

The thickness of the boundary layer on a stretching filament moving in a fluid at rest

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THE LAMINAR stationary boundary layer on an infinite filament, moving along its axis in a resting fluid, is considered. The filament issues from a circular orifice, with an initial velocity w_0 . Due to stretching, the velocity of the filament increases, and its radius decreases at increasing distance from the origin. The thickness of the boundary layer was investigated using the Kármán-Pohlhausen method. It has been found that for intense stretching the thickness of the boundary layer varies nonmonotonically along the filament.

Rozważana jest laminarna stacjonarna warstwa przyścienna na nieskończonym włóknie o przekroju kołowym, poruszającym się wzdłuż osi w spoczywającym płynie. Włókno wydobywa się z prędkością w_0 z kołowego otworu. Wskutek rozciągania prędkość włókna narasta w kierunku ruchu, zaś promień maleje. Stosując metodę parametryczną Kármána-Pohlhausena badano numerycznie grubość warstwy przyściennej. Stwierdzono, że przy intensywnym rozciąganiu grubość włókna zmienia się wzdłuż włókna w sposób istotnie niemonotoniczny.

Рассматривается ламинарный стационарный пограничный слой на бесконечном осесимметрическом волокне круглого сечения, движущемся вдоль своей оси в покоящейся жидкости. Волокно выдвигается из круглого отверстия с начальной скоростью w_0 . Вследствие растяжения волокна, скорость его по направлению движения увеличивается, а радиус — уменьшается. В работе с помощью параметрического метода Кармана-Польгаузена рассчитана толщина пограничного слоя, для некоторой модели растяжения волокна. Установлено, что при интенсивном растяжении толщина волокна изменяется по длине волокна немонотонно.

1. Introduction

SAKIADIS [1-3] in 1961 investigated a special class of boundary layer flows on continuously formed surfaces moving in a fluid at rest. Flows of this type represent a new class of boundary layer problems, with solutions substantially different from those with usual boundary conditions. Using the Kármán-Pohlhausen method, SAKIADIS analysed the boundary layer on a plane sheet [2] and a circular cylinder [3] issuing from a slit or an orifice, respectively. Later the boundary-layer flow on a continuously formed cylinder was investigated also by KOLDENHOF [4], CRANE [5] and CLARK and DOMBROWSKI [6].

In this paper the laminar stationary axisymmetric flow of an incompressible fluid, generated by a stretching filament issuing from a circular orifice into a half-space, is considered (Fig. 1). The coordinates x and y denote the distance to the bordering plane and to the body (the filament), respectively. The initial radius of the filament is r_0 , and the initial velocity of its surface — w_0 . The velocity of the body surface increases, and its

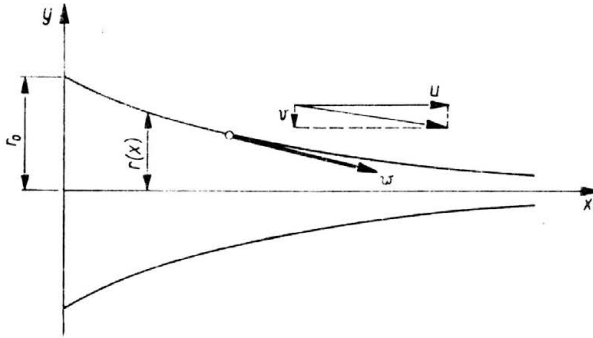


FIG. 1. A general view of a stretching filament.

radius decreases at increasing distance from the origin. An assumption is made that the radius $r(x)$ varies slowly along the filament:

$$(1.1) \quad \left| \frac{dr}{dx} \right| \ll 1.$$

Far from the body the fluid is at rest. Near the body the viscous forces generate a flow which can be described by the boundary-layer approximation. The Kármán-Pohlhausen method of solution of boundary-layer equations is used.

2. The boundary layer equations

The stationary axisymmetric boundary layer is described by Eqs. (2.1):

$$(2.1) \quad \begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{v}{y} &= 0, \\ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \nu \left(\frac{\partial^2 u}{\partial y^2} + \frac{1}{y} \frac{\partial u}{\partial y} \right), \end{aligned}$$

where u , v denote the velocity components in the x - and y -direction respectively; ρ is the density; and ν the kinematic coefficient of viscosity. The pressure derivative $\partial p / \partial x$ has been omitted in Eq. (2.1)₂ because the fluid is at rest at infinity.

