Archives of Mechanics • Archiwum Mechaniki Stosowanej • 28, 5-6, pp. 1073-1075, Warszawa 1976

On the unique solvability of the initial value problem for viscous incompressible inhomogeneous fluids

O. A. LADYZHENSKAYA (LENINGRAD)

THE PAPER contains a short discussion of the results concerning uniqueness and existence of the solution of the system of equations describing two- and three-dimensional flows of inhomogeneous viscous fluids. The results presented are in agreement with the author's earlier results obtained for Navier-Stokes equations at $\varrho =$ const.

I SHOULD like to discuss the principal results of the joint work of V. A. SOLONNIKOV and myself which will be published in "Notes of the scientific seminars of LOMI", vol. 52.

We have considered the problem

$$\varrho\Big[\mathbf{v}_t+\sum_{k=1}^{n} v_k\mathbf{v}_{x_k}\Big]-\nu\Delta\mathbf{v} = -\nabla p+\varrho f,$$

$$\operatorname{div}\mathbf{v} = 0, \tag{1}$$

$$\varrho_t+\sum_{k=1}^{n} v_k\varrho_{x_k} = 0,$$

$$\mathbf{v}|_{S_T} = 0, \quad \mathbf{v}|_{t=0} = \mathbf{v}^0(x), \quad \varrho|_{t=0} = \varrho^0(x).$$

The fluid occupies a bounded region Ω of the Euclidean space R^n ($n = 2$ or 3; $n = 2$ — is the case of the two-dimensional flows; $n = 3$ — the general three-dimensional flows), $t\in[0, T)$, $(T\leqslant\infty)$, $Q_T = \Omega\times(0, T)$, $S = \partial\Omega$, $S_T = S\times[0, T]$. The functions $\mathbf{v}^0(x) = (v_1^0(x), \ldots, v_n^0(x))$, $\varrho^0(x)$, $\mathbf{f}(x, t)$ — are known and $\mathbf{v}(x, t)$, $p(x, t)$, $\varrho(x, t)$ (the velocity-vector, the pressure and the density) must be found. The coefficient of the viscosity ν is a known positive constant.

Prior to this work most complet rigorous results concerning the problem (1_k) were obtained by A. V. KAZHICHOV (this is published in the book of S. N. ANTONZEV and A. V. KAZHICHOV "The mathematical questions of the dynamics of the inhomogeneous fluids", Novosibirsk, 1973). He has proved the solvability of the problem (1_k) in some functional spaces. But these spaces are so large (i.e., his solutions are so inadequate) that the theorem of uniqueness for them is not proved even in the two-dimensional case.

We have proved the existence of a smoother solution of the problem (1_k) in the classes in which the theorem of uniqueness is valid. The principal results are as follows:

THEOREM 1. *Let $n = 2$, $\varrho^0(x)$ be a smooth positive function of $x\in\bar{\Omega}$* (i.e., $\varrho^0\in C^1(\bar{\Omega})$), $\mathbf{v}^0(x)\in \mathring{J}_q^{(2-2/q)}(\Omega)$, $\mathbf{f}(x, t)\in L_q(Q_T)$ $(\forall T<\infty)$ *and* $q>2$. *Then, the problem* (1_k) *has* *a unique solution which has the following properties*: $\mathbf{v}(x, t) \in \mathring{J}_q^{2,1}(Q_T)$, $p(x, t) \in W_{q,0}^{1,0}(Q_T)$ *and* $\varrho(x, t)$ *is a positive smooth function of* $(x, t) \in \bar{Q}_T$ (*i.e.*, $\varrho \in C^1(\bar{Q}_T)$).

The boundary S must be a smooth curve of the class C^2.

