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Iterative methods in the analysis of dynamic processes of plastic forming of metals

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THIS work presents iterative procedures enabling to solve a relatively wide class of axially-symmetric problems of plastic forming under conditions of plane stress. In all solutions the Huber-Mises yield criterion and the associated flow law have been assumed. A procedure leading us to solutions for the strain-hardening and strain-rate sensitive materials is also proposed. As the first approximation in the iterative procedure we assume a quasi-static solution or a dynamic solution for the Tresca yield criterion and the associated flow law. A number of solutions to specific problems illustrated the methods proposed.

W pracy przedstawiono algorytmy iteracyjne dla dynamicznych zagadnień osiowo-symetrycznych w płaskim stanie naprężenia. Sformułowanie obejmuje równanie ruchu, warunek plastyczności Hubera-Misesa, stowarzyszone z nim prawo płynięcia oraz warunek nieściśliwości. Przeanalizowano również wpływ prędkości na wzrost oporu plastycznego, a także zjawisko wzmocnienia materiału. Za punkt wyjścia procesu iteracyjnego przyjmuje się bądź rozwiązanie quasi-statyczne, bądź uproszczone rozwiązanie dynamiczne, otrzymane dla warunku plastyczności Treski i stowarzyszonego z nim prawa płynięcia. Efektywność proponowanych metod rozwiązań zilustrowano na przykładach liczbowych.

В работе представлены итерационные алгоритмы для динамических осе-симметричных задач в плоском напряженном состоянии. Формулировка охватывает: уравнение движения, условие пластичности Губера-Мизеса, ассоциированный с ним закон течения, а также условие несжимаемости. Проанализированы тоже влияние скорости на рост пластического сопротивления, а также явление упрочнения материала. За исходную точку итерационного процесса принимаются или квазистатическое решение, или упрощенное динамическое решение полученное для условия пластичности Треска и ассоциированного с ним закона течения. Эффективность предлагаемых методов иллюстрирована на числовых примерах.

1. Introduction

IN many cases of plastic forming processes a quasi-static approach to the theoretical analysis which neglects inertia terms in the equations of motion and assumes a rigid (perfectly-plastic model of the material) gives satisfactory results [1, 2]. In numerous cases the strain-hardening effect can be taken into account [3, 4]. There exist, however, many problems of plastic forming in which the dynamic effects play a significant role and cannot be omitted in the theoretical analysis. In general, two groups of such processes can be distinguished. In the first group we find processes where, in spite of relatively slow speeds of the tool, the strain rates reach great values; the compression of a thin plastic layer between two rigid plates may be mentioned as a typical example. In the second group we have high energy processes of plastic forming characterized by very fast deformation speeds. Here, explosive forming may be mentioned as an example. In both cases the increase of the resistance of the metal to plastic deformation due to the high rate of strains, and the inertia terms in the equations of motion should be taken into consideration. Solving such dynamic problems is, however, a difficult to ask. We know of only a few solutions which account for both effects (see, for example [7, 8]). The dynamic solutions for perfectly plastic material may be found in [5, 9, 10, 14].

This work presents iterative procedures enabling to solve a relatively wide class of axially-symmetric problems of plastic forming under conditions of plane stress. In all solutions the Huber-Mises yield criterion and the associated flow law have been assumed. A procedure leading us to solutions for the strain-hardening and strain-rate sensitive materials is also proposed. As the first approximation in the iterative procedure we assume a quasi-static solution or a dynamic solution for the Tresca yield criterion and the associated flow law. A number of solutions to specific problems illustrated the methods is proposed.

2. Formulation of the problem

The basic system of equation for a dynamic axially-symmetric plane stress has the form

$$\sigma_{r,r}+\sigma_r\frac{1}{h}h_{,r}+\frac{1}{r}(\sigma_r-\sigma_\theta)=\varrho(v_{,t}+vv_{,r}), \tag{2.1}$$

$$\sigma_r^2-\sigma_r\sigma_\theta+\sigma_\theta^2=3k^2, \tag{2.2}$$

$$v_{,r}(2\sigma_\theta-\sigma_r)=v\frac{1}{r}(2\sigma_r-\sigma_\theta), \tag{2.3}$$

$$\frac{1}{h}(h_{,t}+vh_{,r})+v_{,r}+\frac{1}{r}v=0. \tag{2.4}$$

The state of stress is determined by the radial σ_r and the circumferential σ_θ components, while two magnitudes — the radial velocity v and the thickness of the wall h describe the kinematics of deformation. ϱ and k are material constants, connected with the density and the yield locus in shear. In the introductory analysis a perfectly-plastic material (k = const) will be assumed. The constitutive equation (2.3) associated with the yield condition (2.2) and the condition of incompressibility together with the equation of motion (2.1) constitute the system of four equations in four unknowns.

