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On the circular crack problem

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IN THE FRAMES of the mathematical theory of defects, the problem of a crack is reduced to the solution of a boundary integral equation for the displacement discontinuity. The equation for the circular crack is worked out.

1. Introduction

THREE-DIMENSIONAL crack problems are difficult from the mathematical point of view. The method of integral transforms is effective only for problems with the simplest geometry. In the frames of the mathematical theory of defects, the crack is represented as the elastic potential of a double layer. This approach allows to reduce the crack problem to the solution of a boundary integral equation for the displacement discontinuity function over the crack surface. No important limitations, either on the shape of the crack or on the character of external stresses, are necessary. However, even for finite, plane cracks, other than circular or elliptic, the integral equation thus obtained can be solved only numerically. Moreover, the numerical solution of this equation is troublesome as, in general, we deal with the strongly singular integro-differential equation for the function of two variables.

We present below one of the methods of exact solution of the integral equation for the circular crack. Another method was presented in [7].

2. Equation of equilibrium of a crack

The static Lamé equation

$$c_{iklm}\nabla_k\nabla_m u_l(\mathbf{x}) = -X_i(\mathbf{x}); \qquad \nabla_k = \frac{\partial}{\partial x_k} \tag{2.1}$$

for the isotropic body, for which

$$c_{iklm} = \mu\left[\frac{2\nu}{1-2\nu}\,\delta_{ik}\delta_{lm}+\delta_{il}\delta_{km}+\delta_{im}\delta_{kl}\right] \tag{2.2}$$

has the basic solution $\mathbf{G}$:

$$c_{iklm}\nabla_k\nabla_m G_{jl}(\mathbf{x}-\mathbf{x}') = -\delta_{ij}\delta_3(\mathbf{x}-\mathbf{x}'), \tag{2.3}$$

$$G_{ik}(\mathbf{x}-\mathbf{x}') = \frac{1}{4\pi\mu}\left\{\frac{\delta_{ik}}{r}-\frac{1}{4(1-\nu)}\,\nabla_i\nabla_k r\right\}, \qquad \mathbf{r} = \mathbf{x}-\mathbf{x}'. \tag{2.4}$$

The expression called the elastic potential of a double layer is related to the Lamé equation (see [1]) and has the following properties. Let S be the smooth surface having the normal vector $\mathbf{n}$, and $\mathbf{u}$ be the displacement field given by the expression

$$(2.5)\qquad u_i(\mathbf{x}) = -\int_S ds' U_n n_b c_{nbrs} \nabla_s G_{ir}(\mathbf{x}-\mathbf{x}'), \quad \mathbf{x}' \in S, \quad \nabla_s \equiv \frac{\partial}{\partial x_s}.$$

To the field $\mathbf{u}$ corresponds the stress field $\boldsymbol{\sigma}$:

$$(2.6)\qquad \sigma_{ik}(\mathbf{x}) = -c_{iklm} \int_S ds' U_n n_b c_{nbrs} \nabla_s \nabla_l G_{mr}(\mathbf{x}-\mathbf{x}').$$

The following equalities are established [1, 3]:

$$(2.7)\qquad |[u_i]| = U_i,$$

$$(2.8)\qquad n_k |[\sigma_{ik}]| = 0,$$

where $|[\ldots]|$ denotes the difference of the one-sided limits of the appropriate functions at the surface S, i.e. the jump discontinuity of a function across S. Obviously outside S, $\mathbf{u}$ satisfies the homogeneous Lamé equation.

In the mathematical theory of defects, basing on the linear theory of elasticity, the potential of a double layer describes a surface defect to which corresponds the displacement discontinuity $\mathbf{U}$ across the defect's surface S, the displacement field $\mathbf{u}$ given by Eq. (2.5), the self-stress $\boldsymbol{\sigma}$ given by Eq. (2.6), and which is self-equilibrated [3, 4].

