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Slip theory and inelastic deformations; relations between the theory and the experimental results

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THE PAPER presents a theory based upon the concept of independent, thermally active slip processes. The results are compared with experimental data. The modifications of the slip theory by S. B. Batdorf and B. Budiansky introduced here lead to considerable differences in the results obtained by means of the slip theory.

W pracy tej została przedstawiona teoria oparta na koncepcji niezależnych termicznie aktywnych procesów poślizgów. Porównano wyniki obliczeń uzyskanych przy zastosowaniu tej koncepcji z wynikami eksperymentalnymi. Wprowadzone modyfikacje w stosunku do teorii poślizgów S. B. Batdorfa i B. Budianskyego spowodowały zasadnicze zmiany w rezultatach uzyskiwanych przy zastosowaniu teorii poślizgów.

В этой работе представлена теория, опирающаяся на концепции независимых термически активных процессов скольжений. Сравнены результаты расчетов, полученных при применении этой концепции, с экспериментальными результатами. Введенные модификации, по отношению к теории скольжений С. Б. Батдорфа и Б. Будянского, вызвали принципиальные изменения в результатах, получаемых при применении теории скольжений.

1. Introduction

INVESTIGATION of the deformation processes in solids indicate that the plastic behaviour of such bodies depends upon the path along which the given state of stress and deformation has been achieved. This phenomenon is manifested by different forms of the yield surfaces obtained by various authors, owing to different histories of the preliminary inelastic deformation. It is most easily seen in the case of the analysis performed in the $\sigma-\tau$ plane, [1–3].

Most of the theories which attempted to describe the deformation process are unable to take this relation into account since the inelastic behaviour, for instance the yield surface, is usually uniquely determined by means of the stress, strain and strain rate tensors and by the values of the functionals with arguments which represent the histories of such tensor invariants, like the dissipation energy. The introduction of such magnitudes as the only state variables necessitates the introduction of equivalence classes for such processes which, in view of the aforementioned criteria, are indistinguishable. From the experimental investigations it follows that for two different preliminary loading paths belonging to the same class, the forms of the corresponding yield surfaces may be completely different. It is seen that the description of the phenomenon coinciding, within a certain range, with experimental results may be obtained by means of applying a concept of independent and thermally active slip processes. This concept is based on the observations of the experimentally determined yield surfaces (A. Phillips, J. L. Tang) and the theoretical results obtained from the slip theory (S. B. Batdorf, B. Budiansky [4, 5]).

2. Slip theory

It has been shown by W. T. Koiter [6] that the slip theory presented by S. B. Batdorf and B. Budiansky may be treated as a method of describing of the deformation in such materials in which plastic deformation is produced by the simultaneous reaching of an infinite number of elementary yield conditions of the type of Eq. (2.1). Each of the conditions corresponds to a different orientation of the slip surfaces in space; each of them may correspond to a different limit stress.

$$f_0 = \sigma_{ij} \cdot n_{ij} - \tau_0 = 0. \tag{2.1}$$

Here $n_{ij} = \frac{1}{2}(c_{(n)i} c_{(s)j} + c_{(n)j} c_{(s)i})$, **c** is the tensor of direction cosines with n denoting the normal, and s — the tangential directions. The system orientation is shown in Fig. 1.

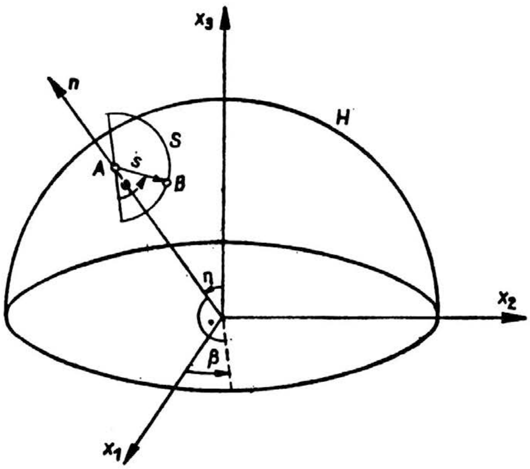

Fig. 1.

The normal vector n pierces the hemisphere H at point A, the semicircle S lies in the plane tangent to the hemisphere at A. The tangent vector s determines the position of point B on S.