The yield condition (2.2) can be automatically satisfied when the stresses σ_r and σ_θ are expressed by the function ω

$$\left.\begin{matrix}\sigma_r\\ \sigma_\theta\end{matrix}\right\}=2k\cos\left(\omega\mp\frac{\pi}{6}\right). \tag{2.5}$$

Thus the system of equations reduces to three non-homogeneous partial quasi-linear differential equations in three unknown functions, ω, h and v in two independent variables, r and t:

$$\sin\left(\omega-\frac{\pi}{6}\right)\omega_{,r}-\cos\left(\omega-\frac{\pi}{6}\right)\frac{1}{h}h_{,r}-\frac{1}{r}\sin\omega=\frac{\varrho}{2k}(v_{,t}+vv_{,r}), \tag{2.6}$$

$$v_{,r} = \frac{1}{r}v\frac{2\cos\left(\omega-\frac{\pi}{6}\right)-\cos\left(\omega+\frac{\pi}{6}\right)}{2\cos\left(\omega+\frac{\pi}{6}\right)-\cos\left(\omega-\frac{\pi}{6}\right)}, \tag{2.7}$$

$$\frac{1}{h}h_{,t}+\frac{v}{h}h_{,r}+v_{,r} = -\frac{1}{r}v. \tag{2.8}$$

According to Courant's classification [15] this system is not entirely hyperbolic. Along the characteristics of three families, a double one $t = \text{const}$ and the other $dr-vdt = 0$, only two differential relations can be established

for $t = \text{const}$

$$\frac{dv}{dr} = \frac{1}{r}v\frac{2\cos\left(\omega-\frac{\pi}{6}\right)-\cos\left(\omega+\frac{\pi}{6}\right)}{2\cos\left(\omega+\frac{\pi}{6}\right)-\cos\left(\omega-\frac{\pi}{6}\right)} \tag{2.9}$$

and for $dr-v\,dt = 0$

$$\frac{dh}{dr} = -\frac{h}{r}\left[\frac{2\cos\left(\omega-\frac{\pi}{6}\right)-\cos\left(\omega+\frac{\pi}{6}\right)}{2\cos\left(\omega+\frac{\pi}{6}\right)-\cos\left(\omega-\frac{\pi}{6}\right)}+1\right]. \tag{2.10}$$

The lack of the third relation along the characteristics does not allow to apply the standard technique used in problems which are described by hyperbolic sets of equations. Thus, we propose iteration procedures for solving the dynamic problems of the plane axially-symmetric state of stress.

3. Iterative method of characteristics

Two basic iteration procedures will be distinguished, depending on the choice of the first approximation:

a) a method, when we begin with the functions obtained from the dynamic solution, in which the Tresca yield condition and the flow law associated with it were assumed,

b) a method, when we begin with the values of all functions obtained from the quasi-static solution, $(v_{,t}+vv_{,r} = 0)$.

In the first case the basic system of equations contains: the equation of motion (2.1), the Tresca yield condition

$$\begin{aligned} \sigma_r-\sigma_\theta = 2k \quad &\text{for} \quad \sigma_r\sigma_\theta \leqslant 0,\\ \sigma_\theta = 2k \quad &\text{for} \quad \sigma_r\sigma_\theta \geqslant 0, \end{aligned} \tag{3.1}$$

the associated flow rule

$$\begin{aligned} v_{,r}+\frac{v}{r} = 0 \quad &\text{for} \quad \sigma_r\sigma_\theta \leqslant 0,\\ v_{,r} = 0 \quad &\text{for} \quad \sigma_r\sigma_\theta \geqslant 0 \end{aligned} \tag{3.2}$$

and the incompressibility condition (2.8).

Solutions of this system for particular boundary and initial value problems in the two cases of the linear yield condition (3.1) have been presented in the book [12].

The second method consists in neglecting in the first approximation the substantial derivative $\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}\right)$ which appears in Eq. (2.6). Such a simplified system (2.6)-(2.8) describes the quasi-static flow, in which the inertia forces are disregarded. The quasi-static problems have been analysed in Ref. [3], and the equations of characteristics for the first approximation have the form

for $t = \text{const}$

$$\frac{d\omega}{dr}\sin\left(\omega-\frac{\pi}{6}\right)-\frac{1}{h}\frac{dh}{dr}\cos\left(\omega-\frac{\pi}{6}\right)-\frac{1}{r}\sin\omega = 0, \tag{3.3}$$

$$\frac{dv}{dr} = \frac{1}{r}v\,\frac{2\cos\left(\omega-\frac{\pi}{6}\right)-\cos\left(\omega+\frac{\pi}{6}\right)}{2\cos\left(\omega+\frac{\pi}{6}\right)-\cos\left(\omega-\frac{\pi}{6}\right)} \tag{3.4}$$

and for $dr-vdt = 0$

$$\frac{dh}{dr} = -\frac{h}{r}\left[\frac{2\cos\left(\omega-\frac{\pi}{6}\right)-\cos\left(\omega+\frac{\pi}{6}\right)}{2\cos\left(\omega+\frac{\pi}{6}\right)-\cos\left(\omega-\frac{\pi}{6}\right)}+1\right]. \tag{3.5}$$

In most practical problems the dynamic flow is defined by the kinematic boundary conditions, when the velocity of the tool or velocity of the material at certain sections is given. In such cases it is reasonable to expect that the flow velocities $v(r, t)$, and thickness distribution $h(r, t)$ obtained from the solution of the system (2.1), (3.1), (3.2), or of the system for the quasi-static formulation, will only slightly differ from those resulting from the solution of the basic system (2.6)–(2.8). The iteration procedure will be based on this observation. By substituting the values of the functions $v(r, t)$ and $h(r, t)$, which are obtained from the first approximations computed according to the first or the second method, into the equation of motion (2.6), we get the lacking second differential relation for the function $\omega(r, t)$ which has to be satisfied along the characteristics $t = \text{const}$

$$\frac{d\omega}{dr}\sin\left(\omega-\frac{\pi}{6}\right)=\cos\left(\omega-\frac{\pi}{6}\right)\left\langle\frac{1}{h}h_{,r}\right\rangle+\frac{1}{r}\sin\omega-\frac{\varrho}{2k}\langle v_{,t}+vv_{,r}\rangle. \tag{3.6}$$

The expressions $\left\langle\frac{1}{h}h_{,r}\right\rangle$ and $\langle v_{,t}+vv_{,r}\rangle$ are assumed to be known from the foregoing approximation. If in the first approximation Eqs. (3.1) and (3.2) are used, we obtain simple analytic expression for $v(r, t)$ and $h(r, t)$. In the successive approximations we solve subsequently (3.6), (2.9), (2.10), introducing at each subsequent iteration into (3.6) the values of the functions v and h as obtained from the previous iteration. This procedure is repeated until two consecutive approximations give sufficiently close results.