Here we use the following notations: $\mathring{J}_q^{2,1}(Q_T)$ — the Banach space of all solenoidal vector-functions $\mathbf{v}(x, t)$ from the $L_q(Q_T)$ which have the (generalized) derivatives $\frac{\partial}{\partial x_i}$, $\frac{\partial^2}{\partial x_i \partial x_j}$, $\frac{\partial}{\partial t}$ from $L_q(Q_T)$ and which are zero at the S_T. The norm in $\mathring{J}_q^{2,1}(Q_T)$ is determined by the equality

$$||\mathbf{v}||_{q,Q_T}^{(2,1)} = ||\mathbf{v}_t||_{q,Q_T} + ||\mathbf{v}_{xx}||_{q,Q_T} + ||\mathbf{v}_x||_{q,Q_T} + ||\mathbf{v}||_{q,Q_T},$$

where

$$||u||_{q,Q_T} = \Big(\int_{Q_T} |u|^q dxdt\Big)^{1/q}, \qquad ||\mathbf{v}||_{q,Q_T} = ||\,|\mathbf{v}|\,||_{q,Q_T},$$

$$|\mathbf{v}| = \Big(\sum_{k=1}^n v_k^2\Big)^{1/2}, \quad ||\mathbf{v}_x||_{q,Q_T} = ||\,|v_x|\,||_{q,Q_T}, \quad |\mathbf{v}_x| = \Big(\sum_{k,j=1}^n v_{kx_j}^2\Big)^{1/2},$$

$$||\mathbf{v}_{xx}||_{q,Q_T} = ||\,|\mathbf{v}_{xx}|\,||_{q,Q_T}, \quad |\mathbf{v}_{xx}| = \Big(\sum_{k,i,j=1}^n v_{kx_ix_j}\Big)^{1/2}.$$

$\mathring{J}_q^{(2-2/q)}(\Omega)$ is the Banach space of all solenoidal vector-functions $\mathbf{v}(x)$ from $L_q(\Omega)$ which are equal to zero on S, which have the derivatives $\frac{\partial}{\partial x_i}$ from $L_q(\Omega)$ and for which the following norm is finite:

$$||\mathbf{v}||_{q,\Omega}^{(2-2/q)} = ||\mathbf{v}||_{q,\Omega} + ||\mathbf{v}_x||_{q,\Omega} + \Big(\int_\Omega \int_\Omega |\mathbf{v}_x(x) - \mathbf{v}_y(y)|^q \frac{dx\,dy}{|x-y|^{n-2+q}}\Big)^{1/q},$$

$W_{q,0}^{1,0}(Q_T)$ is the Banach space of functions $p(x, t)$ from $L_q(Q_T)$ which satisfy the conditions $\int_\Omega p(x, t)dx = 0$ for almost all $t \in [0, T]$ and which have the derivatives $\frac{\partial}{\partial x_i}$ from $L_q(Q_T)$. The norm in $W_{q,0}^{1,0}(Q_T)$ is determined by the equality

$$||p||_{q,Q_T}^{(1,0)} = ||\nabla p||_{q,Q_T}.$$

Thus Theorem 1 guarantees the unique solvability "in the large" of the problem (1_k) without any restriction on values of the known functions.

Let us pass to the three-dimentional case ($n = 3$). Denote the pair of vector-functions $\{\mathbf{f}(x, t), \mathbf{v}^0(x)\}$ by $\mathscr{F}$ and the value $||\mathbf{f}||_{q,Q_T} + ||\mathbf{v}^0||_{q,\Omega}^{2-2/q}$ by $|\mathscr{F}|_{B_T}$. We have proved the following theorem:

THEOREM 2. *Let* S, ϱ^0, $\mathbf{v}^0(x)$ *and* $\mathbf{f}(x, t)$ *satisfy the conditions of Theorem* 1 *with* $q > 3$. *Then, for the arbitrary* $R > 0$ *we guarantee the existence of positive* $T(R) \in (0, T)$ *such that the problem* (1_k) *has a unique solution for* $t \in [0, T(R)]$ *with the properties* $\mathbf{v}(x, t) \in \mathring{J}_q^{2,1}(Q_{T(R)})$, $p(x, t) \in W_{q,0}^{1,0}(Q_{T(R)})$, $\varrho(x, t) \in C^1(\bar{Q}_{T(R)})$ *if the known function* $\mathscr{F}$ *satisfies the condition* $|\mathscr{F}|_{B_T} \leqslant R$.

