Archives of Mechanics • Archiwum Mechaniki Stosowanej • 28, 4, pp. 607–614, Warszawa 1976

# The Stüssi-Kollbrunner paradox in the light of the concept of decohesive carrying capacity

TRAN-LE BINH and M. ŻYCZKOWSKI (KRAKÓW)

THE Stüssi-Kollbrunner paradox consists in the independence of the classic limit carrying capacity of a beam shown in Fig. 1 of the length of the beam span $l_1$; at $l_1 \to \infty$ we obtain the result which is different than the result obtained for the carrying capacity of the free-supported beam. It was shown in this paper that the paradox mentioned above does not occur for the perfectly-elastic-plastic material if continuous displacement fields or fields with admissible discontinuities are the only considered. The classical scheme of limit carrying capacity cannot be reached; the work of a beam which is assumed as a continuous system ends when the first plastic hinge under the force appears. According to SZUWALSKI-ŻYCZKOWSKI proposition the corresponding load was called the decohesive carrying capacity of a beam and is continuous function of the geometric parameter $k$ in a whole interval $0 \leqslant k \leqslant 1$, Fig. 4. The value of the non-admissible discontinuities of the displacements corresponding to the limit carrying capacity is also determined.

Paradoks Stüssi-Kollbrunnera polega na niezależności klasycznej nośności granicznej belki pokazanej na rys. 1 od długości przęsła $l_1$; przy $l_1 \to \infty$ otrzymujemy wynik różniący się od nośności granicznej belki swobodnie podpartej. W pracy pokazano, iż dla materiału idealnie sprężysto-plastycznego przy ograniczeniu się do pól przemieszczeń ciągłych lub wykazujących dopuszczalne nieciągłości powyższy paradoks nie istnieje. Klasyczny schemat nośności granicznej nie może być osiągnięty, praca belki jako ustroju ciągłego kończy się przy powstaniu pierwszego przegubu plastycznego pod siłą. Zgodnie z propozycją K. SZUWALSKIEGO i M. ŻYCZKOWSKIEGO odpowiednie obciążenie nazwano nośnością rozdzielczą belki: jest ona ciągłą funkcją geometrycznego parametru $k$ w całym przedziale $0 \leqslant k \leqslant 1$, rys. 4. Określono również wartość niedopuszczalnych nieciągłości przemieszczeń, odpowiadających nośności granicznej.

Парадокс Стисси-Кольбруннера заключается в независимости классической предельной нагрузки балки, указанной на рис. 1, от длины пролета $l_1$; при $l_1 \to \infty$ получается результат отличающийся от предельной нагрузки свободно подпертой балки. В работе показано, что для идеально упруго-пластического материала, ограничиваясь полями непрерывных перемещений или обладающих допускаемыми разрывами, вышеупомянутый парадокс не существует. Классическая схема предельной нагрузки не может быть достигнутой, работа балки как сплошного устройства кончается при возникновении первого пластического шарнира под силой. Согласно предложению К. Шувальского и М. Жичковского соответствующая нагрузка названа несущей способностью расцепления балки: она является непрерывной функцией геометрического параметра $k$ в целом интервале $0 \leqslant k \leqslant 1$, рис. 4. Определено тоже значение недопускаемых разрывов перемещений, отвечающих предельной нагрузке.

## 1. Introduction

THE classical theory of the plastic limit analysis is based on two intuitive axioms. Within the frames of perfect plasticity it is assumed that for a given structure: (1) there exists at least one mechanism of plastic collapse (infinitesimal motion at a constant loading parameter), (2) if it exists for a rigid-perfectly plastic body, then it may be reached by an elastic-perfectly-plastic structure as well. The corresponding stresses and velocities (or displacements) should be continuous or exhibit such discontinuities which are admissible from the viewpoint of a continuous medium.

