Application of the asymptotic expansion method for singularly perturbed equations of the resonance type in the kinetic theory

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THE SINGULARLY perturbed systems of ordinary differential equations are treated by the asymptotic expansion method in two forms: standard and newly developed by the authors. Both procedures are applied to the Carleman model of the Boltzmann equation and are shown to be related to the Hilbert and Chapman–Enskog expansions.

Osobliwie perturbowane układy zwyczajnych równań różniczkowych potraktowano za pomocą metody rozwinięć asymptotycznych w dwóch postaciach: metody standardowej i metody od nowa opracowanej przez autorów. Obydwie procedury zastosowano do modelu Carlemana równania Boltzmanna oraz wykazano, że odpowiadają one rozwinięciom Hilberta i Chapmana–Enskoga.

Особенно пертурбированные системы обыкновенных дифференциальных уравнений трактуются при помощи асимптотических разложений в двух видах: стандартного метода и метода вновь разработанного авторами. Обе процедуры применены к модели Карлемана уравнения Больцмана, а также показано, что отвечают они разложениям Гильберта и Чепмена–Энскога.

1. Introduction

IN THE PREVIOUS paper [1] we developed a new algorithm of the asymptotic expansion method for singularly perturbed systems of ordinary differential equations:

$$\varepsilon \frac{dx}{dt} = f(x, y);$$

(1.1)

$$\frac{dy}{dt} = g(x, y).$$

As it was indicated in that paper, the analysis can be extended to situations when f and g depend in a smooth way on ε.

The motivation for the new algorithm came from considering singularly perturbed equations of the resonance type

(1.2)
$$\frac{dz}{dt} = h(z) + \frac{1}{\varepsilon} j(z);$$

in which the function $j(z)$ is singular. Such equations are closely related to discrete models of the Boltzmann equation.

In this paper we formulate the properties of the functions appearing in Eq. (1.2) which

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enable us to translate it to the form (1.1) and then apply the standard [2, 3] or new [1] algorithms of the asymptotic expansion.

The results of the analysis will be used for the Carleman model of the Boltzmann equation (see [4]) to shed some light on the relationship between the asymptotic expansion method for ordinary differential equations and the Hilbert and Chapman–Enskog asymptotic procedures (see, e.g. [5] or [6]).

2. Singularly perturbed equations of the resonance type

In this section we shall consider singularly perturbed differential equations of the resonance type (1.2) and formulate the conditions under which such equations may be reduced to systems of the form (1.1) and then treated by the asymptotic expansion method.

To that purpose introduce the following notation:

$$t \in T = [0, t_0], \quad t_0 > 0;$$

$$\varepsilon \in E_0 = (0, \varepsilon_0), \quad \varepsilon_0 > 0, \quad \bar{E}_0 = [0, \varepsilon_0];$$

(2.1)
$$\Omega \subset R^m, \quad m > 1, \quad \Omega \text{ open and connected;}$$

$$z : T \to \Omega;$$

$$h : \Omega \to \Omega;$$

$$j : \Omega \to \Omega;$$

$$\gamma : \bar{E}_0 \to \Omega.$$

By $\| \cdot \|$ we denote any of the (equivalent) norms in R^m or in any of its subspaces and by $(\cdot, \cdot)$ the scalar product.

Consider the initial value problem for Eq. (1.2)

(2.2)
$$\frac{dz}{dt} = h(z) + \frac{1}{\varepsilon} j(z), \quad z(0) = \gamma(\varepsilon), \quad t \in T.$$

The above equation will be said to be of the resonance type if the next assumption is valid:

A.1. There exists a set of vectors $\psi_i \subset R^m$; $i = 1, ..., p$; $1 \leqslant p < m$ such that

$$(\psi_i, \psi_k) = \delta_{ik}, \quad 1 \leqslant i, k \leqslant p,$$

and for all $z \in \Omega$

$$(\psi_i, j(z)) = 0, \quad i = 1, ..., p.$$

With A.1. we can write Eq. (2.2) as a system of equations. To do this we introduce the projectors