The total plastic strain rate is a sum of the slips produced by reaching the elementary yield conditions. In the case of a finite sum

$$\dot{\varepsilon}_{ij} = \sum_k \lambda^{(k)} \frac{\partial f^{(k)}}{\partial \sigma_{ij}} = \sum_k \lambda^{(k)} n_{ij}^{(k)}. \tag{2.2}$$

In view of the continuous distribution of the possible slip system orientations in space, the sum takes the integral form

$$\dot{\varepsilon}_{ij} = \int_H d\Omega \int_S \hat{\gamma} n_{ij} d\varphi = \int_{\beta=0}^{2\pi} \int_{\eta=0}^{\pi/2} \int_{\varphi=0}^{\pi} \dot{\gamma} n_{ij} \sin\eta d\beta d\eta d\varphi. \tag{2.3}$$

Here β, η, φ are the Euler angles (Fig. 1), $\dot{\gamma}$ is the strain rate density [$\mathrm{rad}^{-3}\ \mathrm{s}^{-1}$].

If the deformation is assumed to be described by the slip theory, then the yield surface observed experimentally must constitute the envelope (or a part of an envelope) of the family of functions representing the elementary yield conditions. S. B Batdorf and B. Budiansky assumed a deformational form of the equation determining the limit stress $\tau_0 = F^{-1}(\gamma)$, [5]; when the stress $\tau = \sigma_{ij} n_{ij}$ was less than τ_0, the slip did not occur.

3. Slip theory and experimental results

Observation of the evolution of directions tangent to the yield surface may lead to conditions which must be satisfied by the constitutive equations for a single slip system in order that the inelastic deformations be described by the slip theory without violating the experimental results.

Figure 2 demonstrates the evolution of the yield surface in aluminium according to [2]; the system α shown here corresponds to the yield condition written in the form

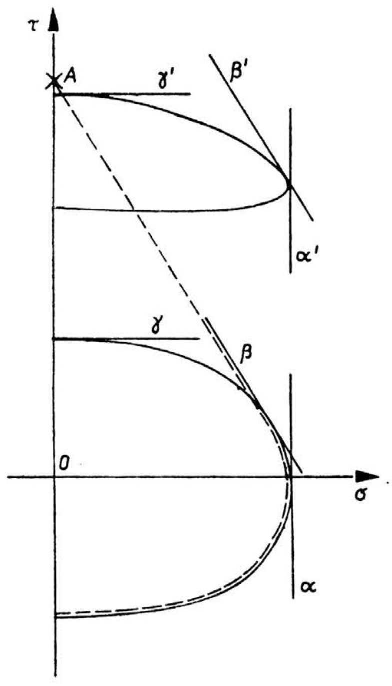

FIG. 2.

$O \cdot \tau + \frac{1}{2}\sigma - \tau_\alpha = 0$. During the loading process from 0 to A the projection of the stress increments on the directions of the system α was equal to zero, $\Delta\sigma_{ij}n_{ij} = 0$. After the loading process the condition is represented by the tangent α' described by the same equation. Hence the critical stress remains unchanged, $\tau_\alpha = \tau_{\alpha'}$, what complies with the Batdorf-Budiansky slip theory. According to that theory, only if the tangential stress corresponding to the system $\tau = \sigma_{ij}n_{ij}$ equals τ_α, the slip (and hence the change of the limit stress) is possible. In the case considered the stress τ was equal to zero and the limit stress remained unchanged.

The behaviour of the system β in Fig. 2 contradicts however, the results of the slip theory mentioned above. During the loading process τ was always smaller than τ_β and thus, according to the theory by Batdorf and Budiansky, hardening was not possible. On the other hand, from the equations for the straight lines β and β' it follows that hardening took place, $\tau_{\beta'} > \tau_\beta$. This reasoning leads us to the following conclusions as to the properties which have to be taken into account in the constitutive equations of the elementary slip system:

(3a) if the stress tangent to the given system is zero, the limit stress remains unchanged,

(3b) if the stress tangent to the given system $\tau = \sigma_{ij}n_{ij}$ is different from zero, then — in spite of, for exemple $\tau < \tau_0$ — the limit stress is changed according to the sign of τ. The conditions must be applied to such materials which were not subject to any previous deformation processes.

4. Strain rate

In order to fulfill the conditions (3a) and (3b) let us assume the following equation for the inelastic strain rate density:

$$\dot{\gamma} = \gamma_0 e^{-\frac{u_1}{kT}} - \gamma_0 e^{-\frac{u_2}{kT}}. \tag{4.1}$$

This equation may be derived by applying the generally known concept of Arrhenius used also in the analysis of various phenomena in physics and chemistry. The concept consists in the assumption that the velocity of a given process is proportional to the frequency of attaining certain limiting values of energy (activation energy) by the particles subject to thermal motions. The frequency is given by the formula $\nu = \nu_0 \exp(-u/kT)$, Fig. 3. Additional assumptions make it possible to express the activation energy and stresses in terms of the actual state of deformation.

