpha)$$

gives the production velocity of that gas portion which is characterized by the relaxation variable α.

In the conservation equations (2.1) there are integrals of the form $\int_0^R f(X, r) r\,dr$. We solve these integrals using the linear approach

$$f(X, r) = f_A(X) + [f_R(X) - f_A(X)]\frac{r}{R(X)}. \tag{2.2}$$

In this way we obtain approximate equations involving R, which is the distance between the axis and boundary, and flow variables along the axis (indicated by index A) and along the boundary (indicated by index R) as unknown functions of the coordinate in the direction of the axis.

$$
\begin{aligned}
& R^2(\varrho_A w_A + 2\varrho_R w_R \cos\vartheta_R) = c_1, \\
(2.3) \qquad & R^2(\varrho_A w_A^2 + p_A + 2\varrho_R w_R^2 \cos^2\vartheta_R - p_R) = 3c_2 - 3p_R R_N^2, \\
& R^2 \varrho_R w_R^2 \sin 2\vartheta_R + 3\int_0^X R(p_R - p_A)\,dX = 3c_3.
\end{aligned}
$$

The energy equation, state equation and rate equation

$$
\begin{aligned}
& \frac{w^2}{2} + h = \text{const}, \\
(2.4) \qquad & h = h(\varrho, \alpha, p), \\
& \mathbf{w}\,\mathrm{grad}\,\alpha = L(\varrho, p, \alpha)
\end{aligned}
$$

are valid both along the axis and the boundary of the jet.

By means of the simple relationship between the distance, between the axis and boundary and the inclination of the boundary of the jet

$$
(2.5) \qquad \frac{dR}{dx} = \mathrm{tg}\,\vartheta_R
$$

one can deduce an ordinary differential equation system which can be solved by well-known numerical methods. In earlier papers [2, 3] some examples were calculated in this way.

In the present investigation the system (2.3), (2.4), (2.5) is solved through linearization.

We assume that the pressure p_R on the boundary of the jet is only slightly smaller than the pressure p_N at the exit of the nozzle; in special cases p_R equals p_N.

We adopt for pressure, density, velocity, relaxation variable and distance between the axis and boundary of the jet the perturbation variables μ, σ, u, ξ and η; x is the dimensionless coordinate in the direction of the axis.

$$
\begin{aligned}
& \frac{p}{w_N^2 \varrho_N} = \frac{1}{\varkappa M_N^2} - \mu, \qquad \frac{\varrho}{\varrho_N} = 1 + \sigma, \\
(2.6) \qquad & \frac{w}{w_N} = 1 + u, \qquad \frac{\alpha}{\alpha_N} = 1 + \xi, \\
& \frac{R}{R_N} = 1 + \eta, \qquad x = \frac{X}{R_N}.
\end{aligned}
$$

The index N refers to the undisturbed uniform flow in the nozzle. M_N is the Mach-number, $\varkappa$ is the ratio of specific heats. α_N is the value of the relaxation variable in the nozzle, it need not be the equilibrium value. ξ and its derivative in the x-direction can be small, both near the equilibrium and near the frozen state.

By introducing the perturbation approach (2.6) in the system (2.3), (2.4), (2.5) and neglecting terms of the 2nd and higher order in the perturbation variables and their derivatives, one obtains an equation system for the values of the perturbation variables both along the axis (indicated by index A) and along the boundary (indicated by index R):

$$
\begin{aligned}
\sigma_A+u_A+2\sigma_R+6\eta &= -2M_N^2u_R,\\
\mu_A &= u_A,\\
\frac{d^2\eta}{dx^2}+\frac{3}{2}u_A &= \frac{3}{2}u_R,\\
\frac{d\xi_A}{dx} &= L_N+\left(\frac{\partial L}{\partial\alpha}\right)_N\xi_A+\left(\frac{\partial L}{\partial\varrho}\right)_N\sigma_A-\left(\frac{\partial L}{\partial p}\right)_N u_A,\\
\frac{d\xi_R}{dx} &= L_N+\left(\frac{\partial L}{\partial\alpha}\right)_N\xi_R+\left(\frac{\partial L}{\partial\varrho}\right)_N\sigma_R-\left(\frac{\partial L}{\partial p}\right)_N u_R,\\
\left[1-\left(\frac{\partial h}{\partial p}\right)_N\right]u_A+\left(\frac{\partial h}{\partial\varrho}\right)_N\sigma_A+\left(\frac{\partial h}{\partial\alpha}\right)_N\xi_A &= 0,\\
\left[1-\left(\frac{\partial h}{\partial p}\right)_N\right]u_R+\left(\frac{\partial h}{\partial\varrho}\right)_N\sigma_R+\left(\frac{\partial h}{\partial\alpha}\right)_N\xi_R &= 0.
\end{aligned}
\tag{2.7}
$$