(2.3)
$$Pu = \sum_{i=1}^{P} (\psi_i, u) \psi_i,$$

$$Qu = u - Pu, \quad u \in R^m.$$

Operating on both sides of Eq. (2.2) with P and Q we get

$$\frac{d(Qz)}{dt} = Qh(Pz+Qz) + \frac{1}{\varepsilon}j(Pz+Qz),$$

(2.4)
$$\frac{d(Pz)}{dt} = Ph(Pz+Qz),$$

$$(Qz)(0) = Q\gamma, \quad (Pz)(0) = P\gamma.$$

Denoting

$$Qz = x, \quad Pz = y,$$

$$Q\gamma = \mu, \quad P\gamma = \eta,$$

(2.5)
$$j(Pz+Qz) + \varepsilon Qh(Pz+Qz) = f(x, y, \varepsilon),$$

$$Ph(Pz+Qz) = g(x, y),$$

we get from Eq. (2.4) the initial value problem

$$\varepsilon\frac{dx}{dt} = f(x, y, \varepsilon), \quad x(0) = \mu(\varepsilon),$$

(2.6)
$$\frac{dy}{dt} = g(x, y), \quad y(0) = \eta(\varepsilon).$$

This is the system identical with that considered in [1] except for the fact that f depends explicitly on ε. However, it follows from Eq. (2.5) that this dependence is smooth so that the analysis in [1] can be straitghforwardly adapted to include Eq. (2.6).

We shall now make assumptions concerning the functions appearing in Eq. (2.2) such that the requirements concerning Eq. (2.6) and set up in [1] are satisfied.

A.2. The functions h and j belong to $C^{2n+2}(\Omega)$ and $\gamma \in C^{n+1}(\overline{E}_0)$ for some E_0. The latter means that we admit the expansion

(2.7)
$$\gamma(\varepsilon) = \sum_{n=1}^{k} \varepsilon^k\gamma_k + 0(\varepsilon^{n+1}).$$

A.3. There exists an open, bounded, and connected set $\theta \subset P\Omega$ and a continuous function $\varrho:\overline{\theta} \to \Omega$ such that for all $y\in\overline{\theta}$, $j(\varrho(y)) = 0$. The root $\varrho(y)$ is isolated in the sense that there exists $\delta > 0$ such that for all $y \in \overline{\theta}$ and $z \in Q\Omega$ such that $0 < ||z|| < \delta$ we have

$$j(z+\varrho(y)) \neq 0.$$

A.4. For every $y \in \overline{\theta}$ the kernel of the matrix operator $D_z j(\varrho(y))$ coincides with PR^m and the nonzero eigenvalues $\lambda_i(y)$ of $D_z j(\varrho(y))$ satisfy the inequality

(2.8)
$$\text{Re } \lambda_i(y) < -\alpha < 0.$$

A.5. The initial value problem

(2.9)
$$\frac{dy_0}{dt} = Ph(\varrho(y_0)), \quad y_0(0) = \eta_0 \in \theta$$

has a unique, bounded solution $y_0: T \to \theta$. The initial value η_0 is defined from Eq. (2.7) as $P\gamma_0$.

Consider the equation

(2.10)
$$\frac{du}{d\tau} = j(\varrho(y)+u), \quad \tau \geqslant 0, \quad y \in \bar{\theta},$$

for $u \in Q\mathbf{R}^m$. It follows from (A.4) that $u = 0$ is an asymptotically stable fixed point of Eq. (2.10) (see Proposition 3 of [1]).

A.6. For each $y \in \bar{\theta}$ the point $Q(\gamma_0 - \varrho(y))$ belongs to the region of attraction of the fixed point $u = 0$ of Eq. (2.10). Additionally,

$$\varrho(y) + Mu \in \Omega,$$

where u is the solution of Eq. (2.10) and M is any diagonal matrix such that $0 \leqslant M_{ii} \leqslant 1$, $i = 1, \ldots, m-p$.

It can now be checked that if all the assumptions (A.2–A.6) are satisfied, then the hypotheses (H1–5) of [1] are valid for the system (2.6). Observing that the assumptions of [2] are included in those of [1], we conclude that the system (2.6), and hence Eq. (2.2), can be treated by both standard and new asymptotic expansion procedures.

In the next section we take the Carleman model of the Boltzmann equation which is of the form (2.2) and apply both procedures to show their relationship to the Hilbert and Chapman–Enskog approaches.

