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On the fluctuations of total pressure associated with a general type of hydromagnetic turbulence

H. P. MAZUMDAR (TRONDHEIM)

CORRELATION of fluctuations of total pressure associated with a general type of MHD turbulence is formulated in the wave number space. The resultant expression, as modified here for the case of homogeneous and isotropic MHD turbulence, reduces to an elegant expression for $\overline{(\text{grad}\,\varpi)^2}$, ϖ being the total pressure.

Sformułowano zasady korelacji fluktuacji całkowitego ciśnienia związanej z ogólnym przypadkiem turbulencji magneto-hydrodynamicznej w przestrzeni liczb falowych. Wynik końcowy, zmodyfikowany tutaj do przypadku jednorodnej i izotropowej turbulencji MHD, sprowadza się do eleganckiego wyrażenia dla $\overline{(\text{grad}\,\varpi)^2}$, gdzie ϖ oznacza ciśnienie całkowite.

Сформулированы принципы корреляции флуктуации полного давления, связанной с общим случаем магнитогидродинамической турбулентности в пространстве волновых чисел. Остаточный результат, модифицированный здесь для случая однородной и изотропной магнитогидродинамической турбулентности, сводится к элегантному выражению для $\overline{(\text{grad}\,\varpi)^2}$, где ϖ обозначает полное давление.

1. Introduction

IN MANY hydrodynamical problems it is important to know the distribution of pressure fluctuations in addition to velocity fluctuations and their correlations. The study of fluctuations of pressure in ordinary hydrodynamic turbulence were initiated by OBUKHOV [1] and HEISENBERG [2]. Batchelor [3] gave a detailed analysis of pressure fluctuations in the case of homogeneous and isotropic turbulence and obtained an expression for $\overline{(p-\overline{p})^2}$ in the wave number space, which is analogous to Heisenberg's expression for $\overline{(\text{grad}\,p)^2}$ (cf. MONIN and YAGLOM [4]). UBEROI [5] studied both theoretically and experimentally the quadruple velocity correlations associated with the correlations of pressure fluctuations in homogeneous turbulence). CHANDRASEKHAR [6] worked out a few correlation functions involving fluctuations of total pressure in the case of homogeneous and isotropic MHD turbulence. His calculation for the double correlation of the fluctuations of total pressure in MHD turbulence is a straightforward generalization of BATCHELOR'S [3] calculation for $\overline{pp'}$ in ordinary homogeneous and isotropic turbulence.

In the present paper our aim is to derive the spectral equation for the double correlation of the fluctuations of total pressure associated with a general type of MHD turbulence as pictured in Sect. 2. The theory of such a general type of turbulence for the case of a turbulence velocity field and for the case of turbulence scalar fields (e.g. temperature and pressure) has been discussed respectively by GHOSH [7] and by the present author

[8, 9]. The spectral expression for the correlation (double) of fluctuations of total pressure associated with the MHD turbulence, as obtained herein, is simplified for the case of homogeneous and isotropic turbulence.

2. Spectral formulation of the correlation of fluctuations of total pressure in MHD turbulence

The equations governing viscous, incompressible hydromagnetic turbulent flows are

$$\text{(2.1)} \qquad \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_l}(u_i u_l - h_i h_l) = -\frac{\partial \overline{\omega}}{\partial x_i} + \nu \nabla_X^2 u_{i\ldots},$$

$$\text{(2.2)} \qquad \frac{\partial h_i}{\partial t} + \frac{\partial}{\partial x_l}(h_i u_l - u_i h_l) = \lambda \nabla_X^2 h_{i\ldots},$$

$$\text{(2.3)} \qquad \frac{\partial u_i}{\partial x_i} = 0, \qquad \frac{\partial h_i}{\partial x_i} = 0,$$

where u_i, h_i denote respectively the turbulence components of the velocity and magnetic fields at the point $P(\mathbf{X}, t)$; ν the kinematic viscosity; λ the magnetic diffusivity; $\overline{\omega} = \frac{p}{\varrho} + \frac{1}{2}|\overline{h}|^2$ the turbulent fluctuation of total pressure; p the turbulent fluctuation of static pressure and ϱ the fluid density. Taking the divergence of Eq. (2.1) and multiplying the resultant equation by $\overline{\omega}'$ at $P'(X', t)$, we obtain on averaging

$$\text{(2.4)} \qquad \frac{\partial^2 F^{(0),(0)}_{\omega\,.\,\omega}(X, X', t)}{\partial x_i \partial x_i} = \frac{\partial^2}{\partial x_i \partial x_l}\{F^{(11),(0)}_{il\,.\,\omega}(X, X', t) - F^{(22),(0)}_{il\,.\,\omega}(X, X', t)\},$$