(4a) Activation energy is approximately proportional to the stresses corresponding to the slip system orientation

$$\begin{cases} U_1 = U_1^* - r \cdot \tau, \\ U_2 = U_2^* + r \cdot \tau. \end{cases}$$

(4b) During the process of deformation the energies U_1^*, U_2^* are changed. The corresponding increments may be referred to the initial state of equilibrium

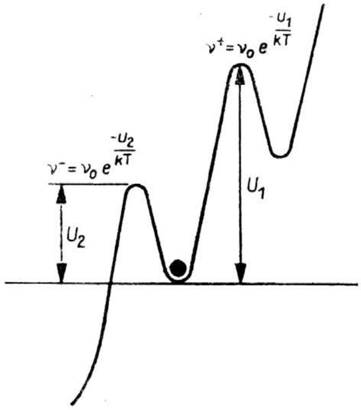

FIG. 3.

$$\begin{cases} U_1^* = U_0 + \Delta U_1^*, \\ U_2^* = U_0 + \Delta U_2^*. \end{cases}$$

Assuming $\Delta U_1^* = -\Delta U_2^*$ and referring this energy variation to the stresses, we obtain

$$\begin{cases} U_1^* = U_0 + r\tau_u, \\ U_2^* = U_0 - r\tau_u. \end{cases}$$

Here τ_u is the parameter determining the variation of the activation energy during the deformation process. In the considerations concerning the motion of dislocations it corresponds to the long range stresses treated as such components of stresses which do not affect the activation energy and which are transmitted by these atoms for which the frequency of attaining the activation energy is negligibly small.

From the assumptions (4a) and (4b) it follows that

$$\begin{aligned} U_1 &= U_0 - r(\tau - \tau_u), \\ U_2 &= U_0 + r(\tau - \tau_u). \end{aligned} \tag{4.2}$$

Substitution of Eq. (4.2) into Eq. (4.1) yields

$$\dot{\gamma} = \gamma_0 e^{-\frac{u_0}{kT}} \left[e^{\frac{r(\tau-\tau_u)}{kT}} - e^{-\frac{r(\tau-\tau_u)}{kT}} \right]. \tag{4.3}$$

The equations of evolution for τ_u, r follow from the behaviour of the selected tangent of the yield surface. In the theory presented here the description of a single line tangent to the yield surface resulting from an arbitrary loading process generates the form of the yield surface from the test considered and also all the forms of the yield surfaces corresponding to the all other loading tests. An example of the set of equations determining τ_u, r has been based on the evolution of the limit stresses for the system α in the process of extension of an aluminium specimen [3] (points lying on the σ-axis were taken into account). This type of evolution of the position of tangents to the yield surface is used in the following Sect. 5 which presents the results of numerical calculation of the yield surface in the tension test and in some other tests.

In the case of a cross effect some additional information would be needed concerning the behaviour of the yield conditions corresponding to other slip systems, first of all those orthogonal to the loading direction, in order to enable us to determine the dependence of the activation energy on the deformations occurring in other slip systems. This can be done by means of introducing various types of influence functions h determining the influence of the distribution of magnitudes $x(\beta, \eta, \varphi)$ (e.g. the deformation density γ) on the value of the parameter y (e.g. the coefficient r) corresponding to the direction $\beta_0, \eta_0, \varphi_0$

$$y(\beta_0, \eta_0, \varphi_0) = \int\limits_{\beta=\beta_0}^{\beta_0+\pi} \int\limits_{\eta=\eta_0}^{\eta_0+\pi} \int\limits_{\varphi=\varphi_0}^{\varphi_0+\pi} h(\beta-\beta_0, \eta-\eta_0, \varphi-\varphi_0) x(\beta, \eta, \varphi) \cdot \sin\eta d\beta d\eta d\varphi.$$

The same may be achieved directly by expressing the activation energy in terms of the invariants, for instance $r = r(\gamma, \sigma_{ij} \times \varepsilon_{ij})$.