3. Carleman model

Consider the Carleman equations with periodic boundary conditions [4]:

(3.1)
$$\frac{\partial X_1}{\partial t} + \frac{\partial X_1}{\partial r} = \frac{1}{4\varepsilon}(X_2^2 - X_1^2),$$

$$\frac{\partial X_2}{\partial t} - \frac{\partial X_2}{\partial r} = \frac{1}{4\varepsilon}(X_1^2 - X_2^2),$$

$$X_i(0, t) = X_i(a, t),$$

$$X_i(r, 0) = \overset{(0)}{X_i}(r, \varepsilon), \quad i = 1, 2,$$

$$t \in T = [0, t_0], \quad 0 \leqslant r \leqslant a, \quad a > 0.$$

To reduce Eq. (3.1) to the system of ordinary differential equations we discretize the space variable r such that

$$0 = r_0 < r_1 < \ldots < r_s = a;$$

and replace the functions X_1 and X_2 by the vector functions w_1 and w_2, respectively. Thus we take

(3.2)
$$w_i(t) = \{w_i^1(t), \ldots, w_i^s(t)\};$$

$$w_i(t) = X_i(r_k, t), \quad i = 1, 2, \quad k = 1, \ldots, s.$$

The value of X_i at $r = 0$ is eliminated on account of the periodic boundary conditions in Eq. (3.1).

The differentiation with respect to r is replaced by the finite differencing. Thus, instead of $\left. \dfrac{\partial X_i}{\partial r}\right|_{r=r_k}$, we take

$$\frac{w_i^{k+1}-w_i^{k-1}}{r_{k+1}-r_{k-1}}.$$

Taking into account the boundary conditions we may replace the derivative with respect to r by the matrix operator

(3.3)
$$G = \left\{\begin{matrix} 0 & b_1 & 0 & ... & 0 & 0 & -b_1 \\ -b_2 & 0 & b_2 & ... & 0 & 0 & 0 \\ \hline & & & & & & \\ 0 & 0 & 0 & ... & -b_{s-1} & 0 & b_{s-i} \\ b_s & 0 & 0 & ... & 0 & -b_s & 0 \end{matrix}\right\},$$

where

$$b_k = \begin{cases} (r_{k+1}-r_{k-1})^{-1}, & k = 1, ..., s-1; \\ (r_s - r_{s-1} + r_1 - r_0)^{-1}, & k = s. \end{cases}$$

In this paper we shall treat the vectors as functions of a discrete variable and perform with the vectors operations similarly as with functions. Thus, for example ab stands for the vector with the components $(ab)^i = a^i b^i$, $a^2 = aa$, and $(\exp a)^i = \exp a^i$.

With Eqs. (3.2) and (3.3) we may replace Eq. (3.1) by the following system of ordinary differential equations:

(3.4)
$$\frac{dw_1}{dt}+Gw_1 = \frac{1}{4\varepsilon}(w_2^2-w_1^2),$$

$$\frac{dw_2}{dt}-Gw_2 = \frac{1}{4\varepsilon}(w_1^2-w_2^2),$$

$$w_i(0) = \overset{(0)}{w_i}(\varepsilon), \quad i = 1, 2, \quad t \in T.$$

Finally we introduce the vector function $z(t)$ whose components are consecutively the components of the vectors $w_1(t)$ and $w_2(t)$ and define the block matrix

$$\hat{G} = \left\{\begin{matrix} G & 0 \\ 0 & -G \end{matrix}\right\}.$$

With this notation we can write Eq. (3.4) in the form

(3.5)
$$\frac{dz}{dt}+\hat{G}z = \frac{1}{\varepsilon}\, j(z),$$

$$z(0) = \gamma(\varepsilon),$$

where

(3.6)
$$j(z) = \{j^1, ..., j^{2s}\},$$

$$j^k = \begin{cases} \dfrac{1}{4}[(z^{s+k})^2-(z^k)^2], & k = 1, ..., s, \\ \dfrac{1}{4}[(z^{k-s})^2-(z^k)^2], & k = s+1, ..., 2s \end{cases}$$

and

$$\gamma(\varepsilon) = \{\overset{(0)}{w}{}^1_1, \ldots, \overset{(0)}{w}{}^s_1, \overset{(0)}{w}{}^1_2, \ldots, \overset{(0)}{w}{}^s_2\}.$$

Now, if $-\hat{G}z$ is identified with $h(z)$, Eq. (3.5) takes exactly the form (2.2).

The finite discrete version of the Carleman model (3.5) is an approximation to the original system (3.1) in which the basic features of the model are preserved. This is due to the choice of the periodic boundary conditions. For other boundary conditions we would have to cope with the boundary layer for which the proposed discretization would be inadequate.

Take the following system of orthonormal vectors in R^{2s}:

$$\psi_i = \{\psi_i^1, \ldots, \psi_i^{2s}\}, \quad i = 1, \ldots, s,$$

(3.7)

$$\psi_i^k = \begin{cases} \dfrac{1}{\sqrt{2}}, & k = i, s+i, \\ 0, & \text{otherwise.} \end{cases}$$

It is seen that

$$(\psi_i, j(z)) = 0, \quad z \in R^{2s}, \quad i = 1, \ldots, s,$$

and (A.1) is fulfilled.