In many cases both these assumptions are justified; however, some exceptions prove that they are not quite obvious. In general, no mechanism of plastic collapse may exist, or, if it exists for a rigid-plastic body, it may be unreachable by an elastic-plastic structure without violating the required continuity conditions. The existence theorem for the elastic-perfectly plastic bodies fails (G. DEL PIERO [1]), and the papers by K. SZUWALSKI and M. ŻYCZKOWSKI [14, 19] demonstrate several examples of the non-existence of any mechanism of plastic collapse: earlier, in the elastic-plastic range, inadmissible discontinuities appear and a continuous solution ceases to exist. The corresponding loading parameter was called in [14] the "decohesive carrying capacity" of the structure. In the light of the concept of decohesive carrying capacity the problem of a half-plane discussed by S. S. GOLUSHKEVITCH [4] and V. O. GEOGDZHAYEV [3], is quite clear: the limit carrying capacity cannot be reached here, since it is preceded by inadmissible discontinuities, connected with infinitely large strains. Similar objections were raised by E. M. SHOEMAKER [10, 11] and R. H. WOOD [16].

Of course, the conclusions as regards the existence or non-existence of a continuous solution may depend on basic assumptions of the theory. In [14] the small-strain-theory of elastic-perfectly plastic bodies was used. Some deviations from perfect plasticity were also discussed (asymptotically perfect plasticity). The simplest finite-strain-theory (NADAI-DAVIS) for an annular disk joined with a rigid central shaft was applied in [18] with the geometrical changes taken into account; this approach leads to some minor differences, but a certain impassable limit of the elastic-plastic solution was determined as well.

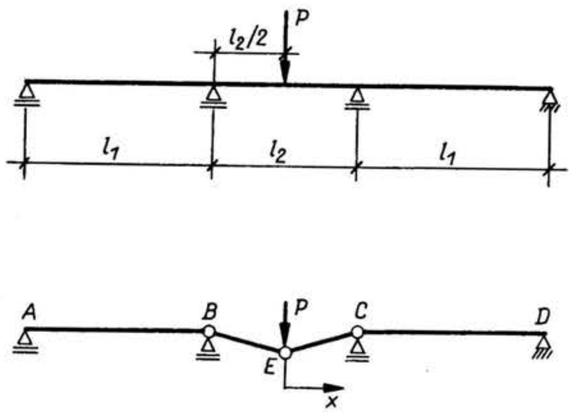


FIG. 1.

On the other hand, there exist many structures for which a certain mechanism of plastic collapse may be found, but this mechanism may not be reached by an elastic-perfectly plastic body. Several examples of statically indeterminate beams were shown in [15]. The present paper is also devoted to a statically indeterminate beam, namely that discussed by F. STÜSSI and C. F. KOLLBRUNNER [12], Fig. 1. These authors analysed the dependence of the limit carrying capacity on the ratio $l_1/l_2$ and pointed out the following paradox:

the limit load $\bar{\bar{P}}$ equals $\bar{\bar{P}} = 8\bar{\bar{M}}/l_2$ (where $\bar{\bar{M}}$ is the limit bending moment for the cross-section) and is independent of $l_1/l_2$. However, for $l_1 \to \infty$ the middle span is practically unaffected by the outer spans and we should obtain $\bar{\bar{P}} = 4\bar{\bar{M}}/l_2$ as in the case of a simply supported beam. So the case $l_1 \to \infty$ leads to a certain discontinuity of the result, which is not justified from the physical point of view.

The experimental tests, carried out by STÜSSI and KOLLBRUNNER [12], as well as by H. MAIER-LEIBNITZ [6], do not agree with the classical limit analysis: real carrying capacity depends on the ratio $l_1/l_2$. A. M. FREUDENTHAL [2] noticed that these results lie between the elastic and the limit carrying capacity, $\bar{P}$ and $\bar{\bar{P}}$ respectively, and proposed — quite arbitrarily — to assume the arithmetic mean of $\bar{P}$ and $\bar{\bar{P}}$ as the real carrying capacity of the beam.

Several attempts have been made to clarify the Stüssi-Kollbrunner paradox. P. S. SYMONDS and B. G. NEAL [13] calculated the deflections assuming ideal I cross-section and found an infinite increase of central deflection in the limiting case $l_1 \to \infty$. Their calculations were developed by K. A. RECKLING [8], who tried to introduce a "distributed plastic hinge" with the length $0.1l_2$. However, these calculations admitted a finite jump of the rotation angle $\alpha$ under the force, what is in contradiction with continuity requirements. It turns out that the paradox discussed disappears if we adopt, in a consistent manner, the assumptions of a continuous medium and perform the calculations until only the first inadmissible discontinuity is formed.