5. Numerical analysis

The set of constitutive equations consisted of Eqs. (2.3), (4.3) and the equations determining the parameters τ_u and r. In order to compare the results of the concept presented here with experimental results let us assume the equations in the simplest form leading, however, to results complying with the behaviour of the selected tangents of the yield surfaces derived by A. Phillips and J. L. Tang. The equations are written in the deformational form

$$\tau_u = A \ln\left(\frac{|\gamma|}{B} + 1\right) \operatorname{sgn}\gamma, \tag{5.1}$$
$$r = \frac{(\gamma^2 + D)^{D1}}{C},$$

where $A = 0.47$ lb/in², $B = 2.10^{-6}$, $CkT = 0.15$ in/lb, $D = 1.5 \times 10^{-9}$, $D1 = 0.1$, $\gamma_0 \exp(-u_0/kT) = 8 \times 10^{-12}$ 1/rad³ s. The equations are selected according to the material the properties of which are to be described, and to the effects which are to be taken into account in the deformation process.

5.1. Details of numerical calculations

The integrals to be evaluated in Eq. (2.3) are approximated by finite sums corresponding to the discrete distribution of the slip systems orientation in space. In Fig. 4 the points are shown in which the normal directions pierce the unit sphere. Each of the points corresponds to six elementary slip systems resulting from the six consecutive rotations of the system about the normal by the angle $\Delta\varphi = \pi/6$. One hundred and seventy four slip systems were considered and Eq. (2.3) assumed the form

$$\varepsilon_{ij} = \sum_{k=1}^{174} \gamma^{(k)} n_{ij}^{(k)} \Delta S. \tag{5.2}$$

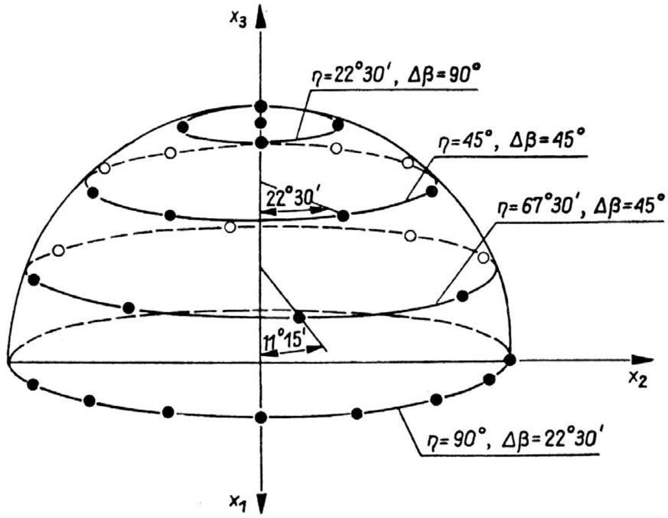

FIG. 4.

Here $\Delta S = \sin\eta\Delta\beta\Delta\eta\Delta\varphi$, $\Delta\beta$, $\Delta\eta$, $\Delta\varphi$ denote the Euler angles increments corresponding to a single slip system. The distribution of the slip system orientations was selected so that each of them corresponded to the same set measure $\Delta S = 0.113$ rad^3. In numerical calculations the loading process of the specimen is designed in the following manner: for a given step the stress increment is prescribed, for instance $\Delta\sigma = 0.15$ lb/in^2 or $\Delta\tau = 0.11$ lb/in^2; then the strain increment is evaluated for each of the slip systems, and finally all the increments are summed up according to Eq. (5.2). In the preliminary loading process the time difference between the consecutive loadings is 3 minutes, and 1 minute in the analysis of a single yield surface.

The yield surface is determined by means of evaluating the index $w = \sum_i \sum_j |\varepsilon_{ij}|$. The surfaces determined correspond to the index reaching the value of $1 \cdot 10^{-6}$. Consecutive tests within the yield surface are performed from the same state of deformation. In experimental investigations the tests are performed consecutively for different directions from the state reached by the material after the preceding test.

5.2. Comparison of numerical and experimental results

Figure 5a presents the experimental results obtained by A. PHILLIPS and J. L. TANG [3] who tested the yield surface in specimens subject to initial tension, while Fig. 5b demonstrates the numerical results concerning an analogous loading process. Similar comparisons are presented in Figs. 6a, 6b and 7a, 7b. Dashed lines indicate the yield surface fragments obtained for the index reaching the value 3×10^{-6}.