Performing the projections defined by Eq. (2.3) and introducing the notation given by Eq. (2.5), we get

$$\frac{dx}{dt} + \hat{G}y + \frac{1}{\varepsilon}xy = 0,$$

(3.8)

$$\frac{dy}{dt} + \hat{G}x = 0,$$

$$x(0) = \mu(\varepsilon), \quad y(0) = \eta(\varepsilon).$$

We shall now check the remaining assumptions listed in the previous section.

First we observe that both $h(z)$ and $j(z)$ are infinitely many times differentiable in the whole R^{2s}. Hence (A.2) is fulfilled whenever we take $\gamma(\varepsilon)$ sufficiently smooth.

As the set θ we take

(3.9) $\theta = \{y \in P R^{2s}: y^k > \alpha > 0; \ k = 1, \ldots, 2s\}$

and $\varrho(y) \equiv y$ for all $y \in \bar{\theta}$. With this (A.3) is fulfilled. From the relations (3.6) it is easy to verify that

$$(D_z j)(z) = \frac{1}{2}\begin{pmatrix} -z^1 & 0 & 0 & \ldots & 0 & 0 & z^{s+1} & 0 & \ldots & 0 & 0 \\ 0 & -z^2 & 0 & \ldots & 0 & 0 & 0 & z^{s+2} & \ldots & 0 & 0 \\ \hline 0 & 0 & 0 & \ldots & 0 & -z^s & 0 & 0 & \ldots & 0 & z^{2s} \\ z^1 & 0 & 0 & \ldots & 0 & 0 & -z^{s+1} & 0 & \ldots & 0 & 0 \\ \hline 0 & 0 & 0 & \ldots & z^{s-1} & 0 & 0 & 0 & \ldots & -z^{2s-1} & 0 \\ 0 & 0 & 0 & \ldots & 0 & z^s & 0 & 0 & \ldots & 0 & -z^{2s} \end{pmatrix}.$$

By direct calculations we check that

(3.10)
$$(D_z j)(y)\psi_i = 0,$$
$$(D_z j)(y)\hat{\psi}_i = -y^i\hat{\psi}_i, \quad i = 1, ..., s, \quad y \in \bar{\theta},$$

where ψ_i are defined in the relations (3.7) and

$$\hat{\psi}_i = \{\hat{\psi}_i^1, ..., \hat{\psi}_i^{2s}\}, \quad i = 1, ..., s,$$

(3.11)

$$\hat{\psi}_i^k = \begin{cases} \dfrac{1}{\sqrt{2}}, & k = i, \\ -\dfrac{1}{\sqrt{2}}, & k = s+i, \\ 0, & \text{otherwise.} \end{cases}$$

From Eqs. (3.10) and (3.9) it follows that (A.4) is satisfied with α defined in the relations (3.9).

The initial value problem (2.9) reduces to

(3.12)
$$\frac{dy_0}{dt} = 0, \quad y_0(0) = \eta_0,$$

since for all $y \in \theta$

$$Ph\big(\varrho(y)\big) = -P\hat{G}\varrho(y) = -P\hat{G}y \equiv 0$$

and (A.5) is trivially fulfilled.

Equation (2.10) has in the present case the form

(3.13)
$$\frac{du}{d\tau} = -yu, \quad \tau \geqslant 0, \quad y \in \bar{\theta}, \quad u \in QR^{2s},$$

which shows that the region of attraction of the fixed point $u = 0$ is all QR^{2s} and (A.6) is valid for any γ_0.

We see that all the assumptions (A.1–A.6) are satisfied and we may apply both standard and new algorithms of the asymptotic expansion.

From the definition of the projectors P and Q we see that

$$x^k = -x^{s+k},$$
$$y^k = y^{s+k}, \quad k = 1, ..., s.$$

Obviously it is true also for $\mu(\varepsilon)$ and $\eta(\varepsilon)$. At the same time it is seen that Eqs. (3.8) for the first s components of x and y are identical to those for the second s components. This shows that x and y are fully described by vectors from R^s which again will be denoted by x and y. Now Eqs. (3.8) may be replaced by the system of equations in $R^s \times R^s$:

$$\frac{dx}{dt} + Gy + \frac{1}{\varepsilon}xy = 0,$$

(3.14)
$$\frac{dy}{dt} + Gx = 0,$$

$$x(0) = \mu(\varepsilon), \quad y(0) = \eta(\varepsilon),$$

which formally differs from Eqs. (3.8) only by G substituted for $\hat{G}$.