The macroscopic crack in the infinite, elastic, continuous medium is a defect, described by the function $\mathbf{U}$ which is given over the crack surface and has a meaning of the relative displacement of the crack faces at a given point, caused by the external stress $\mathring{\boldsymbol{\sigma}}$. The displacement discontinuity $\mathbf{U}$ is to be found from the condition of equilibrium of the crack surface:

$$(2.9)\qquad n_k \sigma_{ik}(\mathbf{r}) + n_k \mathring{\sigma}_{ik}(\mathbf{r}) = 0 \quad \text{for} \quad \mathbf{r} \in S,$$

where S is the crack surface having the normal vector $\mathbf{n}$, $\boldsymbol{\sigma}$ is the self-stress of the crack and $\mathring{\boldsymbol{\sigma}}$ is the external stress.

The equation of equilibrium of a plane crack, opened by the tensile stress p normal to its surface, has the form

$$(2.10)\qquad -\frac{\mu}{4\pi(1-\nu)} \nabla_k \nabla_k \int_S ds' U(\mathbf{r}') \frac{1}{|\mathbf{r}-\mathbf{r}'|} = p,$$

$$\mathbf{r}, \mathbf{r}' \in S; \quad k = 1, 2,$$

$$\mathbf{r} = [x_1, x_2], \quad \mathbf{r}' = [x_1', x_2'].$$

The crack surface is the region of the plane (x_1, x_2), U is the displacement discontinuity in the direction of the x_3 axis; (see [5]). The function U has to be zero at the boundary of the crack surface:

$$(2.11)\qquad U(\mathbf{r}') = 0 \quad \text{for} \quad \mathbf{r}' \in \partial S.$$

If we perform in Eq. (2.10) the differentiation of the integral kernel with respect to the variables x_1, x_2, and then the integration by parts with respect to the variables x_1', x_2', making use of Eq. (2.11), we obtain the equation

$$\frac{\mu}{4\pi(1-\nu)}\int\limits_{S} ds'\, U_{,k}(\mathbf{r}')\frac{r_k-r'_k}{|\mathbf{r}-\mathbf{r}'|^3} = p, \qquad k = 1,2. \tag{2.12}$$

This is the strongly singular integro-differential equation for the unknown function U. After the function U satisfying Eq. (2.12) has been found, the displacement and stress field of the crack can be calculated from the fromulae (2.5) and (2.6).

3. The circular crack

We consider the axisymmetrical problem of a circular plane crack opened by the normal stress p, depending in the cylindrical coordinates r, φ on the variable r only. The opening function U depends on r only. The crack has the unit radius.

The equation of equilibrium of the crack has the form

$$-\frac{\mu}{4\pi(1-\nu)}\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right]\int\limits_0^1 dr'r'U(r')\int\limits_0^{2\pi} d\varphi\,\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\varphi}}, \qquad r, r' \leqslant 1, \tag{3.1}$$

whereas the condition (2.11) for the function U to be zero at the boundary of the crack's surface has the form

$$U(r=1) = 0. \tag{3.2}$$

To be able to make use of some important tricks, we do not perform now the differentiation with respect to r in Eq. (3.1).

From the formulae $A1$, $A2$, the following representation for the internal integral with respect to the variable φ results:

$$\int\limits_0^{2\pi} d\varphi\,\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\varphi}} = 2\pi\int\limits_0^{\infty} dz J_0(zr)J_0(zr'). \tag{3.3}$$

The following substitution for the function U plays the essential role in what follows:

$$U(r) = 2(1-\nu)\int\limits_r^1 dt\,\frac{q(t)}{\sqrt{t^2-r^2}}. \tag{3.4}$$

The function $U(r)$ sought for in this form automatically satisfies the condition (3.2).

Equation (3.1) takes now the following form:

$$-\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right]\int\limits_0^1 dr'r'\int\limits_{r'}^1 dt\,\frac{q(t)}{\sqrt{t^2-r'^2}}\int\limits_0^{\infty} dz J_0(zr)J_0(zr') = \frac{p}{\mu}. \tag{3.5}$$

Interchanging the order of integration with respect to the variables r' and t and afterwards with respect to the variables r' and t, we obtain the equation

$$-\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right]\int\limits_0^1 dtq(t)\int\limits_0^{\infty} dz J_0(zr)\int\limits_0^{t} dr'\,\frac{r'}{\sqrt{t^2-r'^2}}J_0(zr') = \frac{p}{\mu}. \tag{3.6}$$