A similar iterative procedure can be used in problems where the static boundary conditions are given. In such cases the tractions exerted by the tool on the plastically-deformed material are usually known. We may expect that an appropriate stress field chosen in advance will have a decisive influence on the efficiency of the iteration procedure. The equations written according to the sequence of computations have now the form

for t = const

$$\frac{dv}{dr} = \frac{v}{r}\left\langle\frac{2\cos\left(\omega-\frac{\pi}{6}\right)-\cos\left(\omega+\frac{\pi}{6}\right)}{2\cos\left(\omega+\frac{\pi}{6}\right)-\cos\left(\omega-\frac{\pi}{6}\right)}\right\rangle \tag{3.7}$$

and for $dr-vdt = 0$

$$\frac{dh}{dr} = -\frac{h}{r}\left\langle\frac{2\cos\left(\omega-\frac{\pi}{6}\right)-\cos\left(\omega+\frac{\pi}{6}\right)}{2\cos\left(\omega+\frac{\pi}{6}\right)-\cos\left(\omega-\frac{\pi}{6}\right)}+1\right\rangle \tag{3.8}$$

and for t = const

$$\frac{d\omega}{dr}\sin\left(\omega-\frac{\pi}{6}\right)+\frac{1}{r}\sin\omega-\cos\left(\omega-\frac{\pi}{6}\right)\left\langle\frac{1}{h}h_{,r}\right\rangle = -\frac{\varrho}{2k}\langle v_{,t}+vv_{,r}\rangle. \tag{3.9}$$

The expressions in angular brackets should be computed on the basis of the foregoing approximation. We begin the iteration procedure by solving the system of equations (2.1), (3.1) and (3.2) for the dynamic problem with the Tresca yield criterion, or the system for the quasi-static problem, depending on which method of computation [a) or b)] has been chosen. Now, we introduce the obtained function $\omega(r, t)$ into Eqs. (3.7), (3.8), and compute the next approximation of the velocity $v(r, t)$ and the wall thickness $h(r, t)$. Substituting them into (3.9), we find the second approximation of the function ω, and so on. We stop the computations when the difference between the consecutive approximations is sufficiently small.

If the velocity of deformation is very high the differences between the consecutive approximations may be too large. In such cases we may gradually increase the value of the velocity at the boundary for each approximation until the prescribed value is reached.

4. Expanding of a flat ring

Consider now the operation of expanding of a flat ring loaded by pressure applied at the inner rim (Fig. 1). An analogous quasi-static problem has been examined in [3]. Now,

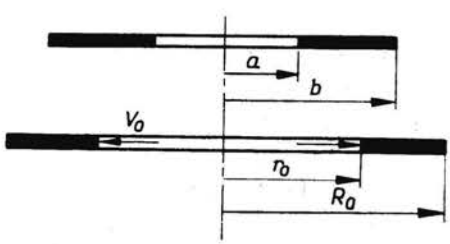

FIG. 1.

the dynamic problem for the Huber-Mises yield criterion will be solved by the iteration procedure. As the first approximation we will assume the dynamic solution for the Tresca yield criterion (3.1).

The radial stress is compressive and the circumferential stress is tensile. Thus we have the case $\sigma_r\sigma_\theta \leqslant 0$ and the Tresca yield criterion is described by the first expression in (3.1). The velocity $v(r, t)$ and the thickness $h(r, t)$ are determined by the first equation in (3.2) and by (2.4). The following initial value condition

$$h(r, 0) = h_0 = \text{const} \tag{4.1}$$

and the boundary condition

$$v(r_0, t) = v_0(t) \tag{4.2}$$

have to be satisfied.

The value of the radius of the inner rim at any arbitrary instant is

$$r_0(t) = a + \int_0^t v(\tau)d\tau, \tag{4.3}$$

where $a = r_0(0)$. Thus, we finally obtain

$$v(r, t) = v_0(t)\frac{1}{r}\left(a + \int_0^t v_0(\tau)\,d\tau\right), \tag{4.4}$$

$$h(r, t) = h_0. \tag{4.5}$$

Introducing (4.4) and (4.5) into (3.6) we obtain

$$\frac{d\omega}{dr} = \frac{1}{r\sin\left(\omega - \frac{\pi}{6}\right)}\left\{\sin\omega - \frac{\varrho v_0^2}{2k}\left[1 - \frac{1}{r^2}(a + v_0 t)\right]\right\}. \tag{4.6}$$

Equation (4.6) together with (2.9) and (2.10) constitute a set of three equations in three unknowns ω, v, h, from which the first iteration can be computed. Since the outer rim R_0 is assumed to be stress-free ($\sigma_r = 0$), we obtain from (2.5) the boundary condition for

$$\omega(R_0, t) = \frac{5}{3}\pi. \tag{4.7}$$

Solving numerically (4.6) and then (2.9) and (2.10) with the conditions (4.1), (4.2) and (4.7), we obtain the first approximation for ω and the second approximation for v and h.