In the next sections we shall analyze directly the system (3.14).

4. Asymptotic expansion for the Carleman model

In accordance with the general procedure we introduce the stretched (local) variable $\tau = t/\varepsilon$ and represent the functions x and y in the system (3.14) as sums of functions of t and τ, separately. Thus, writing

$$x(t) = \bar{x}(t) + \tilde{x}(\tau),$$

$$y(t) = \bar{y}(t) + \tilde{y}(\tau),$$

we get from the system (3.14) by separating the dependence on t and τ

(4.1)
$$\varepsilon \frac{d\bar{x}}{dt} + \varepsilon G \bar{y} + \bar{x}\bar{y} = 0,$$

(4.1')
$$\frac{d\bar{y}}{dt} + G\bar{x} = 0,$$

and

$$\frac{d\tilde{x}}{d\tau} + \varepsilon G \tilde{y} + \tilde{j} = 0,$$

(4.2)
$$\frac{d\tilde{y}}{d\tau} + \varepsilon G\tilde{x} = 0,$$

where

$$\tilde{j} = \left(\bar{x}(\varepsilon\tau) + \tilde{x}(\tau)\right)\left(\bar{y}(\varepsilon\tau) + \tilde{y}(\tau)\right) - \bar{x}(\varepsilon\tau)\bar{y}(\varepsilon\tau).$$

In the standard approach we replace all the functions in Eqs. (4.1) and (4.2) by power expansions in ε. Thus we write for a particular n

(4.3)
$$\bar{x} = \sum_{k=0}^{n} \varepsilon^k \bar{x}_k, \quad \bar{y} = \sum_{k=0}^{n} \varepsilon^k \bar{y}_k,$$

(4.3')
$$\tilde{x} = \sum_{k=0}^{n} \varepsilon^k \tilde{x}_k, \quad \tilde{y} = \sum_{k=0}^{n} \varepsilon^k \tilde{y}_k.$$

From this, by expansing the functions of $\varepsilon\tau$ into Taylor series, we get

(4.4) $\bar{x}\bar{y} = \bar{x}_0\bar{y}_0 + \varepsilon(\bar{x}_0\bar{y}_1 + \bar{x}_1\bar{y}_0) + \dots,$

$$\tilde{j} = \tilde{x}_0(\tau)\tilde{y}_0(\tau) + \bar{x}_0(0)\tilde{y}_0(\tau) + \tilde{x}_0(\tau)\bar{y}_0(0)$$

$$+ \varepsilon\left[\left(\bar{y}_0(0) + \tilde{y}_0(\tau)\right)\tilde{x}_1(\tau) + \left(\bar{x}_0(0) + \tilde{x}_0(\tau)\right)\tilde{y}_1(\tau)\right.$$

$$\left. + \tilde{x}_0(\tau)\left(\tau\frac{d\bar{y}_0}{dt}(0) + \bar{y}_1(0)\right) + \left(\tau\frac{d\bar{x}_0}{dt}(0) + \bar{x}_1(0)\right)\tilde{y}_0(\tau)\right] + \dots \equiv \tilde{j}_0 + \varepsilon\tilde{j}_1 + \dots.$$

Taking in Eqs. (4.3) $n = 1$, we obtain from all the above equations

$$\bar{x}_0\bar{y}_0 = 0,$$

(4.5)
$$\frac{d\bar{y}_0}{dt} + G\bar{x}_0 = 0,$$

(4.5)
[cont.]

$$\frac{d\tilde{x}_0}{d\tau} + \tilde{j}_0 = 0,$$

$$\frac{d\tilde{y}_0}{d\tau} = 0,$$

and

$$\frac{d\bar{x}_0}{dt} + G\bar{y}_0 + \bar{y}_0\bar{x}_1 + \bar{x}_0\bar{y}_1 = 0,$$

$$\frac{d\bar{y}_1}{dt} + G\bar{x}_1 = 0,$$

(4.6)

$$\frac{d\tilde{x}_1}{d\tau} + G\tilde{y}_0 + \tilde{j}_1 = 0,$$

$$\frac{d\tilde{y}_1}{d\tau} + G\tilde{x}_0 = 0.$$

From Eqs. (2.7) and (2.5) it follows that

(4.7)
$$\mu(\varepsilon) = \mu_0 + \varepsilon\mu_1 + 0(\varepsilon^2),$$
$$\eta(\varepsilon) = \eta_0 + \varepsilon\eta_1 + 0(\varepsilon^2).$$