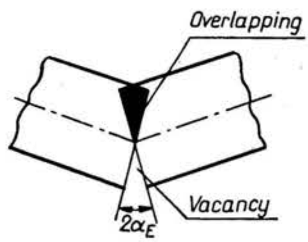


FIG. 2.

Such a discontinuity is introduced by the plastic hinge if it at a finite deflection of the beam is achieved (the corresponding discussion is given by A. R. RZHANITSYN [9] — e.g., this case occurs if the bending moment reaches its strong maximum with a simultaneous jump in the derivative, under concentrated force). Indeed, the plastic hinge may be understood to correspond to the limit carrying capacity of the cross-section: an infinitesimal rotation at a constant bending moment. For a rigid-plastic multi-span beam three such infinitesimal rotations describe a certain mechanism of plastic collapse (Fig. 1), but, in general, this mechanism will not be reached by an elastic-plastic beam, since the formation of the first hinge under the concentrated force will terminate the process and the decohesive carrying capacity is reached. Any finite rotation angle $\alpha$ in the hinge is namely impossible, since it cannot be described by a displacement field which is continuous or contains admissible discontinuities. It leads to vacancies on the tensile side and overlapping of the material on the compressive side of the beam, Fig. 2.

The present paper is devoted to the determination of the decohesive carrying capacity of the Stüssi-Kollbrunner beam. This quantity depends continuously on the length of the outer span $l_1$ and in this statement of the problem no paradox occurs.

## 2. Decohesive carrying capacity

Because of the symmetry of the beam we only consider its right-hand side, Fig. 1. The bending moment under the force will be equal to the limit carrying capacity of the cross-section, $\overline{M}$. Therefore, assuming there is no rotation angle at this point we can determine the decohesive carrying capacity, whereas admitting rotation and assuming that the subsequent moment $|M_c|$ is equal to $\overline{M}$ we can estimate the inadmissible discontinuity, which is determined by the magnitude of $\alpha$ under the force, corresponding to the classical mechanism of plastic collapse.

For a beam of the rectangular cross-section $b \times h$ we introduce a dimensionless force $p$, dimensionless coordinate $\xi$, and dimensionless deflection $v$:

$$p \overset{\text{def}}{=} \frac{l_2}{\overline{M}} P = \frac{4l_2}{bh^2\sigma_0} P, \quad \xi \overset{\text{def}}{=} \frac{x}{L} = \frac{x}{\frac{l_2}{2}+l_1}, \quad v \overset{\text{def}}{=} \frac{Eh}{3\sigma_0 l_2^2} w, \tag{2.1}$$

where $w$ denotes the physical deflection and $\sigma_0$ the yield-point stress. The geometry of the beam will be characterized by the ratio

$$k \overset{\text{def}}{=} \frac{l_2}{2L} = \frac{l_2}{l_2+2l_1}, \quad 0 < k < 1. \tag{2.2}$$

The dimensionless bending moment in the inner span, $m_{\mathrm{I}}$, and in the outer span, $m_{\mathrm{II}}$, equals

$$m_{\mathrm{I}} = \frac{M_{\mathrm{I}}}{\overline{M}} = 1 - \frac{p}{4k}\xi, \quad 0 \leqslant \xi \leqslant k, \tag{2.3}$$

$$m_{\mathrm{II}} = \frac{M_{\mathrm{II}}}{\overline{M}} = -\frac{p-4}{4(1-k)}(1-\xi), \quad k \leqslant \xi \leqslant 1. \tag{2.4}$$

Within the elastic range we integrate easily the two corresponding differential equations [without the assumption $m(0) = 1$] and equating the maximal stress (under the force $P$) to the yield-point stress $\sigma_0$ determine the elastic carrying capacity of the beam:

$$\bar{p} = \frac{16}{3}\frac{1+2k}{2+k}. \tag{2.5}$$

The number of intervals for the integration of differential equations of bending in the elastic-plastic range must be higher, since we have to separate elastic bending from elastic-plastic one. For the rectangular cross-section the dimensionless elastic carrying capacity equals $\overline{m} = 2/3$, so the first, elastic-plastic interval is given by $0 < \xi < \xi_1$, where $\xi_1 = 4k/3p$. The rest of the beam may remain elastic, or other elastic-plastic zones appear near the support $C$. Equating $m_{\text{I}}$ to $(-2/3)$ we find the second boundary coordinate $\xi_2 = 20k/3p$, if $p > 20/3$. Finally equating $m_{\text{II}}$ to $(-2/3)$ we find

$$\xi_3 = \frac{3p+8k-20}{3(p-4)}, \quad \text{also if} \quad p > 20/3. \tag{2.6}$$

Thus, for $p > 20/3$ we have five intervals for integration, and for $p \leqslant 20/3$ — three.