The constant disturbance velocity at the boundary u_R is calculated from the given pressure at the boundary. The value of the rate function and the values of its partial derivatives and those of the enthalpy at the end of the nozzle are assumed to be dimensionless and are defined in a manner analogous to the variables.

We consider the relationship for the Mach-number (with frozen speed of sound)

$$
\mathsf{M}_N^2=\frac{1-\left(\dfrac{\partial h}{\partial p}\right)_N}{\left(\dfrac{\partial h}{\partial\varrho}\right)_N}
$$

(see, e.g., [4]) and introduce the following abbreviations:

$$
\begin{aligned}
A &= \frac{\left(\dfrac{\partial h}{\partial\alpha}\right)_N}{\left(\dfrac{\partial h}{\partial\varrho}\right)_N},\\
\nu &= A\left(\frac{\partial L}{\partial\varrho}\right)_N-\left(\frac{\partial L}{\partial\alpha}\right)_N,\\
c &= -\left(\frac{\partial L}{\partial p}\right)_N-\mathsf{M}_N^2\left(\frac{\partial L}{\partial\varrho}\right)_N.
\end{aligned}
\tag{2.8}
$$

The equations $(2.7)_5$ and $(2.7)_7$ can be treated separately and one can obtain a solution along the boundary of the jet:

$$
\begin{aligned}
\xi_R &= \frac{cu_R+L_N}{\nu}(1-e^{-\nu x}),\\
\sigma_R &= A\xi_R-\mathsf{M}_N^2u_R.
\end{aligned}
\tag{2.9}
$$

This solution means: At the boundary of the jet the disturbances of the relaxation variable and the density are not influenced by the processes in the inner jet; they tend exponentially towards their equilibrium values. It may be easily shown that in the case of non-linear exact equations, the relaxation variable and the density along the boundary can also be treated separately from the inner jet.

Now, from the other equations of the set (2.7) we can extract a single differential equation for the disturbance of the radius of the jet η. Using the abbreviation $k = 3/\sqrt{M_N^2-1}$, we get

$$\eta''' + \left[\nu + \frac{Ack^2}{9}\right]\eta'' + k^2\eta' + \nu k^2 \eta = u_R\left[\frac{3}{2}\nu + \frac{Ack^2}{2}\right] + \frac{AL_N k^2}{2}. \tag{2.10}$$

The initial conditions are

$$\begin{aligned} \eta(o) &= 0, \\ \eta'(o) &= \frac{3}{k}u_R, \\ \eta''(o) &= \frac{3}{2}u_R. \end{aligned} \tag{2.11}$$

The second condition determines the inclination of the boundary of the jet at the edge of the nozzle exit due to a Prandtl-Meyer-expansion. The third initial condition results from Eq. $(2.7)_3$ and $u_A(0) = 0$.

3. Solution in general terms

The solution of the ordinary differential equation (2.10) is in principle very simple. The solution of the inhomogeneous equation is

$$\hat{\eta} = u_R\left(\frac{3}{2k^2} + \frac{Ac}{2\nu}\right) + \frac{AL_N}{2\nu}. \tag{3.1}$$

In order to obtain the solution of the homogeneous equation one has to solve the characteristic algebraic equation of third order,

$$\varepsilon^3 + \left[\nu + \frac{Ack^2}{9}\right]\varepsilon^2 + k^2\varepsilon + \nu k^2 = 0. \tag{3.2}$$

The discriminant of this equation

$$D = (k^2+\nu^2)^2 + k^2\frac{Ac}{9}\left[3\nu^2\left(\nu + \frac{Ack^2}{9}\right) + \frac{A^2c^2k^4\nu}{81} - k^2\left(\frac{\nu}{2} + \frac{Ac}{36}k^2 + \frac{9}{2\nu}\right)\right]$$

is difficult to analyse. We assume that D is positive. However, in the case of a negative discriminant ($D < 0$) a free jet without wave structure could evolve. Whether such a flow could be physically possible, and under what conditions, has not yet been examined.