Denoting

$$\bar{x}_k(0) = \bar{\mu}_k, \quad \tilde{x}_k(0) = \tilde{\mu}_k,$$
$$\bar{y}_k(0) = \bar{\eta}_k, \quad \tilde{y}_k(0) = \tilde{\eta}_k,$$

we get the relationships

(4.8)
$$\bar{\mu}_k + \tilde{\mu}_k = \mu_k,$$
$$\bar{\eta}_k + \tilde{\eta}_k = \eta_k.$$

These conditions do not suffice to solve uniquely Eqs. (4.5) and (4.6), so additionally we require that

(4.9)
$$\lim_{\tau \to \infty} \tilde{y}_k(\tau) = 0$$

since $\tilde{y}_k$ are local functions and should vanish for large τ.

We shall now solve Eqs. (4.5) and (4.6) with the conditions (4.8) and (4.9). First we see that Eq. (4.5) reduces to

(4.10)
$$\bar{x}_0(t) \equiv 0, \quad \tilde{y}_0(\tau) \equiv 0,$$

(4.10′)
$$\frac{d\bar{y}_0}{dt} = 0, \quad \bar{y}_0(0) = \eta_0,$$

(4.10″)
$$\frac{d\tilde{x}_0}{d\tau} + \eta_0\tilde{x}_0 = 0, \quad \tilde{x}_0(0) = \mu_0.$$

From the last two equations we get

(4.10‴)
$$\bar{y}_0(t) \equiv \eta_0, \quad \tilde{x}_0(\tau) = e^{-\eta_0\tau}\mu_0.$$

The system (4.6) is simplified to give

(4.11)
$$\bar{x}_1(t) = -\eta_0^{-1}G\eta_0,$$

$$\tilde{y}_1(\tau) = \int_\tau^\infty ds\, G\tilde{x}_0(s) = G(\eta_0^{-1}e^{-\eta_0\tau}\mu_0)$$

and

(4.11′) $\dfrac{d\bar{y}_1}{dt} = -G\bar{x}_1 = G(\eta_0^{-1}G\eta_0), \quad \bar{y}_1(0) = \eta_1 - \tilde{\eta}_1 = \eta^1 - G(\eta_0^{-1}\mu_0),$

(4.11″) $\dfrac{d\tilde{x}_1}{d\tau} + \eta_0\tilde{x}_1 = e^{-\eta_0\tau}\mu_0\big(G(\eta_0^{-1}\mu_0(1-e^{-\eta_0\tau})) - \eta_1\big), \quad \tilde{x}_1(0) = \eta_0^{-1}G\eta_0 + \mu_1.$

Solving the last equations we get

$$\bar{y}_1(t) = \eta_1 - G(\eta_0^{-1}\mu_0) + tG(\eta_0^{-1}G\eta_0);$$

(4.11‴) $\tilde{x}_1(\tau) = e^{-\eta_0\tau}\big(\eta_0^{-1}G\eta_0 + \mu_1 + \tau\mu_0 G(\eta_0^{-1}\mu_0 - \eta_1) - \mu_0 G(\eta_0^{-2}\mu_0(1-e^{-\eta_0\tau}))\big).$

Defining the functions

(4.12)
$$x^{(1)}(t) = \tilde{x}_0(t/\varepsilon) + \varepsilon\big(\bar{x}_1(t) + \tilde{x}_1(t/\varepsilon)\big),$$
$$y^{(1)}(t) = \bar{y}_0(t) + \varepsilon\big(\bar{y}_1(t) + \tilde{y}_1(t/\varepsilon)\big),$$

we get from the theorem in [2] that the functions (4.12) represent an asymptotic solution of the first order to the system (3.14) uniformly valid on T. The zeroth order is given by first terms in the RHS of the functions (4.12).

If, by coincidence,

(4.13) $\mu_0 = 0, \quad \mu_1 = -\eta_0^{-1}G\eta_0,$

then

$$\tilde{x}_0(\tau) = \tilde{x}_1(\tau) = \tilde{y}_1(\tau) \equiv 0,$$

and the uniformly valid solution is given in terms of the bulk approximation

(4.14)
$$x^{(1)}(t) = -\varepsilon\eta_0^{-1}G\eta_0,$$
$$y^{(1)}(t) = \eta_0 + \varepsilon\big(\eta_1 + tG(\eta_0^{-1}G\eta_0)\big).$$

The standard algorithm described above is such that the approximate equations at each step k are independent of the final order of approximation. This is not so in case of the new algorithm described in [1].

For $n = 0$ the new algorithm leads to same equations (4.10) as in the standard case. For $n = 1$ both approaches are different.