Now, introduce the obtained approximation of the functions $v(r, t)$ and $h(r, t)$ into (3.6), from which the second approximation for ω can be computed. Then, from (2.9) and (2.10), the third approximation for v and h can be found, and so on.

The numerical example shows that the iteration procedure is quickly converging. The difference between the second and the third approximations is less than 1 per cent. The velocity of the inner rim was assumed to be constant and equal to $v_0 = 10$ m/sec. Figure 2 shows the net of characteristics for the third approximation as calculated for the following data: $k = 13.3$ kp/cm^2 (mild steel), $a = 20$ mm, $b = 40$ mm. In Fig. 3 we see the distrib-

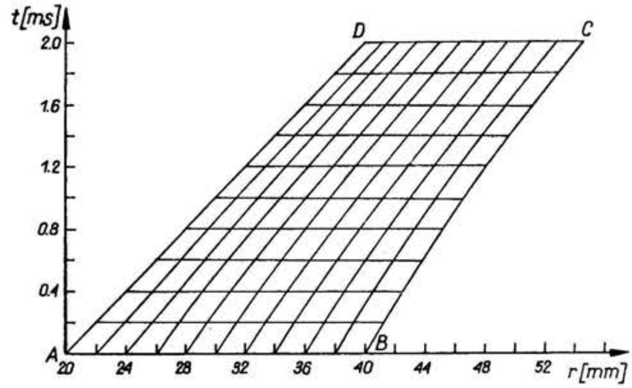

Fig. 2.

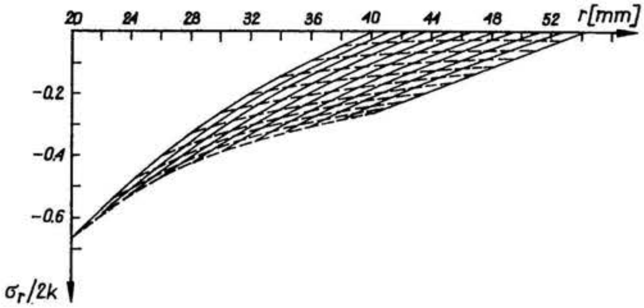

Fig. 3.

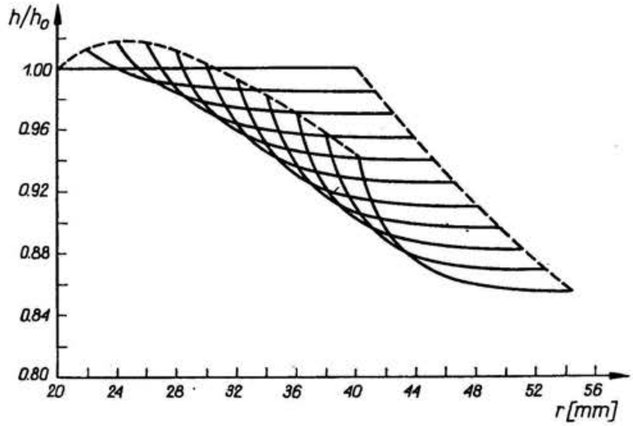

Fig. 4.

ution of the radial stress σ_r, and in Fig. 4 the variation of the wall thickness. Continuous lines represent distribution along the radius for given instants t = const. Broken lines indicate how the radial stress and thickness change at a given particle of the ring. The solution can be extended for arbitrarily large deformations. It has, however, physical sense until the decohesion or loss of stability occurs.

The influence of the velocity v_0 on the radial stress at the inner rim is presented in Fig. 5. Computations have been carried out by means of the iteration procedure b) — (see Sect. 3). As the first approximation we took the quasi-static solution for the Huber-Mises yield condition [see Eqs. (3.3)–(3.5)]. In the consecutive approximations the velocity v_0 was assumed to have consecutively the values $v_0 = 10, 30, 50$ and 100 m/sec. Results for the velocity $v_0 = 10$ m/sec are very close to those shown in Fig. 3. Thus the two iteration methods lead to the same results. In further considerations the iteration procedure a) will be used.

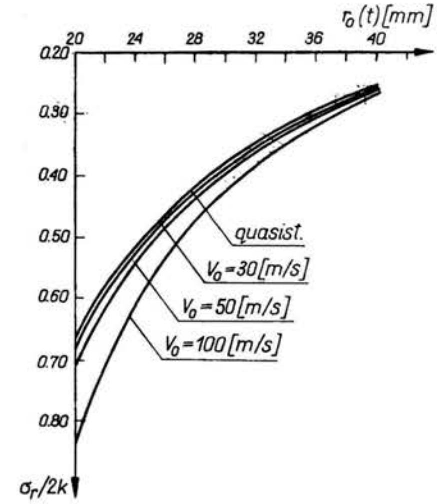

FIG. 5.

It is seen that inertial effects are significant for velocities greater than $v_0 = 50$ m/sec. One can expect that for smaller velocities the effect of the increase of the resistance of material to deformation will be more visible with an increasing rate of deformation.