If D is greater than zero, then the characteristic equation has one real solution and two conjugate complex solutions. These solutions have the form

$$\varepsilon_1 = \gamma - \frac{1}{3}\left(\nu + \frac{Ack^2}{9}\right),$$

$$\varepsilon_2 = -\frac{\gamma}{2} - \frac{1}{3}\left(\nu + \frac{Ack^2}{9}\right) + i\beta,$$

$$\varepsilon_3 = -\frac{\gamma}{2} - \frac{1}{3}\left(\nu + \frac{Ack^2}{9}\right) - i\beta.$$

γ and β are real constants and are given by cubic roots of expressions involving the coefficients of the differential equation.

Thus we arrive at the following solution of the differential equation (1.10):

$$(3.3) \qquad \eta = \frac{u_R}{2}\left(\frac{3}{k^2} + \frac{Ac}{\nu}\right) + \frac{AL_N}{2\nu} + C_1 e^{-\left(\frac{N}{3}+\gamma\right)x} + C_2 e^{-\left(\frac{N}{3}+\frac{\gamma}{2}\right)x}(\sin\beta x + C_3\cos\beta x),$$

N is the abbreviation for $N = \nu + \frac{Ack^2}{9}$. The constants C are given by the initial conditions (2.11)

$$C_1 + C_2C_3 = \frac{3u_R}{k^2}\left(1 - \frac{3}{2}\frac{N}{\nu}\right) - \frac{AL_N}{2\nu},$$

$$\left(\gamma - \frac{N}{3}\right)C_1 - \left(\frac{\gamma}{2} + \frac{N}{3}\right)C_2C_3 + \beta C_2 = \frac{3u_R}{k},$$

$$\left(\gamma - \frac{N}{3}\right)^2 C_1 + \left[\left(\frac{\gamma}{2} + \frac{N}{3}\right)^2 - \beta^2\right]C_2C_3 - 2\beta\left(\frac{\gamma}{2} + \frac{N}{3}\right)C_2 = \frac{3}{2}u_R.$$

From the solution (3.3) for the disturbance of the boundary of the jet, the solution for the other flow variables can be derived. Thus, the solution for the disturbance of the relaxation variable is given:

$$(3.4) \qquad \xi_A = \frac{6}{Ak^2}\left\{C_1\left(k^2 + \left[\gamma - \frac{N}{3}\right]^2\right)e^{-\left(\frac{N}{3}-\gamma\right)x} + e^{-\left(\frac{N}{3}+\frac{\gamma}{2}\right)x}[B_1\sin\beta x + B_2\cos\beta x]\right\} + \frac{cu_R + L_N}{\nu}\left(1 + 2e^{-\frac{\nu x}{3}}\right)$$

with

$$B_1 = C_2\left(\left[\frac{\gamma}{2} + \frac{N}{3}\right]^2 + 2\beta C_3\left[\frac{\gamma}{2} + \frac{N}{3}\right] - \beta^2 + k^2\right),$$

$$B_2 = C_2\left(C_3\left[\frac{\gamma}{2} + \frac{N}{3}\right]^2 - 2\beta\left[\frac{\gamma}{2} + \frac{N}{3}\right] + C_3[k^2 - \beta^2]\right).$$

It is difficult to discuss the solutions (3.3), (3.4) in general terms. For instance, γ and β are cubic roots of the expressions of coefficients of the differential equation. If one of the three exponents in Eqs. (3.3), (3.4) is positive, one obtains an instable solution. The question under what physical conditions this is going to happen cannot be answered.

Instead of discussing our solution in general we shall consider special cases.