We now perform integration with respect to r', making use of (A.4):

$$\int_0^t dr' \frac{r'}{\sqrt{t^2-r'^2}} J_0(zr') = \frac{\sin zt}{z}, \tag{3.7}$$

and afterwards integration with respect to z, making use of $A5$:

$$\int_0^\infty dz J_0(zr) \frac{\sin(zt)}{z} = \int_0^\infty d\zeta J_0(\zeta) \frac{\sin\left(\zeta \frac{t}{r}\right)}{\zeta} = \begin{cases} \frac{\pi}{2} & \text{for} \quad t > r, \\ \arcsin \frac{t}{r} & \text{for} \quad t < r. \end{cases} \tag{3.8}$$

Our equation then takes the form

$$-\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\right]\left[\int_0^r dt q(t) \arcsin \frac{t}{r} + \frac{\pi}{2}\int_r^1 dt q(t)\right] = \frac{p}{\mu}. \tag{3.9}$$

Performing one differentiation with respect to r (see A.6), we obtain the equation

$$\frac{\partial}{\partial r}\int_0^r dt \frac{tq(t)}{\sqrt{r^2-t^2}} = \frac{p}{\mu} r. \tag{3.10}$$

This is the integral equation of the Abel type, which has the solution (see [2])

$$q(t) = \frac{2}{\pi\mu}\int_0^t du \frac{up(u)}{\sqrt{t^2-u^2}}. \tag{3.11}$$

The solution for $U(r)$ takes thus the form

$$U(r) = \frac{4(1-\nu)}{\pi\mu}\int_r^1 dt \frac{1}{\sqrt{t^2-r^2}}\int_0^t du \frac{up(u)}{\sqrt{t^2-u^2}}. \tag{3.12}$$

For the constant stress p,

$$\begin{aligned} q(t) &= \frac{2pt}{\pi\mu}, \\ U(r) &= \frac{4(1-\nu)p}{\pi\mu}\sqrt{1-r^2}. \end{aligned} \tag{3.13}$$

Here we are not going to derive and discuss in details the expressions for the displacement and stress fields for the circular crack as they are well known and can be found in [2]. We only wanted to present the application of elastic potentials and the corresponding integral equation method to the three-dimensional crack problem.

4. Conclusions

The problem of a macroscopic crack in the elastic medium under the external stress field can be formulated as a problem of a surface defect, to which corresponds the appropriate displacement discontinuity function U. The equation of equilibrium of a crack is the strongly singular integro-differential equation for the function U.

For the circular crack under the axially-symmetric external stress field, we can find the exact solution of the integral equation for the displacement discontinuity function U.

5. Appendix

The formulae for the integrals series and derivatives we have used in the present paper (references to [6]) are

(A.1)
$$\int_0^\infty dx J_0(bx) = \frac{1}{b}, \quad b > 0, \quad \text{G.R. } 6.511.1,$$

(A.2)
$$J_0(z\sqrt{r^2+r'^2-2rr'\cos\varphi}) = J_0(zr)J_0(zr') + 2\sum_{k=1}^\infty J_k(zr)J_k(zr')\cos k\varphi, \quad \text{G.R. } 8.531.1,$$

where J_0, J_k are the Bessel functions.

(A.3)
$$\int_0^1 dx\, x J_0(xy)\frac{1}{\sqrt{1-x^2}} = \frac{\sin y}{y}, \quad \text{G.R. } 6.554.2.$$

It follows that

(A.4)
$$\int_0^a dx\, x J_0\left(\frac{x}{a}y\right)\frac{1}{\sqrt{a^2-x^2}} = a\frac{\sin y}{y}.$$

(A.5)
$$\int_0^\infty dx J_0(x)\frac{\sin\beta x}{x} = \frac{\pi}{2} \quad [\beta > 1], \quad \text{G.R. } 6.693.7,$$
$$= \arcsin\beta \quad [\beta^2 < 1],$$
$$= -\frac{\pi}{2} \quad [\beta < -1].$$

(A.6)
$$\frac{d}{dx}\left[\arcsin\frac{a}{x}\right] = -\frac{a}{x}\frac{1}{\sqrt{x^2-a^2}}.$$

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