where

$$F^{(0),(0)}_{\omega\,.\,\omega}(\mathbf{X}, \mathbf{X}', t) = \overline{\overline{\omega}\overline{\omega}'};$$

$$F^{(11),(0)}_{il\,.\,\omega}(\mathbf{X}, \mathbf{X}', t) = \overline{u_i u_l \overline{\omega}'};$$

$$F^{(22),(0)}_{il\,.\,\omega}(\mathbf{X}, \mathbf{X}', t) = \overline{h_i h_l \overline{\omega}'}$$

The equation for u_j' at the point $P'(\mathbf{X}', t)$ can be written as

$$\text{(2.5)} \qquad \frac{\partial u_j'}{\partial t} + \frac{\partial}{\partial x_m'}(u_j' u_m' - h_j' h_m') = -\frac{\partial \overline{\omega}'}{\partial x_j'} + \nu \nabla_{X'}^2 u_j'.$$

Taking the divergence of Eq. (2.5) and multiplying the resultant equation by $u_i u_k''$, we obtain on averaging

$$\text{(2.6)} \qquad \frac{\partial^2}{\partial x_j' \partial x_j'} F^{(1),(0),(1)}_{i\,.\,\omega\,.\,k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t)$$
$$= \frac{\partial}{\partial x_j'} \frac{\partial}{\partial x_m'}\{F^{(1),(11),(1)}_{i\,.\,jm\,.\,k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t) - F^{(1),(22),(1)}_{i\,.\,jm\,.\,k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t),$$

where u_k'' is the turbulent velocity component at the point $P''(\vec{\mathbf{X}}'', t)$;

$$F^{(1),(11),(1)}_{i\,.\,jm\,.\,k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t) = \overline{u_i u_j' u_m' u_k''}, \qquad F^{(1),(22),(1)}_{i\,.\,jm\,.\,k}(\mathbf{X}', \mathbf{X}'', t) = \overline{u_i h_j' h_m' u_k''}.$$

Similarly, taking the divergence of Eq. (2.5) and multiplying the resultant equation by $h_i h_k''$, we get on averaging

$$(2.7)\qquad -\frac{\partial^2}{\partial x_j'\partial x_j'} F^{(2)\cdot(0)\cdot(2)}_{i\,\cdot\,\omega\,\cdot\,k}(\mathbf{X}', \mathbf{X}', \mathbf{X}'', t)$$

$$= \frac{\partial}{\partial x_j'}\frac{\partial}{\partial x_m'}\{F^{(2)\cdot(11)\cdot(2)}_{i\,\cdot\,jm\,\cdot\,k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t) - F^{(2)\cdot(22)\cdot(2)}_{i\,\cdot\,jm\,\cdot\,k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t)\},$$

where h_k'' is the turbulence component of the magnetic field at $P''(\mathbf{x}'', t)$

$$F^{(2)\cdot(0)\cdot(2)}_{i\,\cdot\,\omega\,\cdot\,k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t) = \overline{h_i\overline{\omega}' h_k''},$$

$$F^{(2)\cdot(11)\cdot(2)}_{i\,\cdot\,jm\,\cdot\,k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t) = \overline{h_i u_j' u_m' h_k''},$$

$$F^{(2)\cdot(22)\cdot(2)}_{i\,\cdot\,jm\,\cdot\,k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t) = \overline{h_i h_j' h_m' h_k''}.$$

Now, introducing Fourier transforms of various correlation tensors as in [7], we obtain the reduced versions of Eqs. (2.4), (2.6) and (2.7) respectively in the wave number space as

$$(2.8)\qquad K^2\psi^{(0)\cdot(0)}_{\omega\,\cdot\,\omega}(\mathbf{K}, K', t) = -K_iK_l\psi^{(11)\cdot(0)}_{il\,\cdot\,\omega}(\mathbf{K}, \mathbf{K}', t) + K_iK_l\psi^{(22)\cdot(0)}_{il\,\cdot\,\omega}(\mathbf{K}', \mathbf{K}', t),$$

where

$$\psi^{(0)\cdot(0)}_{\omega\,\cdot\,\omega}(\mathbf{K}', \mathbf{K}', t), \qquad \psi^{(11)\cdot(0)}_{il\,\cdot\,\omega}(\mathbf{K}, \mathbf{K}', t)$$