Figure 8a presents the experimental results concerning the yield surface obtained after the process of initial loading which consisted of two stages: tension and torsion of the specimen. Figure 8b presents the numerical results for an analogous loading process; the final deformation stage corresponds to the strain tensor shown in the figure. In order to

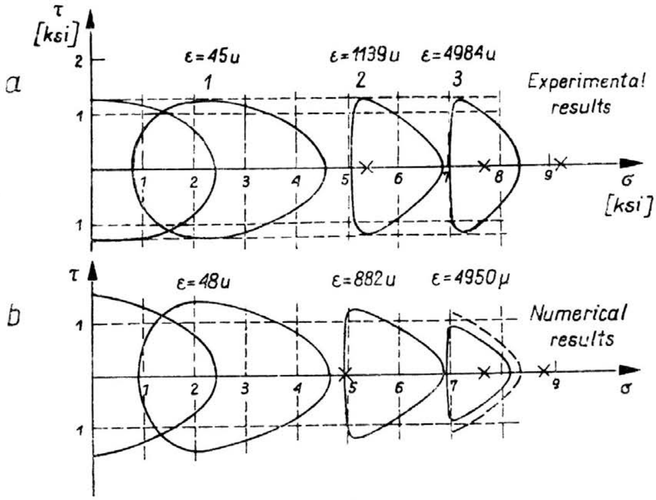

FIG. 5.

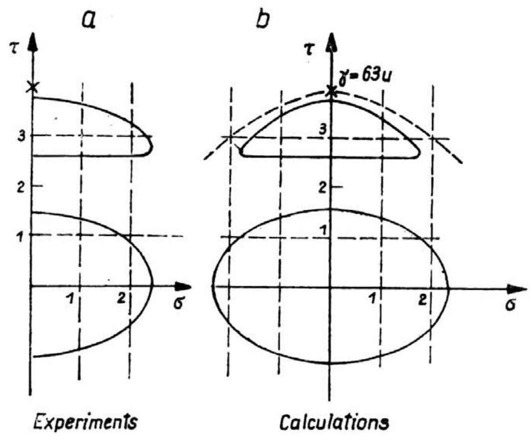

FIG. 6.

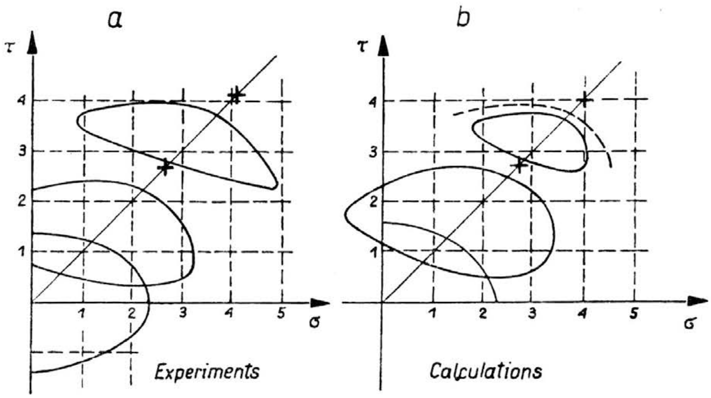

FIG. 7.

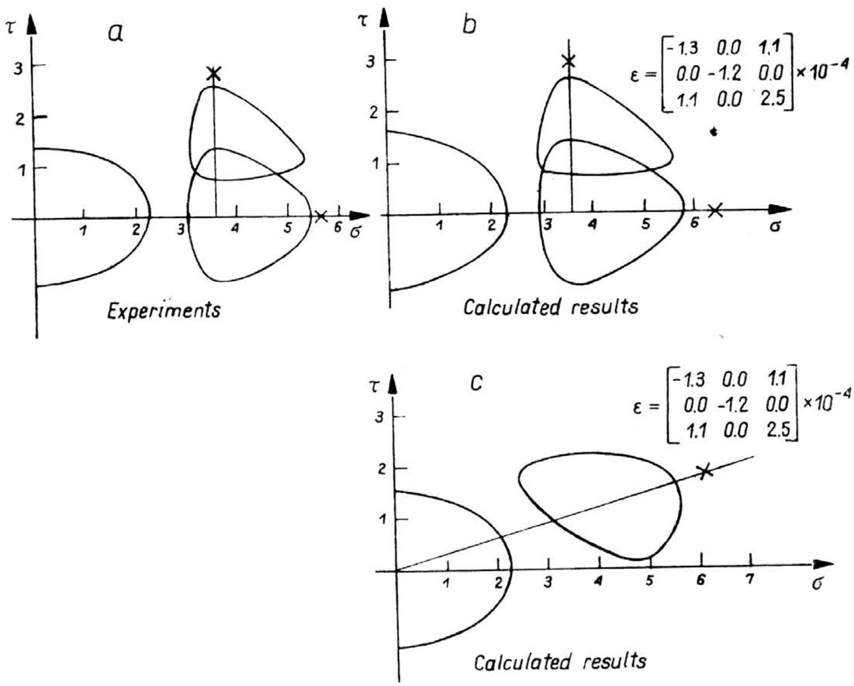

FIG. 8.