5. Solutions for the dynamic yield condition

A yield condition in which both the strain-hardening effect and strain-rate sensitivity of the material are accounted for will be called a dynamic yield condition. The dynamic yield criterion based upon the classic Huber-Mises condition takes the form (see [11])

$$\sqrt{J_2} = \varkappa(W_p)\left[1+\Phi^{-1}\left(\frac{\sqrt{I_2^p}}{\gamma}\right)\right], \tag{5.1}$$

where J_2 and I_2^ϱ stand for the invariants of the stress deviator and strain rate tensor respectively. For the axially-symmetric plane state of stress they are expressed as follows:

$$(5.2)\qquad \begin{aligned} J_2 &= \frac{1}{2}S_{ij}S_{ij} = \frac{1}{3}(\sigma_r^2 - \sigma_r\sigma_\theta + \sigma_\theta^2),\\ I_2^\varrho &= \frac{1}{2}\varepsilon_{kl}\varepsilon_{kl} = \frac{1}{2}(\varepsilon_r^2 + \varepsilon_\theta^2 + \varepsilon_h^2). \end{aligned}$$

In expression (5.1) $\varkappa$ is the strain-hardening parameter, and γ is the viscosity parameter. The yield function $\Phi(F)$ for a linear isotropic hardening has the form

$$(5.3)\qquad \Phi(F) = \frac{\sqrt{J_2}}{\varkappa} - 1.$$

Now, instead of (5.1) we can write

$$(5.4)\qquad \sigma_r^2 - \sigma_r\sigma_\theta + \sigma_\theta^2 = 3\varkappa^2\left(1 + \frac{1}{\sqrt{\gamma}}\sqrt[4]{\frac{\varepsilon_r^2 + \varepsilon_\theta^2 + \varepsilon_h^2}{2}}\right)^2.$$

For the sake of brevity let us introduce the parameter $\alpha(r, t)$ of the strain rate sensitivity of the material

$$(5.5)\qquad \alpha(r, t) = 1 + \frac{1}{\sqrt{\gamma}}\sqrt[4]{\frac{\varepsilon_r^2 + \varepsilon_\theta^2 + \varepsilon_h^2}{2}},$$

where the components of the strain rate tensor are defined as follows

$$(5.6)\qquad \varepsilon_r = v_{,r} \qquad \varepsilon_\theta = \frac{1}{r}v, \qquad \varepsilon_h = \frac{1}{h}(h_{,t} + vh_{,r}).$$

The yield condition (5.4) will be satisfied identically if the stresses are expressed in the following way

$$(5.7)\qquad \left.\begin{matrix}\sigma_r\\ \sigma_\theta\end{matrix}\right\} = 2\alpha\varkappa\cos\left(\omega \mp \frac{\pi}{6}\right).$$

Now, the problem reduces to the solution of the system of equations (2.1), (2.3) and (2.4) complemented by the dynamic yield condition (5.4). Finally, we arrive at the system of equations

$$(5.8)\qquad \sin\left(\omega - \frac{\pi}{6}\right)\omega_{,r} - \left(\frac{1}{\alpha}\alpha_{,r} + \frac{1}{\varkappa}\varkappa_{,r} + \frac{1}{h}h_{,r}\right)\cos\left(\omega - \frac{\pi}{6}\right) + \frac{\varrho}{2\alpha\varkappa}(v_{,t} + vv_{,r}) = \frac{1}{r}\sin\omega,$$

$$(5.9)\qquad v_{,r} = \frac{1}{r}v\,\frac{2\cos\left(\omega - \frac{\pi}{6}\right) - \cos\left(\omega + \frac{\pi}{6}\right)}{2\cos\left(\omega + \frac{\pi}{6}\right) - \cos\left(\omega - \frac{\pi}{6}\right)},$$

$$(5.10)\qquad \frac{1}{h}(h_{,t} + vh_{,r}) + v_{,r} = -\frac{1}{r}v.$$

In the limit case, when $\gamma = \infty$, or equivalently $\alpha(r, t) = 1$, one obtains plastic flow with isotropic strain-hardening only. In the other case, when the strain-hardening parameter is constant and equal to the yield locus in pure shear, only the strain rate sensitivity of the material is accounted for. For $\alpha(r, t) = 1$ and $\varkappa(r, t) = k$ we obtain a case of perfectly plastic material.

The problem can be solved by means of the method of successive approximations. As the first approximation of the velocity $v(r, t)$ and wall thickness $h(r, t)$ functions one can take those obtained from the solution of the system (2.1), (3.1), (3.2) and (2.4) for a dynamic flow, or — of the system (3.3)–(3.5) for a quasi-static flow. Further approximation can be computed by means of the equations of characteristics of the system (5.8)–(5.10) along the characteristics $t = \text{const}$

$$\frac{d\omega}{dr} = \frac{1}{\sin\left(\omega - \frac{\pi}{6}\right)} \left[\left\langle \frac{1}{\alpha}\alpha_{,r} + \frac{1}{\varkappa}\varkappa_{,r} + \frac{1}{h}h_{,r}\right\rangle \cos\left(\omega - \frac{\pi}{6}\right) + \frac{\sin\omega}{r} - \frac{\varrho}{2\langle\alpha\varkappa\rangle}\langle v_{,t} + vv_{,r}\rangle\right], \tag{5.11}$$

$$\frac{dv}{dr} = \frac{1}{r} v \frac{2\cos\left(\omega - \frac{\pi}{6}\right) - \cos\left(\omega + \frac{\pi}{6}\right)}{2\cos\left(\omega + \frac{\pi}{6}\right) - \cos\left(\omega - \frac{\pi}{6}\right)} \tag{5.12}$$

and along the characteristics $dr - v\,dt = 0$

$$\frac{dh}{dr} = -\frac{h}{r}\left[\frac{2\cos\left(\omega - \frac{\pi}{6}\right) - \cos\left(\omega + \frac{\pi}{6}\right)}{2\cos\left(\omega + \frac{\pi}{6}\right) - \cos\left(\omega - \frac{\pi}{6}\right)} + 1\right]. \tag{5.13}$$

By computing a consecutive approximation we substitute the foregoing approximation for the expressions in angular brackets.