The specific feature of the problem considered are unconventional boundary conditions. For the classical boundary layer the velocity vanishes at the body surface, and is nonzero at infinity. In our problem the u -component of the velocity is nonzero at the body surface (on the body, due to inequality (1.1), u can be assumed equal to the velocity of the body surface $w(x)$); and is varying, in general, along the body; while the velocity at infinity is zero. The boundary layer with similar boundary conditions was first considered by Sakiadis. Our problem differs from that analysed by Sakiadis (in [3]) for two reasons:

- 1) the filament radius is not constant,
- 2) the fluid velocity at the body surface varies along the body. These factors can substantially influence the development of the boundary layer; and, as it will be seen later, can in certain cases give quite different general picture of the layer.

3. The parametric method

The strict solutions of the boundary layer equations are known for very special problems only. As a consequence, for practical needs many approximated methods have been developed. One of the best known is the parametric method, associated with the names of Kármán and Pohlhausen. In that method a certain type of functional dependence of u on y is postulated. Namely, it is assumed that u is a known function of the coordinate y and of a certain shape coefficient α , which is a function of x to be determined. It is demanded that u fulfill the boundary conditions and Eqs. (2.1) on the body and on the edge of the boundary layer. Inside the layer, instead of the differential equations (2.1), an overall integral condition is postulated to be fulfilled. This integral condition permits to determine the shape coefficient α , and thereby to find the velocity. To obtain this condition, one multiplies the first and the second equations of the system (2.1) by uy and y , respectively, sums both equations, and there up on integrates across the layer (see [9]). Assuming that the corresponding integral exists and that the operations of integration and differentiation can be interchanged, one obtains

$$(3.1) \quad \frac{d}{dx} \int_r^{\infty} yu^2 dy = \frac{r\tau_w}{\rho},$$

where τ_w is the shear stress at the body surface

$$\tau_w = -\nu\rho \left. \frac{\partial u}{\partial y} \right|_{y=r}.$$

Equation (3.1) expresses the conservation of momentum in projection on the axis of the body. The flow of the fluid is concentrated in a thin (in a sense) layer near the body. Outside the layer the fluid is almost at rest. It follows then that in the left-hand side of Eq. (3.1), the integral over the interval (r, ∞) is almost equal to and may be replaced by the corresponding integral over the interval $(r, r + \delta)$ where δ is a certain thickness of the layer.

Similarly to GLAUERT, LIGHTHILL [7] and SAKIADIS [3], we assume for u

$$(3.2) \quad \frac{u}{w(x)} = 1 - \frac{1}{\alpha(x)} \ln \frac{y}{r(x)}.$$

If u diminishes at increasing distance from the body, then from Eq. (3.2) it follows that

$$(3.3) \quad \alpha > 0.$$

Due to inequality (1.1), the velocity components u and v at the body surface are

$$(3.4) \quad u(x, r) = w(x), \quad v(x, r) = w(x) \frac{dr}{dx}.$$

In the method applied, u should fulfill the differential equations on the body and on the edge of the layer. Since the knowledge of the component v is of less importance, our consideration will be confined to the second equation of the system (2.1).

If u is given by Eq. (3.2), the right-hand side of Eq. (2.1)₂ is identically equal to 0. For the cylindrical filament moving with constant velocity, the left-hand side of Eq. (2.1)₂ vanishes on the body, and Eq. (2.1)₂ is fulfilled locally on the body. However, on the stretching filament with the boundary conditions (3.4), Eq. (2.1)₂ may be fulfilled only approximately. For this reasons both terms in the lefthand side of Eq. (2.1)₂ should be negligible with respect to $\frac{\nu}{\rho} \frac{\partial u}{\partial y}$ on the body surface, or

$$(3.5) \quad \frac{rw}{\nu} \left| \frac{dr}{dx} \right| \ll 1 \quad \text{and} \quad \frac{rw}{\nu} \frac{dw}{dx} \ll \left| \frac{\partial u}{\partial y}(x, r) \right|.$$

The strong inequalities (3.5) together with Eq. (1.1) restrict the admissible model of filament stretching.