In the case of five intervals, the differential equations of bending are as follows

$$\begin{aligned}
v_1''(\xi) &= -\frac{4\sqrt{k}}{3\sqrt{3}\sqrt{p\xi}}, && 0 \leqslant \xi \leqslant \xi_1,\\
v_2''(\xi) &= \frac{p\xi}{4k} - 1, && \xi_1 \leqslant \xi \leqslant \xi_2,\\
v_3''(\xi) &= \frac{4\sqrt{k}}{3\sqrt{3}\sqrt{8k-p\xi}}, && \xi_2 \leqslant \xi \leqslant k,\\
v_4''(\xi) &= \frac{4\sqrt{1-k}}{3\sqrt{3}\sqrt{(p-4)\xi+8-4k-p}}, && k \leqslant \xi \leqslant \xi_3,\\
v_5''(\xi) &= -\frac{p-4}{4(1-k)}(\xi-1), && \xi_3 \leqslant \xi \leqslant 1.
\end{aligned} \tag{2.7}$$

Their general integrals are

$$\begin{aligned}
v_1(\xi) &= -\frac{16\sqrt{k\xi^3}}{9\sqrt{3p}} + A_1\xi + A_2, \quad v_2(\xi) = \frac{p\xi^3}{24k} - \frac{\xi^2}{2} + B_1\xi + B_2,\\
v_3(\xi) &= \frac{16\sqrt{k(8k-p\xi)^3}}{9\sqrt{3}\,p^2} + C_1\xi + C_2,\\
v_4(\xi) &= \frac{16\sqrt{1-k}\sqrt{[(p-4)\xi+8-4k-p]^3}}{9\sqrt{3}\,(p-4)^2} + D_1\xi + D_2,\\
v_5(\xi) &= -\frac{p-4}{8(1-k)}\left(\frac{\xi^3}{3} - \xi^2\right) + E_1\xi + E_2.
\end{aligned} \tag{2.8}$$

The boundary conditions $v(k) = 0$, and $v(1) = 0$, eight continuity conditions, and the additional continuity condition (symmetry) $v_1'(0) = 0$ make it possible to determine ten integration constants and the load parameter — decohesive carrying capacity $\hat{p}$. The final equation takes the form

$$3\sqrt{3(8-\hat{p})}\,[(1+2k)\hat{p}^2 + (4-28k)\hat{p} + 48k] - 80(1-k)\hat{p} = 0. \tag{2.9}$$

It determines $\hat{p}$ in the considered range of five intervals, i.e., if $\hat{p} > 20/3$. Substituting this boundary value to (2.9) we find the corresponding boundary value of $k$, namely $k = 5/11$, and hence (2.9) is valid if $5/11 \leqslant k \leqslant 1$. For $k = 1$ (clamped beam) we obtain $\hat{p} = 8$ and here $\hat{p} = \bar{\bar{p}}$.

If $0 < k \leqslant 5/11$, then the beam should be divided into three intervals only, (1), (2) and (5), since the elastic-plastic intervals (3) and (4) disappear. The differential equations and their general integrals remain without change. The boundary conditions, four continuity conditions and the additional continuity condition $v_1'(0) = 0$ determine six integration constants and the decohesive carrying capacity $\hat{p}$, here in the explicit form