4. Special cases

4.1. Perfect gas

As a simple case of the differential equation (2.1) we consider the equation for the flow of a perfect gas. The rate function in the nozzle equals zero and the relaxation length tends to zero; this implies that ν tends to infinity. Since $\left(\frac{\partial h}{\partial \alpha}\right)_N = 0$, then $A = 0$. So we get the simple differential equation

$$\eta'' + k^2\eta = \frac{3}{2}u_R.$$

With the initial conditions (2.11) the solution is

$$\eta = \frac{3}{2}\frac{u_R}{k^2}(2\sin kx + 1 - \cos kx). \tag{4.1}$$

The jet boundary is a harmonic function with the wavelength

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{3}\sqrt{\mathsf{M}_N^2 - 1}. \tag{4.2}$$

For comparison the Prandtl-formula is given: $\lambda_P = \frac{2\pi}{x_1}\sqrt{\mathsf{M}_N^2 - 1}$ ($x_1 = 2.405$ is the first zero of the Bessel function of zero order).

In an earlier paper of the author [5] it is shown that the distances between the extreme values of the boundary of an axisymmetric free jet are different from each other. The first wavelength is $\lambda = 2.43\sqrt{\mathsf{M}_N^2 - 1}$ [5, 6], but the mean wavelength which can be calculated exactly is $\lambda = \frac{8}{3}\sqrt{\mathsf{M}_N^2 - 1}$ [5]. Hence, our solution (4.2) has the same rough accuracy as the Prandtl-formula.

4.2. Frozen flow

The frozen case is also simple. When the relaxation length is very long as compared to the radius of the nozzle, the dimensionless representation of the rate function and its derivatives equal zero

$$\nu = 0, \quad L_N = 0,$$

$$c = -\left(\frac{\partial L}{\partial p}\right)_N - \mathsf{M}_N^2\left(\frac{\partial L}{\partial \varrho}\right)_N = 0.$$

We then obtain the differential equation

$$\eta''' + k^2\eta' = 0.$$

With the initial conditions for the shape of the jet (2.11) the same solution is obtained as in the case of the perfect gas flow (4.1). Of course, another pressure distribution appears due to the different adiabatic coefficient.

4.3. Non-equilibrium state in the nozzle

It is interesting tc study the special case which results when the pressure on the jet boundary is made equal to the pressure in the nozzle, but it is assumed that a non-equilibrium state exists in the nozzle, that is L_N is not equal to zero. In addition, we assume that the rate function depends only on the relaxation variable.

$$p_R = p_N, \quad L = L(\alpha).$$

The differential equation is then

$$\eta''' + \nu\eta'' + k^2\eta' + \nu k^2 \eta = \frac{Ak^2 L_N}{2}$$

with the initial conditions

$$\eta(0) = \eta'(0) = \eta''(0) = 0.$$

The characteristic equation has the roots

$$\varepsilon_1 = -\nu, \quad \varepsilon_{2,3} = \pm ik.$$

Hence we get the solution

$$\eta = \frac{AL_N}{2\nu(k^2+\nu^2)}[k^2(1-e^{-\nu x}) + \nu^2(1-\cos kx) + \nu k \sin kx],$$

$$\xi_A = \frac{L_N}{\nu}(1-e^{-\nu x}).$$

After a flow length, after which the relaxation variable approaches its equilibrium value, a jet remains, the boundary of which is given by a harmonic function and which has the same wavelength as the jet of a perfect gas. In example (4.1) the amplitude of the boundary disturbance is determinied by the pressure disturbance and the Mach-number, whereas in the present example the non-equilibrium state in the nozzle produces a wave structure of the jet.

4.4. Ideal dissociating gas

The last example deals with the case of an ideal dissociating gas. For this example our general solution is applied without any restrictions.