and

$$\psi^{(22)\cdot(0)}_{il\,\cdot\,\omega}(\mathbf{K}', \mathbf{K}', t)$$

are respectively the Fourier transforms of $F^{(0)\cdot(0)}_{\omega\,\cdot\,\omega}(\mathbf{X}, \mathbf{X}', t)$

$$(2.9)\qquad F^{(11)\cdot(0)}_{il\,\cdot\,\omega}(X, X', t) \quad \text{and} \quad F^{(22)\cdot(0)}_{il\,\cdot\,\omega}(\mathbf{X}, \mathbf{X}', t),$$

$$K'^2\psi^{(1)\cdot(0)\cdot(1)}_{i\,\cdot\,\omega\,\cdot\,k}(\mathbf{K}, \mathbf{K}', \mathbf{K}''\ t) = -K_j'K_m'\psi^{(1)\cdot(11)\cdot(1)}_{i\,\cdot\,jm\,\cdot\,k}(\mathbf{K}, \mathbf{K}', \mathbf{K}'', t)$$

$$+K_j'K_m'\psi^{(1)\cdot(22)\cdot(1)}_{i\,\cdot\,jm\,\cdot\,k}(\mathbf{K}, \mathbf{K}', \mathbf{K}'', t),$$

where

$$\psi^{(1)\cdot(0)\cdot(1)}_{i\,\cdot\,\omega\,\cdot\,k}(\mathbf{K}, \mathbf{K}', \mathbf{K}'', t), \qquad \psi^{(1)\cdot(11)\cdot(1)}_{i\,\cdot\,jm\,\cdot\,k}(\mathbf{K}, \mathbf{K}', \mathbf{K}'', t)$$

and

$$\psi^{(1)\cdot(22)\cdot(1)}_{i\,\cdot\,jm\,\cdot\,k}(\vec{\mathbf{K}}, \vec{\mathbf{K}}', \vec{\mathbf{K}}'', t)$$

are the Fourier transforms of

$$F^{(1)\cdot(0)\cdot(1)}_{i\,\cdot\,jm,\,k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t), \qquad F^{(1)\cdot(11)\cdot(1)}_{i\,\cdot\,jm\,\cdot\,k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t)$$

and

$$F^{(1)\cdot(22)\cdot(1)}_{i\,\cdot\,jm\,\cdot\,k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t)$$

respectively;

$$(2.10)\qquad K'^2\psi^{(2)\cdot(0)\cdot(2)}_{i\,\cdot\,\omega\,\cdot\,k}(\mathbf{K}, \mathbf{K}', \mathbf{K}'', t) = -K_j'K_m'\psi^{(2)\cdot(11)\cdot(2)}_{i\,\cdot\,jm\,\cdot\,k}(\mathbf{K}, \mathbf{K}', \mathbf{K}'', t)$$

$$+K_j'K_m'{}^{(2)\cdot(22)\cdot(2)}_{i\,\cdot\,jm\,\cdot\,k}(\mathbf{K}, \mathbf{K}', \mathbf{K}'', t),$$

where

$$\psi^{(2)\cdot(0)\cdot(2)}_{i\,\cdot\,\omega\,\cdot\,k}(\mathbf{K}, \mathbf{K}', \mathbf{K}'', t), \qquad \psi^{(2)\cdot(11)\cdot(2)}_{i\,\cdot\,jm\,\cdot\,k}(\mathbf{K}, \mathbf{K}', \mathbf{K}'', t)$$

and

$$\psi^{(2),(22),(2)}_{i\;,\;jm\;,\;k}(\mathbf{K}, \mathbf{K}', \mathbf{K}'', t)$$

are respectively the Fourier transforms of

$$F^{(2),(0),(2)}_{i\;,\;\omega\;,\;k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t), \qquad F^{(2),(11),(2)}_{i\;,\;jm\;,\;k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t)$$

and

$$F^{(2),(22),(2)}_{i\;,\;jm\;,\;k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t).$$

At this stage we take an extended view of the lemmas introduced by GHOSH [7] for the present case and obtain two important relations that will be used in the subsequent analysis. Let Y_i, Y'_j and Y''_k be the respective components of fluctuating variables (it may well be velocity fluctuations for magnetic vector fluctions or their combinations, no matter) at $P(\mathbf{X}, t)$, $P'(\mathbf{X}, t)$ and $P''(x'', t)$.