$$\hat{p} = \frac{4}{2+k}\left[1+2k+\sqrt{(1+k)(1+5k)}\right]. \tag{2.10}$$

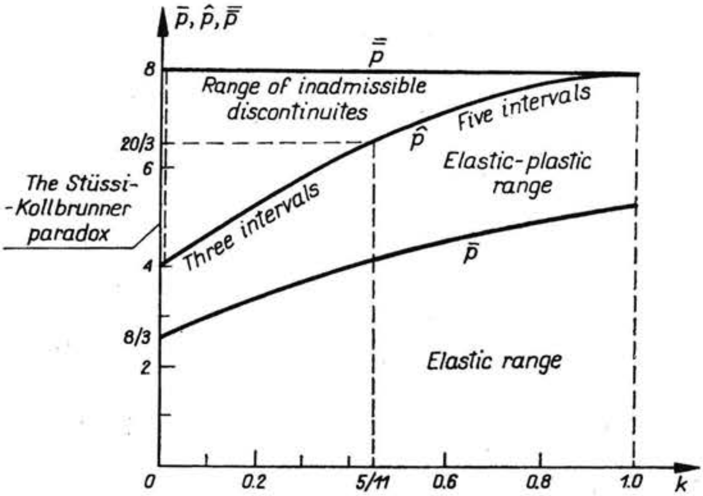


FIG. 3.

In the limiting case $k \to 0$ we obtain $\hat{p} = 4$ and this result corresponds to the classical limit load for a simply supported beam. Inside the interval $0 \leqslant k \leqslant 1$ the dependence $\hat{p} = \hat{p}(k)$ is continuous and no paradox appears. This dependence is shown in Fig. 3. together with $\bar{p} = \bar{p}(k)$ and the paradoxical $\bar{\bar{p}} = \bar{\bar{p}}(k)$.

## 3. Estimation of inadmissible discontinuities corresponding to the classical limit state

The equations (2.7) and their general integrals (2.8) may also be used to estimate the value of inadmissible discontinuities in the classical limit state of the beam. These discontinuities are characterized by the inadmissible finite angle of rotation $\alpha_E$ under the force $P$, for $\xi = 0$. If we reject the continuity condition $v'(0) = 0$ and admit the formation of the subsequent plastic hinges at $B$ and $C$ putting additionally $m(B) = m(C) = -1$, then $\alpha$ may be evaluated. Of course, under these assumptions five intervals of integration must be considered for any value of $k$. Finally, the angle of rotation equals

$$\alpha_E = \left.\frac{dw}{dx}\right|_{x=0} = \frac{40}{27}(1-k)^2 \cdot \frac{\sigma_0 l_2}{Eh} \tag{3.1}$$

and — because of the symmetry — the inadmissible discontinuity is characterized by the angle of mutual rotation of the two adjacent sections $2\alpha_E$. For example, if $\sigma_0/E = 0.001$, $l_2/h = 50$, $l_1 = l_2$, $k = 1/3$, then $2\alpha_E = 0.0658 = 3°46'$.

The dependence of $\vartheta_E = \alpha_E Eh/3\sigma_0 l_2$ on the geometrical parameter $k$ is shown in Fig. 4. For $k = 1$ (clamped beam) $\alpha_E = 0$ and then the limit carrying capacity coincides with the decohesive carrying capacity, as it has been mentioned above.

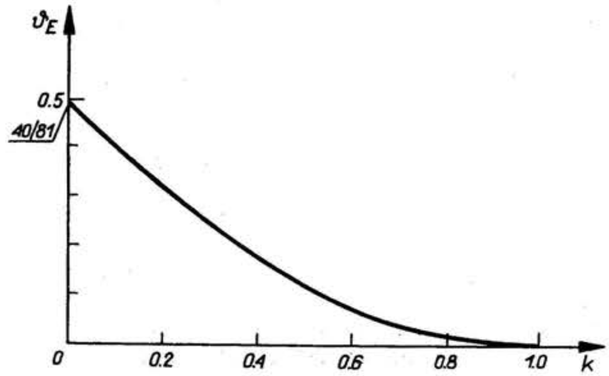


FIG. 4.

## 4. Final remarks

The Stüssi-Kollbrunner paradox has been explained here within the small-deflection theory of perfectly elastic-plastic beams. However, if we replace perfect plasticity by an asymptotically perfect one, the situation may change. The discussion is then similar to that given in [14] for bars in tension, since the type of non-homogeneity of the stress state is also similar. For certain stress-strain diagrams the decohesive carrying capacity terminates the process; for other diagrams (as, e.g., for that proposed by A. YLINEN [17]) the limit carrying capacity theoretically may be reached and further steps must be taken to remove the Stüssi-Kollbrunner paradox. Since the classical limit state is then reached at infinitely large deflections, the finite-deflection or even the finite-strain-theory should then be applied.