As a working example let us consider the expansion of a ring caused by traction applied at its outer rim (Fig. 6). A quasi-static solution to this problem with isotropic strain-hardening has been given by W. Szczepiński [3]. As a first approximation, we assume the solution of the system of Eqs. (2.1), (3.1), (3.2) and (2.4). Since both stresses are evidently of the same sign ($\sigma_r\sigma_\theta > 0$) the second relation in (3.2) holds. Thus, solving (3.2) and (2.4) with the conditions

$$h(r, 0) = h_0 = \text{const}, \quad v(R_0, t) = v_0(t), \tag{5.14}$$

we obtain

$$v(r, t) = v_0(t), \tag{5.15}$$

$$h(r, t) = h_0\left(1 - \frac{1}{r}\int_0^t v_0(\tau)\,d\tau\right) \tag{5.16}$$

Assume a linear relation for a strain-hardening parameter

$$\varkappa(r, t) = k(1 + \varepsilon_i), \tag{5.17}$$

which approximately corresponds to mild steel. The intensity of strain ε_i can be computed from the relation

$$\varepsilon_i(r,t) = \int \sqrt{\frac{1}{2}(\varepsilon_r^2+\varepsilon_\theta^2+\varepsilon_h^2)}\, dt. \tag{5.18}$$

The integral must be computed along the characteristics $dr - vdt = 0$, representing the trajectories of material particles in the r, t-plane.

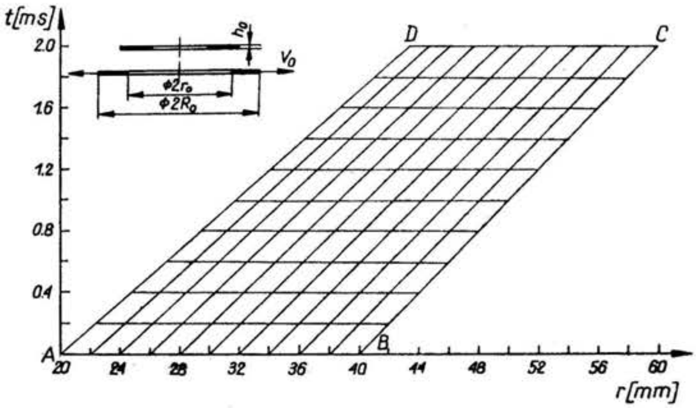

FIG. 6.

Assume that the velocity of the outer rim ($r = R_0$) is constant $v_0(t) = 10$ m/sec. Thus instead of (5.17) and (5.5) we can write

$$\varkappa(r,t) = k\left(1+\frac{v_0}{r}t\right), \tag{5.19}$$

$$\alpha(r,t) = 1+\sqrt{\frac{v_0}{r\gamma}}, \tag{5.20}$$

where t is the time measured from the beginning of the process. Now, introducing (5.15), (5.16) and (5.19), (5.20) into (5.11) we can compute the second approximation for ω, v and h, by solving numerically the system (5.11)–(5.13) with the conditions (5.14) and

$$\omega(r_0,t) = \frac{5}{3}\pi. \tag{5.21}$$

Having found the second approximations of all magnitudes in that manner we can compute the next approximation and so on.

The iteration procedure proved quickly converging. The difference between the second and third approximations is found to be smaller than 1 per cent. Figure 7 illustrates how the radial stress at the outer rim changes with time for four variants of the solution. The parameter γ and the constant k were assumed to be equal $\gamma = 200$ sec^{-1}, $k = 13.3$ kp/cm^2, respectively.

It is clearly seen that the influence of the viscous properties of the material is significant. Figure 8 shows the variation of the wall thickness of the ring. Continuous lines represent the solution for a perfectly-plastic material, while broken lines correspond to the solution in which strain-hardening and strain-rate sensitivity have been taken into account.

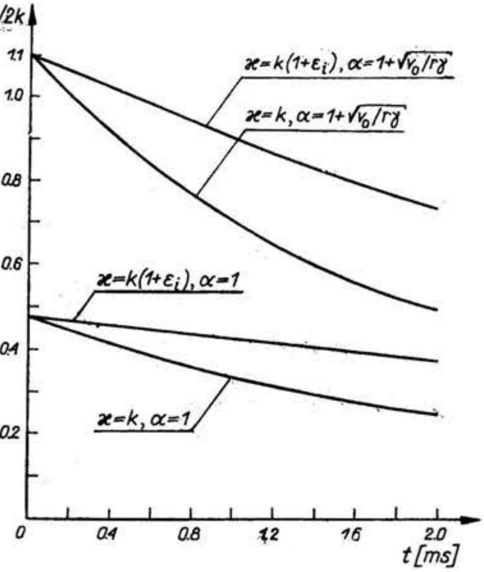

FIG. 7.