Then, following GHOSH [7] we set for the merger of P'' with P a relation of the form

$$\int \varkappa_{i,j,k}(\mathbf{K}-\mathbf{K}'', \mathbf{K}', \mathbf{K}'', t)\, d\mathbf{K}'' = \varkappa_{ik,j}(\mathbf{K}, \mathbf{K}', t), \tag{2.11}$$

where

$$\varkappa_{i,j,k}(\mathbf{k}, \mathbf{k}', \mathbf{k}'', t)$$

is the Fourier transform of the correlation tensor $\overline{Y_i Y'_j Y''_k}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', t)$. Let us consider a fourth point $P'''(\mathbf{X}''', t)$ and the fluctuating component (velocity or magnetic) at this point for the instant t. The quasi-normality hypothesis due to MILLIONSCHIKOV [9] can then be represented in the wave number space as

$$\varkappa_{i,j,k,l} = \varkappa_{i,j}\varkappa_{k,l} + \varkappa_{i,k}\varkappa_{j,l} + \varkappa_{i,l}\varkappa_{j,k}, \tag{2.12}$$

where

$$\varkappa_{i,j,k,l}(\mathbf{K}, \mathbf{K}', \mathbf{K}'', \mathbf{K}''', t), \qquad X_{i,j}(\mathbf{K}, \mathbf{K}', t), \qquad X_{k,l}(\mathbf{K}'', \mathbf{K}''', t) \qquad \text{etc.}$$

are respectively the Fourier transform of

$$\overline{Y_i Y'_j Y''_k Y'''_l}(\mathbf{X}, \mathbf{X}', \mathbf{X}'', \mathbf{X}''', t),$$

$$\overline{Y_i Y'_j}(\mathbf{X}, \mathbf{X}', t), \qquad \overline{Y''_k Y'''_l}(\mathbf{X}'', \mathbf{X}''', t) \qquad \text{etc.}$$

Accordingly, when the fourth point P''' coincides with the first point P and in addition the third point P'' coincides with the second point P', we obtain

$$\varkappa_{il,jk}(\mathbf{K}, \mathbf{K}', t) = \iint [\varkappa_{i,j}(\overline{\vec{K}-\vec{K}'''}, \overline{K'-\vec{K}''}, t)\varkappa_{k,l}(\mathbf{K}'', \mathbf{K}''', t) + \varkappa_{i,k}(\overline{\vec{K}-\vec{K}'''}, \mathbf{K}'', t)\varkappa_{j,l}(\overline{K'-\vec{K}''}, \mathbf{K}''', t)]\, d\mathbf{k}''d\mathbf{k}''' + \varkappa_{il}(\mathbf{K}, t)\varkappa_{jk}(\mathbf{K}', t). \tag{2.13}$$

Now, employing Eqs. (2.9) and (2.10) and making use of the relations (2.11)–(2.13) we derive two expressions

$$\psi^{(11),(0)}_{il,\omega}(\mathbf{K}, \mathbf{K}', t) \quad \text{and} \quad \psi^{(22),(0)}_{il,\omega}(\mathbf{K}', \mathbf{K}', t),$$

which, when substituted in Eq. (2.8), yield the desired spectral equation for the correlation (double) of fluctuations of total pressure associated with a general type of MHD turbulence as

$$\psi^{(0),(0)}_{\omega\;,\;\omega}(\mathbf{K}, \mathbf{K}', t) = H_1(\mathbf{K}, \mathbf{K}', t) + H_2(\mathbf{K}, \mathbf{K}', t), \tag{2.14}$$

where

$$H_1(\mathbf{K},\mathbf{K}',t) = \frac{K_i K_l}{K^2}\frac{K'_j K'_m}{K'^2}\Big[\int\int \{\psi^{(1),(1)}_{i\ ,\ j}(\overrightarrow{K-K'''}, \overrightarrow{K'-K''}, t)\psi^{(1),(1)}_{m\ ,\ l}(\mathbf{K}'', \mathbf{K}''', t$$

$$+\psi^{(1),(1)}_{i\ ,\ m}(\overrightarrow{K-K'''}, \mathbf{K}'', t)\psi^{(1),(1)}_{j\ ,\ l}(\overrightarrow{K'-K''}, K''', t)\}\, d\mathbf{K}'' d\mathbf{K}'''$$

$$-\int\int \{\psi^{(1),(2)}_{i\ ,\ j}(\overrightarrow{K-K'''}, \overrightarrow{K'-K''}, t)\psi^{(2),(1)}_{m\ ,\ l}(\mathbf{K}'', \mathbf{K}''', t)$$

$$+\psi^{(1),(2)}_{i\ ,\ m}(\overrightarrow{K-K'''}, K'', t)\psi^{(2),(1)}_{j\ ,\ l}(\overrightarrow{K'-K''}, \mathbf{K}''', t)\} d\mathbf{K}'' d\mathbf{K}'''$$