The gas is defined by the state equations [7]

$$\frac{p}{\varrho} = (1+\alpha)\mathcal{R}T,$$

$$h = \frac{4+\alpha}{1+\alpha}\frac{p}{\varrho} + \alpha\mathcal{R}T_d$$

and the rate function [8] for the degree of dissociation α

$$L = K_1 \frac{\varrho}{\varrho_d} \sqrt{\frac{T_d}{T}} \left\{(1-\alpha)\frac{T_d}{T} + K_2 \alpha\right\} \left\{(1-\alpha)e^{-\frac{T_d}{T}} - \frac{\varrho}{\varrho_d}\alpha^2\right\},$$

T is the temperature and $\mathcal{R}$ is the gas constant of the undissociated gas. The constants are chosen such that the formulas apply to oxygen ($T_d = 59{,}000°K$, $\varrho_d = 150$ g/cm^3, $K_1 = 6.26 \cdot 10^{14}$ sec^{-1}, $K_2 = 69.1$). For our example we use the following values of the undisturbed reference state in the nozzle and of the pressure along the boundary of the jet:

$$R_N = 0.1\text{ m}, \quad p_N = 1.1 \cdot 10^5\, N,\ w_N = 1500\text{ m/s}, \quad T_N = 3000°K, \quad p_R = 10^5 N.$$

It is assumed that an equilibrium state exists in the nozzle: this means that the degree of dissociation in the nozzle is $\alpha_N = 0.0552$.

For the reciprocal relaxation length ν and the constants c and A, the following values are obtained

$$\nu = 1.42, \quad c = -2.18, \quad A = -0.229.$$

After solving the characteristic equation of third order (3.2) one can get from Eqs. (3.3) and (3.4) the solutions

$$(4.3) \qquad \eta = 0.0115 - 0.00317\, e^{-1.52x} + 0.00951\, e^{-0.193x}[\sin(2.86x) - 0.875\cos(2.86x)],$$

$$(4.4) \qquad \xi_A = 0.105 e^{-1.52x} + e^{-0.193x}[0.00858\sin(2.86x) + 0.0477\cos(2.86x)] - 0.0509(1 + 2e^{-1.42x}).$$

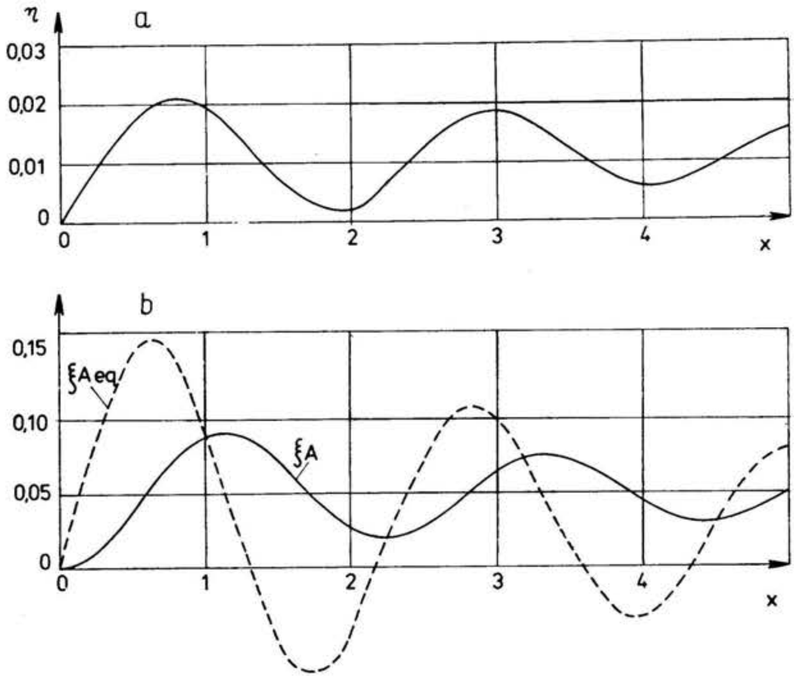

FIG. 2. a) Disturbance of the boundary of the jet η (Eq. (4.3)). b) Disturbance of the degree of dissociation along the axis of the jet ξ_A(Eq. (4.4)).

ξ_{Aeq} is the corresponding equilibrium value.

As one may expect, the boundary of the jet (Fig. 2a) shows the damping effect of the ralaxation. In Fig. 2b the disturbance of the degree of dissociation, Eq. (4.4), is compared with its corresponding local equilibrium value. The actual values of the degree of dissociation differ from the equilibrium values by a phase shift and a damping effect.

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Received November 12, 1975.