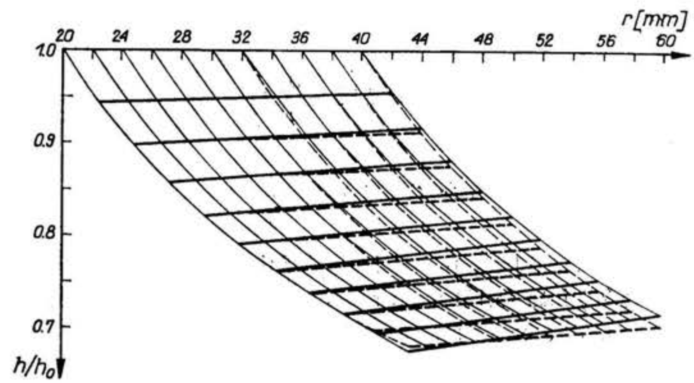

FIG. 8.

6. Rotationally symmetric shells

Consider now a certain class of processes of the plastic forming of thin-walled rotationally symmetric shells, in which only one side of the material is in contact with the tool. Since the contact pressure between the tool and the deformed material is small (provided the radii of the curvature of the shell are sufficiently large as compared with its thickness) the plane state of stress may be assumed with a sufficient degree of accuracy. The equation of motion of the deformed material takes the form (Fig. 9)

$$\sigma_{r,r}+\sigma_r\frac{1}{h}h_{,r}+\frac{1}{r}(\sigma_r-\sigma_\theta)-\frac{\mu p}{h\cos\alpha}-\varrho\dot{v}=0, \tag{6.1}$$

$$\frac{p}{h}-\frac{\sigma_\theta}{\varrho_\theta}-\frac{\sigma_r}{\varrho_\alpha}-\varrho\sin\alpha\dot{v}=0, \tag{6.2}$$

where μ is the coefficient of friction, and ϱ_θ, ϱ_α are the radii of curvature. The dot in the term $\dot{v}$ denotes the substantial derivative.

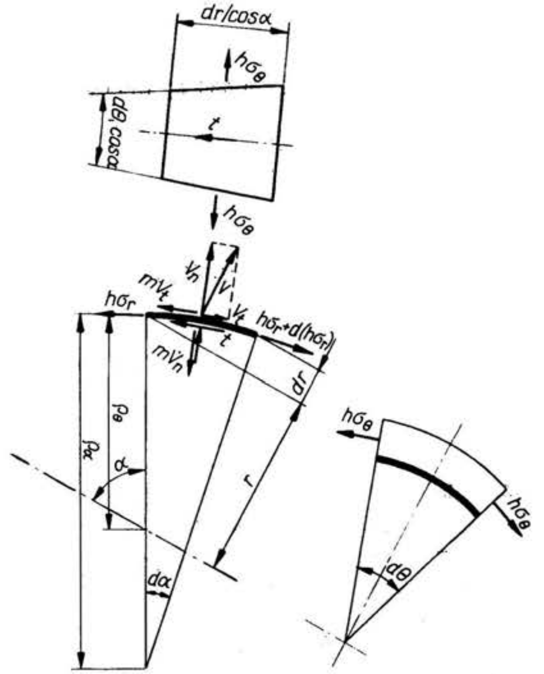

FIG. 9.

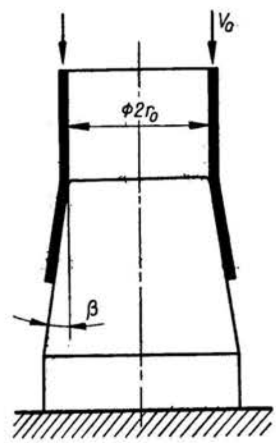

FIG. 10.

As an example let us consider the expansion of a tube on a conical mandrel (Fig. 10). Now, the equation of motion (6.1) may be written in the form

$$\sigma_{r,r}+\sigma_r\frac{1}{h}h_{,r}+\frac{1}{r}\sigma_r-\frac{1}{r}\sigma_\theta(1+\mu\operatorname{ctg}\beta)-\varrho(1+\mu\operatorname{ctg}\beta)(v_{,t}+vv_{,r})=0. \tag{6.3}$$

Assume in (5.1) the function $\Phi(F)$ in the form $\Phi(F)=F^\delta$, where δ is a constant. Condition (5.1) takes the form

$$\sigma_r^2-\sigma_r\sigma_\theta+\sigma_\theta^2=3\varkappa^2\left[1+\left(\frac{1}{\gamma}\sqrt{\frac{1}{2}(\dot\varepsilon_r^2+\dot\varepsilon_\theta^2+\dot\varepsilon_h^2)}\right)^{\frac{1}{\delta+1}}\right]^2 \tag{6.4}$$

In this case also expression (5.7) satisfies identically the condition (6.4). The parameter $\alpha(r, t)$ is now expressed as follows:

$$\alpha(r,t) = 1+\gamma^{\frac{-1}{\delta+1}}\left[\frac{1}{2}(\varepsilon_r^2+\varepsilon_\theta^2+\varepsilon_h^2)\right]^{\frac{1}{2(\delta+1)}}. \tag{6.5}$$

Thus the system of governing equations is similar to the system (5.8)–(5.10). Instead of (5.8), we have now the equation

$$\omega_{,r}\sin\left(\omega-\frac{\pi}{6}\right)-\left(\frac{1}{\alpha}\alpha_{,r}+\frac{1}{\varkappa}\varkappa_{,r}+\frac{1}{h}h_{,r}\right)\cos\left(\omega-\frac{\pi}{6}\right)-\frac{1}{r}\left[\sin\omega -\cos\left(\omega-\frac{\pi}{6}\right)\mu\operatorname{ctg}\beta\right]+\frac{\varrho}{2\alpha\varkappa}(1+\mu\operatorname{ctg}\beta)(v_{,t}+vv_{,r}) = 0 \tag{6.6}$$

obtained from (6.3).