demonstrate the effect of the loading path upon the form of the yield surface, another loading process (Fig. 8c) was performed and the corresponding yield surface is shown. Its form proved to be entirely different in spite of the fact that the strains were identical in the both cases. It may be concluded from the results shown in Figs. 5a and 7a that the form shown in Fig. 8c would comply with the experimental results. The result is of

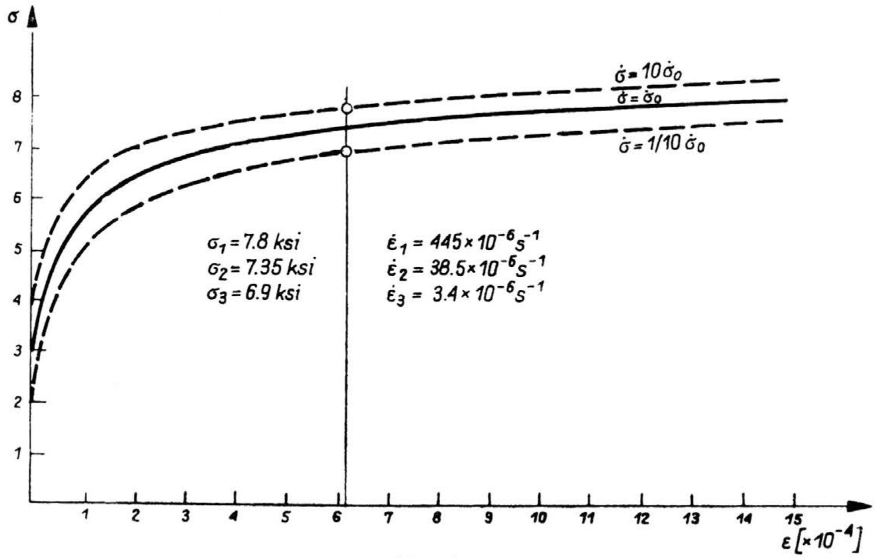

FIG. 9.

particular interest in view of the deformational form of the Eqs. (5.1) assumed. Deformation of the elementary slip system determines the position of the line tangent to the yield surface; however, other loading paths correspond to different distributions of deformations among the individual slip systems and, hence, the yield surface assumes a different form.

In view of the fact that the constitutive equations take into account the rheological properties of aluminium in these strain rate ranges in which the experiments were performed, other numerical calculations were also made which simulated the tension tests under various loading rates. The results were compared with the empirical description of the aluminium specimens tension tests under constant strain rates and temperature 20°C [7],

$$\sigma = k\varepsilon^{1/m}\dot{\varepsilon}^{1/n}. \tag{5.3}$$

In the case of aluminum $n = 35$. Using the formula (5.3) and the data presented in Fig. 9, we may evaluate the expected differences between the processes 1, 2, 3. In the cases 1 and 3 the stresses corresponding to ε_0 at curve *3* may be calculated from the formula $\sigma_3 = \left(\frac{\dot{\varepsilon}_3}{\dot{\varepsilon}_2}\right)^{1/n} \sigma_2 = 0.933\ \sigma_2 = 6.85$, while from the numerical considerations it follows that $\sigma_3 = 6.9$. Similar analysis may be applied to curves 2 and 3 to demonstrate that the choice of rheological properties complies with the experiment.

6. Conclusions

In the slip theory there exists the possibility of describing the behaviour of a slip system. Behaviour of remaining systems is then defined by the same constitutive equations. It is impossible to control the shape of the yield surface, this shape being automatically generated once the behaviour of a single tangent to the surface is prescribed. Comparison of the numerical results with experiments shows that the automatic generation of the yield surface is in good agreement with the experiments.

The slip theory presented by B. Batdorf and B. Budiansky leads to results which are evidently different from the experimental data (in Fig. 2 the surface determined by this theory is marked by dashed lines). The concept presented here is free from that shortcoming without losing the simplicity of interpretation characteristic of the slip theory. The concept may prove to be useful in explaining the phenomena connected with changes of direction of the loadings.

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