Applying the new algorithm we leave the local expansions in the (4.3′) unchanged but replace the relations (4.3) with

(4.15) $\bar{x} = \displaystyle\sum_{k=0}^{n} \varepsilon^k \varphi_k(w), \quad \bar{y} = w,$

which means that $\bar{y}$ remains unexpanded and $\bar{x}$ depends on time only through w. Accordingly,

(4.16) $\dfrac{d\bar{x}}{dt} = \displaystyle\sum_{k=0}^{n} \varepsilon^k D_w \varphi_k \dfrac{dw}{dt} = -\sum_{k=0}^{n} \varepsilon^k D_w \varphi_k G\bar{x} = \sum_{k=0}^{n} \varepsilon^k D_w \varphi_k G \sum_{i=0}^{n} \varepsilon^i \varphi_i(w).$

Now, inserting Eqs. (4.15) and (4.16) into Eq. (4.1) we get up to terms of order ε^2

$$w\varphi_0 = 0,$$

$$D_w\varphi_0 G\varphi_0 + Gw + \varphi_1 w = 0,$$

which gives

(4.17)
$$\varphi_0(w) \equiv 0,$$
$$\varphi_1(w) = -w^{-1}Gw.$$

This substituted into Eq. $(4.1)_2$ yields

(4.18)
$$\frac{dw}{dt} = \varepsilon G(w^{-1}Gw).$$

Making use of the relations (4.3') we get from Eqs. (4.5) and (4.6) as previously

(4.19)
$$\frac{d\tilde{x}_0}{d\tau} + \tilde{j}_0 = 0, \qquad \frac{d\tilde{y}_0}{d\tau} = 0,$$
$$\frac{d\tilde{x}_1}{d\tau} + G\tilde{y}_0 + \tilde{j}_1 = 0, \qquad \frac{d\tilde{y}_1}{d\tau} + G\tilde{x}_0 = 0.$$

The functions $\tilde{j}_0$ and $\tilde{j}_1$ are given by the expressions

$$\tilde{j}_0 = \tilde{x}_0(\tau)\overline{\eta}_0 + \varphi_0(0)\tilde{y}_0(\tau) + \tilde{x}_0(\tau)\tilde{y}_0(\tau),$$
$$\tilde{j}_1 = \left(\overline{\eta}_0 + \tilde{y}_0(\tau)\right)\tilde{x}_1(\tau) + \left(\varphi_0(0) + \tilde{x}_0(\tau)\right)\tilde{y}_1(\tau)$$
$$+ \overline{\eta}_1\tilde{x}_0(\tau) + \left(\tau\frac{d\varphi_0}{dt}(0) + \varphi_1(0)\right)\tilde{y}_0(\tau),$$

where we introduced the notation

$$\varphi_k\big(w(0)\big) = \overline{\mu}_k, \quad \tilde{x}_k(0) = \mu_k,$$
$$w(0) = \overline{\eta} = \overline{\eta}_0 + \varepsilon\overline{\eta}_1 + \dots, \quad \tilde{y}_k(0) = \tilde{\eta}_k.$$

Now we make use of Eqs. (4.8) and (4.9) to obtain the initial conditons for Eqs. (4.18) and (4.19).

First we observe that on account of the relations (4.19) we get

$$\tilde{y}_0(\tau) \equiv 0, \quad \tilde{\eta}_0 = 0, \quad \overline{\eta}_0 = \eta_0.$$

The equation for $\tilde{x}(\tau)$ is

$$\frac{d\tilde{x}_0}{d\tau} + \eta_0\tilde{x}_0 = 0, \quad x_0(0) = \tilde{\mu}_0 = \mu_0 - \varphi_0(\eta_0) = \mu_0,$$

which is identical with Eqs. (4.10''). In a similar way we get for $\tilde{y}_1(\tau)$ the same function as previously (see Eqs. (4.11)), which yields $\tilde{\eta}_1$ and $\overline{\eta}_1 = \eta_1 - \tilde{\eta}_1$.

The function $\tilde{x}_1(\tau)$ is defined as the solution to the equation

$$\frac{d\tilde{x}_1}{d\tau} + \eta_0\tilde{x}_1 = -\overline{\eta}_1\tilde{x}_0 - \tilde{x}_0\tilde{y}_1, \quad \tilde{x}_1(0) = \tilde{\mu}_1 = \mu_1 - \varphi_1(\eta_0) = \mu + \eta_0^{-1}G\eta_0,$$

which again is identical with Eqs. (4.11'').

