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On the lift and drag of a body in viscous fluid steady flow

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THE FORMULAE are derived for the lift and drag of a body in a steady flow of viscous fluid. The formula for the lift is an extension of the Joukowski theorem to the case of a viscous fluid flow around a finite body with a smooth surface. The proof is based on the integral representation of the solution of the Navier-Stokes equations and uses asymptotic behaviour of the solution far from the body.

LET us consider the stationary flow of a viscous incompressible fluid around a finite size body B with a sufficient smooth surface. Such a flow is governed by the Navier--Stokes equations with corresponding boundary conditions.

It is known that there exists a solution of the considered boundary-value problem in the class of functions with the limited Dirichlet integral

$$\int_G |\nabla \mathbf{u}|^2 dx_1 dx_2 dx_3 < \infty$$

for any Reynolds number. Here, $\mathbf{u}$ is the velocity vector, G — the domain occupied by the fluid. However, almost no further information on the behaviour of the solutions of this class has been obtained. Therefore, FINN [1] introduced the class of "physically reasonable" solutions, defined by the requirement that for any $\varepsilon > 0$

$$\mathbf{u}-\mathbf{u}_\infty = 0(R^{-\frac{1}{2}-\varepsilon}),$$

where $R = |\mathbf{x}|$, $\mathbf{x} = (x_1, x_2, x_3)$. For the "physically reasonable" solutions a number of results was obtained. In particular, an asymptotic behaviour of the solutions was found far from the body.

BABENKO [2] has proved that every solution with the limited Dirichlet integral is also a "physically reasonable" solution according to Finn. Thus, all results for "physically reasonable" solutions extend to the solutions with the limited Dirichlet integral.

The results obtained for the "physically reasonable" solutions and hence for the solutions with the limited Dirichlet integral are given below.

It is proved that the lift and drag exerted on the body are written as

$$L = \varrho \mathbf{u}_\infty \times \mathbf{\Gamma} + 0(R^{-\frac{1}{2}+\varepsilon}), \tag{1}$$

$$D = -\boldsymbol{\alpha} \int_{\Sigma_R} (p+\varrho \mathbf{u}_\infty \mathbf{v}) n_1 d\sigma + 0(R^{-\frac{1}{2}+\varepsilon}), \tag{2}$$

where $\mathbf{\Gamma} = \int_{\Sigma_R} (\mathbf{n} \times \mathbf{v})$ is the vector velocity circulation along the sphere Σ_R of the large radius R, ϱ — the density, p — the pressure, $\mathbf{n}$ — the unit exerior directed normal on Σ_R, $\alpha = \mathbf{u}_\infty / |\mathbf{u}_\infty|$.

The formula (1) extends the well-known Joukowski theorem to the case of a viscous fluid flow around a body. In the limit $R \to \infty$ it coincides with the formula given without rigorous proof in the monograph "Theoretical hydrodynamics" by MILNE-THOMPSON.

The starting point of development for the formulas (1) and (2) was an integral representation of the solution — Green's formula. The development was based on a lemma proved in the co-work by K. I. Babenko and by the author. This work was devoted to the study of the asymptotic behaviour of the time-independent solutions of the Navier-Stokes equations far from the body B. We shall refer to the lemma.

Let

$$I(\mathbf{x}) = \int_{R^3} f(\mathbf{y}) F(\mathbf{x}-\mathbf{y}) d\mathbf{y}$$

and suppose f and F admit the majorizations

$$|f(\mathbf{x})| \leqslant C(R+1)^{-\beta}[s(\mathbf{x})+1]^{-\gamma},$$
$$|F(\mathbf{x})| \leqslant CR^{-\mu}[s(\mathbf{x})+1]^{-\nu},$$

where

$$\beta > 0, \quad \gamma \geqslant 0, \quad \beta+\gamma \leqslant 3, \quad \beta-\gamma \geqslant 1,$$
$$1 < \mu \leqslant 2, \quad 1 \leqslant \nu \leqslant 2, \quad s(\mathbf{x}) = |\mathbf{x}|-x_1.$$

Then,

$$|I(\mathbf{x})| \leqslant CR^{\frac{3-\beta-\gamma}{2}-\mu} [s(x)+1]^{\frac{1-\beta-\gamma}{2}} [\Delta_{1,\beta-\gamma}(\Delta_{3,\beta+\gamma}+\Delta_{1,\gamma})+\Delta_{1,\mu-\nu}(\Delta_{2,\mu}+\Delta_{1,\nu})]$$

where

$$\Delta_{a,b} = \begin{cases} \log R, & \text{if} \quad a = b, \\ 1, & \text{if} \quad a \neq b. \end{cases}$$

Finally, it should be noted that the Joukowski theorem for the two-dimensional steady motion of the viscous fluid was considered by FILON [3] and IMAI [4]. But the rigorous proof was given by BABENKO [5].

References

1. R. FINN, *Estimates at infinity for stationary solutions of the Navier-Stokes equations*, Bull. Math. de la Soc. Sci, Math. Phys. de la RPR, 3, 51, 1959.
2. K. I. BABENKO, *On the stationary solutions of the problem flow of a viscous incompressible fluid*, Mat. sb. 91, 1, 1973 [in Russian].
3. L. N. G. FILON, *Forces on a cylinder*, Proc. Roy. Soc., 113A, 1926.
4. J. IMAI, *On the asymptotic behaviour of viscous fluid flow at a great distance from a cylindrical body with special reference to Filon's paradox*, Proc. Roy. Soc., 208A, 1951.
5. K. I. BABENKO, *On the asymptotic behaviour of the vorticity at large distance from a body in the plane flow of viscous fluid*, Prikl. Mat. Meh., 34, 5, 1970 [in Russian].

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