Thus the dynamic problem of the expansion of a tube is determined by the equations

$$\frac{d\omega}{dr} = \frac{1}{\sin\left(\omega-\frac{\pi}{6}\right)}\left[\left\langle\frac{1}{\alpha}\alpha_{,r}+\frac{1}{h}h_{,r}+\frac{1}{\varkappa}\varkappa_{,r}\right\rangle\cos\left(\omega-\frac{\pi}{6}\right)+\frac{1}{r}\left(\sin\omega-\cos\left(\omega-\frac{\pi}{6}\right)\mu\operatorname{ctg}\beta\right)-\frac{\varrho}{2\langle\alpha\varkappa\rangle}(1+\mu\operatorname{ctg}\beta)\langle v_{,t}+vv_{,r}\rangle\right], \tag{6.7}$$

$$\frac{dv}{dr} = \frac{1}{r}v\frac{2\cos\left(\omega-\frac{\pi}{6}\right)-\cos\left(\omega+\frac{\pi}{6}\right)}{2\cos\left(\omega+\frac{\pi}{6}\right)-\cos\left(\omega-\frac{\pi}{6}\right)} \tag{6.8}$$

which hold along the characteristics $t = \text{const}$, and

$$\frac{dh}{dr} = -\frac{h}{r}\left[\frac{2\cos\left(\omega-\frac{\pi}{6}\right)-\cos\left(\omega+\frac{\pi}{6}\right)}{2\cos\left(\omega+\frac{\pi}{6}\right)-\cos\left(\omega-\frac{\pi}{6}\right)}+1\right] \tag{6.9}$$

valid along the characteristics $dr-vdt = 0$. Note that v represents the radial velocity component. These equations will be solved by the procedure of successive approximations. Computing the n-th approximation we substitute in the expressions in angular brackets the magnitudes found in the $(n-1)$th approximation. As the first approximation we can take a dynamic solution for the perfectly-plastic material and the Tresca yield condition with the associated flow law.

Numerical example have been solved for the velocity $v_0 = 5\,\text{m/sec}$ and for the particular values of the constants $\delta = 3$ and $\gamma = 240\,\text{sec}^{-1}$. The net of characteristics is shown in Fig. 11. Along the characteristic AB corresponding to the free edge of the tube, we have $\sigma_r = 0$ and, consequently, $\omega = \frac{5}{3}\pi$. Along the vertical line $AC(r = r_0)$, corresponding to the initial radius we have $v(r_0, t) = v_0\tan\beta$ and $h(r_0, t) = h_0 = \text{const}$. Computations have been performed for $\mu = 0.1$ and $k = 13.3\,\text{kp/mm}^2$.

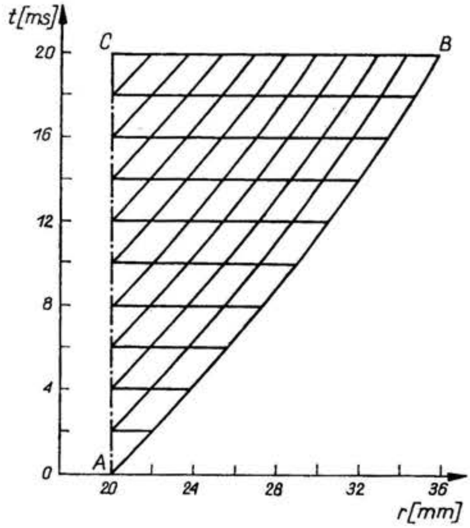

FIG. 11.

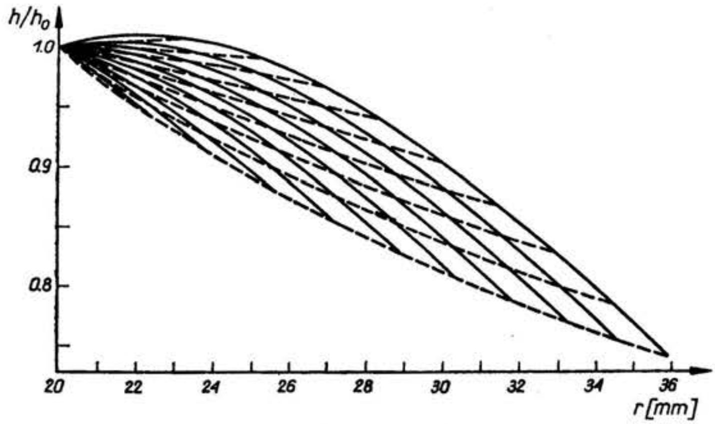

FIG. 12.

Figure 12 shows variation of the wall thickness. Continuous lines indicate how the thickness is distributed along the radius at the fixed instants of the process. Dashed lines show the deformation history of different elements of the tube.

7. Final remarks

All examples presented above indicate that the iterative methods are converging very quickly. This is rather obvious, since the differences between the velocity fields obtained from quasi-static and dynamic solutions are small, if the same kinematic boundary conditions are assumed. Thus, the methods have been proved effective and may be used in the analysis of more advanced practical problems, when the velocities at the edges change according to any law $v_0(t)$.

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