Finally we solve Eq. (4.18) with the initial condition

$$w(0) = \overline{\eta} = \eta_0 + \varepsilon\overline{\eta}_1 = \eta_0 + \varepsilon\left(\eta_1 - G(\eta_0^{-1}\mu_0)\right).$$

The first order approximate solution uniformly valid on T is given by

(4.20)
$$x^{(1)}(t) = \tilde{x}_0(t/\varepsilon) + \varepsilon\left(\varphi_1(w) + \tilde{x}_1(t/\varepsilon)\right),$$
$$y^{(1)}(t) = w(t) + \varepsilon\tilde{y}_1(t/\varepsilon).$$

If the conditions (4.13) are satisfied, we obtain as previously the uniformly valid approximation in a simple form:

(4.21)
$$x^{(1)} = -\varepsilon w^{-1} Gw,$$
$$y^{(1)} = w.$$

5. Relation to the kinetic theory

Any of the discrete models of the Boltzmann equation is of the form (1.2) where $h(z)$ describes streaming and $j(z)$ represents the collision operator.

From the general properties of the collision operator it follows that there exists a p-dimensional subspace of R^m spanned by collision invariants ψ_i as it was assumed in (A.1). It is called the hydrodynamical subspace and the y components of the whole vector z constitute the hydrodynamical moments.

The function ϱ in the assumption (A.3) represents the Maxwellian in the sense that if the vector z_0 is the solution to $j(z) = 0$, then it is uniquely determined by its hydrodynamical moments

$$z_0 = \varrho(y_0);$$

provided y_0 belongs to the properly defined p-dimensional subset of the hydrodynamical subspace.

The linearized collision operator represented here by the matrix $D_z j(\varrho(y))$ has a kernel of the same dimension as the hydrodynamical subspace. For most of the practically relevant models the rest of the spectrum satisfies the inequality (2.8). In general the kernel of the linearized collision operator is not identical with the hydrodynamical subspace contrary to (A.4). However, the latter is valid for the Carleman model.

In view of the above explanations it is now not difficult to see that the bulk approximation obtained with the standard approach is identical with that given by the Hilbert asymptotic expansion procedure. In particular, Eqs. (4.10′) represent the Euler equation and (4.11′) the nonhomogeneous Euler equation. The solutions to the Euler equations give a uniformly valid approximation if Eq. (4.13) is fulfilled or, in other words, if the solution belongs to the Hilbert class.

When applied to the Carleman model, the new algorithm corresponds to the Chapman–Enskog expansion and Eq. (4.18) represents the Navier–Stokes equation of hydrodynamics. As in the previous case the solution to this equation is uniformly approximating the exact solution if Eq. (4.13) is satisfied.

It is seen that the application of the asymptotic expansion method enables us to obtain uniformly valid approximations for arbitrary initial conditions since the local functions contribute to the proper description of the solution in the initial layer and allow to define the appropriate initial conditions for hydrodynamical equations.

It is also suggested that the new algorithm developed by the authors [1] is superior to the standard one similarly as the Navier–Stokes equation is a better description of hydrodynamics than the Euler equations.

Finally, take the general case when, contrary to (A.4), the kernel of the linearized collision operator is not identical with the hydrodynamical subspace. First we observe that any $y \in \bar{\theta}$ can be uniquely represented in terms of collision invariants

$$y = \sum_{i=1}^{p} \beta_i \psi_i,$$

where β_i are scalar coefficients.

Differentiating the identity

$$j\left(\varrho\left(\sum_{i=1}^{p} \beta_i \psi_i\right)\right) = 0$$

with respect to β_i, we get

$$D_z j\left(\varrho\left(\sum_{i=1}^{p} \beta_i \psi_i\right)\right) D_y \varrho\left(\sum_{i=1}^{p} \beta_i \psi_i\right) \psi_i = 0, \quad i = 1, \dots, p.$$

Since the dimensions of the kernel of the linearized collision operator and the hydrodynamical subspace are the same, the last equality shows that for any $y \in \bar{\theta}$ the kernel of the linearized collision operator $D_z j(\varrho(y))$ is spanned by the vectors $D_y \varrho(y) \psi_i$ provided the matrix $D_y \varrho(y)$ is nonsingular.

From above it is seen that (A.4) is satisfied only when the Maxwellian $\varrho(y)$ is a linear function of y which happens to be true for the Carleman model. In the general case the mapping $D_y \varrho(y)$ of the hydrodynamical subspace into the kernel of the linearized collision operator depends explicitly on the solution and this has to be accounted for in the asymptotic analysis.

Acknowledgements

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