Further, for arbitrary $T > 0$ there exists a positive $R(T)$ such that the problem (1_k) is uniquely solvable for $t \in [0, T](T < \infty)$, if $|\mathcal{F}|_{B_T} \leqslant R(T)$. The solution $\mathbf{v}, p, \varrho$ belongs correspondigly to the spaces $\mathring{J}_q^{2,1}(Q_T)$, $W_{q,0}^{1,0}(Q_T)$, $C^1(\bar{Q}_T)$. If $\mathbf{f}(x, t)e^{\gamma t} \in L_q(Q_\infty) \equiv \equiv L_q(\Omega \times (0, \infty))$ with any $\gamma > 0$ and $\|\mathbf{f}e^{\gamma t}\|_{q,Q_\infty} + \|\mathbf{v}^\circ\|_{q,\Omega}^{2-2/q} \leqslant R_\gamma$, where R_γ is certain value, the solution of the problem (1_k) exists for all $t \geqslant 0$ and belong to the spaces indicated above with the arbitrary finite T.

The function $T(R)$ (it is the positive monotonously decreasing function of $R > 0$), the function $R(T)$ (also a positive monotonously decreasing function of $T > 0$) and the number R_γ, introduced in Theorem 2, are defined by some characteristics of the region Ω. As one can see, the conditions of Theorems 1 and 2, do not depend on any quantitative characteristics of $\varrho^0(x)$. Qualitatively these results concerning the solvability of the problem (1_k) coincide with the results concerning the solvability of the initial value problem for the Navier-Stokes equations in the case of homogeneous fluids (i.e, when ϱ is a fixed constant).

We find the solution of problem (1_k) as the limit of the solutions $\{\mathbf{v}^{(m)}, p^{(m)}, \varrho^{(m)}\}$, $m = 1, 2, \ldots$ of the following linear problems:

$$\varrho_t^{(m)} + \sum_{k=1}^{n} v_k^{(m-1)} \varrho_{x_k}^{(m)} = 0, \quad \varrho^{(m)}|_{t=0} = \varrho^0(x), \tag{2}$$

and

$$\begin{gathered} \varrho^{(m)}\Big(\mathbf{v}_t^{(m)} + \sum_{k=1}^{n} v_k^{m-1} \mathbf{v}_{x_k}^{(m)}\Big) - \nu \Delta \mathbf{v}^{(m)} = -\nabla p^{(m)} + \varrho^{(m)} f, \\ \operatorname{div} \mathbf{v}^{(m)} = 0, \quad \mathbf{v}^{(m)}|_{S_T} = 0, \quad \mathbf{v}^{(m)}|_{t=0} = \mathbf{V}^0(x), \end{gathered} \tag{3}$$

where $v^{(0)} = p^{(0)} \equiv 0$. At the beginning it is necessary to solve problem (2) for $m = 1$, then — to solve problem (3) for $m = 1$, afterwards — problem (2) for $m = 2$, etc. The solution of problem (2) is given by the well-known formula. The solution of problem (3) can be found either by the method of least squares or by Galerkin's method with the appropriately chosen coordinate functions.

An inquiry into the properties of the solutions $\{\mathbf{v}^{(m)}, p^{(m)}\}$ of problem (3) including the sharp a priori estimates in the spaces $\{\mathring{J}_q^{2,1}(Q_T), W_{q,0}^{1,0}(Q_T)\}$ is the most dificult analytical part of our work.

The methods developed in this paper allow to follow the increase of the smoothness of the solutions of problem (1_k) under the influence of the increase of the smoothness of the data, in particular to recoqnize when the solutions will be a classical one.

V. A. Solonnikov has considered the boundary value problem for compressible inhomogeneous fluids. He has proved that it is solvable for small t.

INSTITUTE OF MATHEMATICS, LENINGRAD.

Received September 16, 1975.