The edge of the boundary layer is defined by the condition $u = 0$. For u given by Eq. (3.2) it corresponds to $y = re^\alpha$ or

$$(3.6) \quad \delta = r(e^\alpha - 1),$$

where δ is a thickness of the boundary layer. The derivatives of u with respect to y for $y = r + \delta$ are

$$(3.7) \quad \frac{\partial u}{\partial y} \Big|_{r+\delta} = -\frac{w}{\alpha(r+\delta)}, \quad \frac{\partial^2 u}{\partial y^2} \Big|_{r+\delta} = \frac{w}{\alpha(r+\delta)^2}.$$

To fulfill the Eq. (2.1)₂ on the edge of the boundary layer, the derivatives (3.7) should be equal to 0. This condition is not fulfilled but both α and δ increase at increasing distance from the origin so that Eq. (2.1)₂ is satisfied approximately at increasing distance from the origin.

4. The differential equation for the shape coefficient

Replacing ∞ by $r + \delta$ as the upper limit of integration in the left-hand side of Eq. (3.1), and inserting for u the right-hand side of Eq. (3.2), one obtains the following equation:

$$(4.1) \quad \frac{d}{d\xi} (IR^2W^2) = \frac{W}{\alpha},$$

where new dimensionless symbols are used:

$$(4.2) \quad \xi = \frac{x}{r_0 \text{Re}_0} = \frac{x\nu}{r_0^2 w_0}, \quad \text{Re}_0 = \frac{r_0 w_0}{\nu}, \quad R = \frac{r}{r_0}, \quad W = \frac{w}{w_0},$$

$$(4.3) \quad I(\alpha) = \int_r^{1+\frac{\delta}{r}} \frac{y}{r} \left(\frac{u}{w} \right)^2 d\left(\frac{y}{r} \right) = \frac{1}{4\alpha^2} (e^{2\alpha} - 1 - 2\alpha - 2\alpha^2).$$

Upon some transformations of Eq. (4.1), one obtains the differential equation (4.4) for the coefficient of shape α :

$$(4.4) \quad \frac{d\alpha}{d\xi} = \frac{1}{R^2 W J_1(\alpha)} - \frac{J_2(\alpha)}{J_1(\alpha)} \frac{d \ln(RW)}{d\xi},$$

where

$$\begin{aligned}
 (4.5) \quad J_1(\alpha) &= \alpha \frac{dI}{d\alpha} = e^{2\alpha} \left(\frac{1}{2\alpha} - \frac{1}{2\alpha^2} \right) + \left(\frac{1}{2\alpha} + \frac{1}{2\alpha^2} \right), \\
 J_2(\alpha) &= 2\alpha I = \frac{1}{2\alpha} (e^{2\alpha} - 1 - 2\alpha - 2\alpha^2).
 \end{aligned}$$

The initial condition for Eq. (4.4) is $\delta = 0$ or

$$(4.6) \quad \alpha = 0 \quad \text{for} \quad \xi = 0.$$

5. The analysis of solutions

To solve Eq. (4.4), some form of the functions $R(\xi)$ and $W(\xi)$, characterizing the model of filament stretching is needed. In this section it is assumed that the velocity is uniform in the filament cross-section or

$$(5.1) \quad WR^2 = 1.$$

As a consequence of Eq. (5.1), Eq. (4.4) simplifies to the form

$$(5.2) \quad \frac{d\alpha}{d\xi} = \frac{1}{J_1(\alpha)} + \frac{J_2(\alpha)}{J_1(\alpha)} \frac{d \ln R}{d\xi}.$$

In most of this section it is also assumed that the radius decreases along the filament accordingly to the exponential formula (5.3)

$$(5.3) \quad \frac{R-a}{1-a} = e^{-b\xi},$$

where a is a ratio of the asymptotic and initial value of the filament radius, and the positive coefficient b says how quickly the radius decreases along the filament. For the filaments described by Eqs. (5.1) and (5.2), the profile of the velocity W has an S-shape with a deflection point. The R - and W -profile assumed here agree qualitatively with those observed experimentally during fiber formation [8].

The range of validity of the applied method is limited by the strong inequalities (1.1) and (3.5). For u specified by Eq. (3.2), these inequalities become

$$(5.4) \quad \left| \frac{dR}{d\xi} \right| \ll \text{Re}_0, \quad WR^2 \left| \frac{d \ln R}{d\xi} \right| \ll \begin{cases} 1, \\ \frac{1}{2\alpha}. \end{cases}$$