The problem may also be considered as a two-dimensional one (plane stress or plane strain). Exact elastic-plastic analysis is then difficult, but in the case of perfectly elastic-plastic body no major differences are expected: first, W. PRAGER and P. G. HODGE [7] showed that in plastic zones of beams the stress state reduces to uniaxial tension or compression (under the assumption of incompressibility) and, second, in certain two-dimensional problems the necessity of decohesion was found as well (E. H. LEE [5], notched bars). The result of a two-dimensional approach for an asymptotically perfectly plastic body will probably depend on the particular stress-strain diagram and in some cases the finite-strain-theory may be necessary to clarify the Stüssi-Kollbrunner paradox.

## References


1. G. DEL PIERO, *On a mathematical theory of elastic-plastic materials*, Arch. Mech., 27, 2, 253–271, 1975.
2. A. M. FREUDENTHAL, *Inelastic behavior of engineering materials and structures*, Wiley, New York 1950.

3. V. O. GEOGDZHAYEV, *On elastic-plastic equilibrium of a half-plane* [in Russian], Prikl. Mekh., **10**, 1, 127-130, 1974.
4. S. S. GOLUSHKEVITCH, *Statics of limit states of soils* [in Russian], Gos. Izd. Tekhn. Lit., Moskva 1957.
5. E. H. LEE, *Plastic flow in a V-notched bar pulled in tension*, J. Appl. Mech., **19**, 331–336, 1952.
6. H. MAIER-LEIBNITZ, *Versuche zur weiteren Klärung der Frage der tatsächlichen Tragfähigkeit durchlaufender Träger aus Baustahl*, Stahlbau, **9**, 153, 1936.
7. W. PRAGER, P. G. HODGE, Jr., *Theory of perfectly plastic solids*, Wiley, New York 1951.
8. K. A. RECKLING, *Plastizitätstheorie und ihre Anwendung auf Festigkeitsprobleme*, Springer, Berlin—Heidelberg—New York 1967.
9. A. R. RZHANITSYN, *Structural analysis with plastic properties of the materials taken into account* [in Russian], Stroyizdat, Moskva 1954.
10. E. M. SHOEMAKER, *Some paradoxes associated with elastic-plastic limit load analysis*, Arch. Mech. Stos., **20**, 4, 473–483, 1968.
11. E. M. SHOEMAKER, *On velocity discontinuities in elastic-plastic boundary value problems*, Arch. Mech., **26**, 2 327–331, 1974.
12. F. STÜSSI, C. F. KOLLBRUNNER, *Beitrag zum Traglastverfahren*, Bautechnik, **13**, 264, 1935.
13. P. S. SYMONDS, B. G. NEAL, *The interpretation of failure loads in the plastic theory of continuous beams and frames*, J. Aero. Sci., **19**, 1, 15-22, 1952.
14. K. SZUWALSKI, M. ŻYCZKOWSKI, *On the phenomenon of decohesion in perfect plasticity*, Int. J. Solids and Struct., **9**, 1, 85–98, 1973.
15. TRAN-LE BINH, K. SZUWALSKI, *Nośność rozdzielcza idealnie sprężysto-plastycznych belek statycznie niewyznaczalnych*, Czas. Techn. [in print].
16. R. H. WOOD, *A partial failure of limit analysis for slabs, and the consequences for future research*, Mag. Concr. Res., **21**, 67, 79–90, 1969.
17. A. YLINEN, *Theory of continuous beam whose material does not follow Hooke's law*, Arch. Mech. Stos., **14**, 2, 219–231, 1962.
18. M. ŻYCZKOWSKI, K. SZUWALSKI, *On the termination of the process of finite plastic deformations*, Coll. "Finite deformations in plasticity" [Euromech 54], Warsaw 1974 [lecture].
19. M. ŻYCZKOWSKI, K. SZUWALSKI, *Decohesive carrying capacity in thermal stress problems*, Trans. 3rd Int. Conf. "Structural Mechanics in Reactor Technology", **5**, part L, paper L 2/4, London 1975.

INSTITUTE OF MECHANICS AND MACHINE DESIGN,
TECHNICAL UNIVERSITY OF CRACOW.

*Received October* 30, 1975.