When the model of stretching is given by Eqs. (5.1) and (5.3), the conditions (5.4) yield a restriction for b :

$$(5.5) \quad b \ll \begin{cases} \text{Re}_0, \\ 1 \end{cases}.$$

The two asymptotic cases of the formula (5.3) may be distinguished: $a = 1 (R \equiv 1)$ and $a = 0$. In the first case the filament is a circular cylinder moving with a constant

velocity. For this case Eq. (5.2) has been solved numerically by SAKIADIS [3]. KOLDENHOF [4] has given an analytical solution (5.6):

$$(5.6) \quad \xi = \frac{1}{2\alpha} (e^{2\alpha} - 1 - 2\alpha) - \frac{1}{2} [\text{Ei}(2\alpha) - \ln 2\alpha - C],$$

where $\text{Ei}(x)$ is the integral exponent function, and $C = 0.577\dots$ is the Euler constant. The dimensionless thickness of the boundary layer $\Delta = \delta/r_0$, corresponding to the solution (5.6), is shown in Fig. 2. It increases monotonically without limits along the filament.

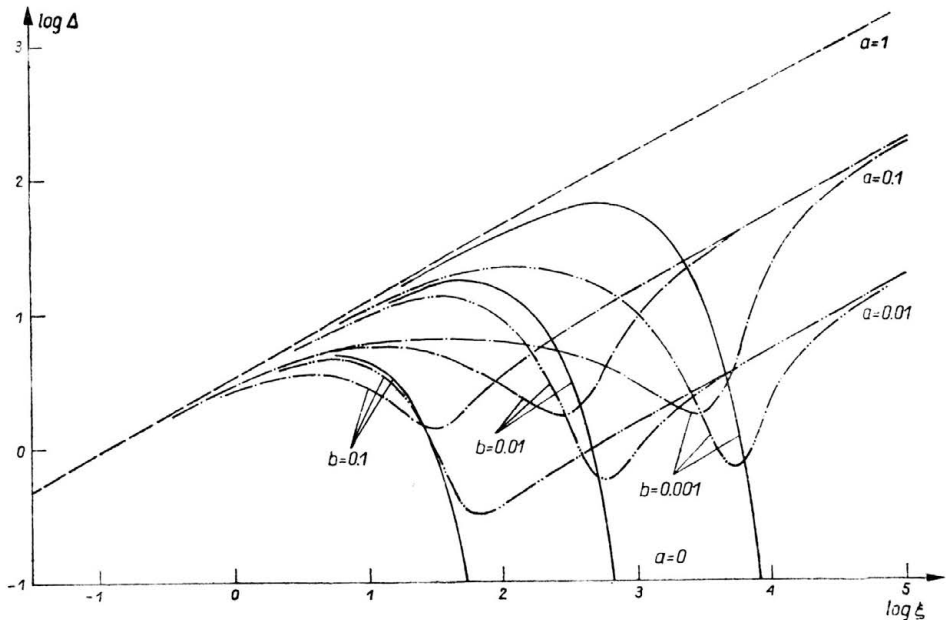


FIG. 2. The dimensionless thickness Δ of the boundary layer.

For $a = 1$, Eq. (5.2) was solved numerically by the author (Fig. 2). In the second (rather theoretical) asymptotic case ($a = 0$, $R = e^{-b\xi}$), the velocity of the filament grows to infinity. The thickness of the boundary layer Δ reaches its maximum, and then decreases to 0. If the asymptotic radius of the filament is finite and small enough, the thickness of the boundary layer first grows along the filament, then after reaching a maximum it begins to drop, and finally it grows once more along this part of the filament, where the radius almost reaches its asymptotic value. On the initial part of the filament, the thickness of the boundary layer increases quickly, and after a relatively short interval (of few r_0) it becomes much larger than the actual radius $r(x)$. In the asymptotic region, the thickness Δ of the boundary layer on a stretching filament is smaller than the corresponding thickness on the circular cylinder, moving with constant velocity. This is due to the fact that Δ is not a real boundary layer thickness, but a ratio δ/r_0 . The discrepancy between curves, corresponding to different values of a , would be, however, slight in the asymptotic region if the asymptotic radius of the filament were taken as a reference thickness, instead of r_0 .

6. Conclusions

The analysis of solutions of Eq. (5.2) has shown that the process of stretching of filaments may have a substantial influence on the thickness of the boundary layer. The local acceleration of the filament counterparts to the process of usual thickening of the boundary layer. If intensive enough, it can cause a deep minimum in the boundary layer thickness. However, when the filament velocity approaches its asymptotic value, the boundary layer begins to thicken once more. Far enough from the origin, the ratio of the boundary layer thickness to the asymptotic radius of the filament is very near to the same ratio for a circular cylinder with constant velocity.

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