$$-\int\int \{\psi^{(2),(1)}_{i\ ,\ j}(\overrightarrow{K-K'''}, \overrightarrow{K'-K''}, t)\psi^{(1),(2)}_{m\ ,\ l}(\mathbf{K}'', \mathbf{K}''', t)$$

$$+\psi^{(2),(1)}_{i\ ,\ m}(\overrightarrow{K-K'''}, \mathbf{K}'', t)\psi^{(1),(2)}_{j\ ,\ l}(\overrightarrow{K'-K''}, \mathbf{K}''', t)\} d\mathbf{K}'' d\mathbf{K}'''$$

$$+\int\int \{\psi^{(2),(2)}_{i\ ,\ j}(\overrightarrow{K-K'''}, \overrightarrow{K'-K''}, t)\psi^{(2),(2)}_{m\ ,\ l}(\mathbf{K}'', \mathbf{K}''', t)$$

$$+\psi^{(2),(2)}_{i\ ,\ m}(\overrightarrow{K-K'''}, \mathbf{K}'', t)\psi^{(2),(2)}_{j\ ,\ l}(\overrightarrow{K'-K''}, \mathbf{K}''', t)\} d\mathbf{K}'' d\mathbf{K}''$$

and

$$H_2(\mathbf{K},\mathbf{K}',t) = \frac{K_i K_l}{K^2}\frac{K_j K_m}{K'^2}[\psi^{(11)}_{il}(\mathbf{K},t)\psi^{(11)}_{jm}(\mathbf{K}',t)$$

$$-\psi^{(11)}_{il}(\mathbf{K},t)\psi^{(22)}_{jm}(\mathbf{K}',t)-\psi^{(22)}_{il}(\mathbf{K}',t)\psi^{(11)}_{jm}(\mathbf{K}',t)+\psi^{(22)}_{il}(\mathbf{K},t)\psi^{(22)}_{jm}(\mathbf{K}',t).$$

3. Reduction of equation (2.14) for the case of homogeneous and isotropic models in MHD turbulence

a) case of homogeneous turbulence:

In homogeneous turbulence, the correlation functions

$$F^{(0),(0)}_{\omega\ ,\ \omega}(\mathbf{X},\mathbf{X}',t), \quad F^{(1),(1)}_{i\ ,\ j}(\mathbf{X},\mathbf{X}',t), \quad F^{(1),(2)}_{i\ ,\ j}(\mathbf{X},\mathbf{X}',t)$$

and

$$F^{(2),(2)}_{i\ ,\ j}(\mathbf{X}',\mathbf{X}',t)$$

composed of fluctuations of relevant physical quantities pertaining to the points $P(\mathbf{X}, t)$ and $P'(\mathbf{X}', t)$ are not separately dependent on $\mathbf{X}$ and $\mathbf{X}'$ but only on $\boldsymbol{\xi} = \mathbf{X}'-\mathbf{X}$. Accordingly, we may introduce a three-dimensional Dirac-delta function of the form $\delta(\mathbf{K}+\mathbf{K}')$ for simplification of Eq. (2.14) to the case of homogeneous turbulence. The spectrum tensors

$$\psi^{(0),(0)}_{\omega\ ,\ \omega}(\mathbf{K},\mathbf{K}',t), \quad \psi^{(1),(1)}_{i\ ,\ j}(\mathbf{K},\mathbf{K}',t), \quad \psi^{(1),(2)}_{i\ ,\ j}(\mathbf{K},\mathbf{K}',t), \quad \psi^{(2),(2)}_{i\ ,\ j}(\mathbf{K},\mathbf{K}',t) \quad \text{etc.}$$

appearing in Eq. (2.14) are to be replaced respectively by

$$\psi^{(0),(0)}_{\omega\ ,\ \omega}(\mathbf{K},\mathbf{K}',t)\,\delta(\mathbf{K}+\mathbf{K}'), \quad \psi^{(1),(1)}_{i\ ,\ j}(\mathbf{K},\mathbf{K}',t)\,\delta(\mathbf{K}+\mathbf{K}'),$$

$$\psi^{(1),(2)}_{i\ ,\ j}(\mathbf{K},\mathbf{K}',t)\,\delta(\mathbf{K}+\mathbf{K}'), \quad \psi^{(2),(2)}_{i\ ,\ j}(\mathbf{K},\mathbf{K}',t)\,\delta(\mathbf{K}+\mathbf{K}')$$

etc. before performing integration over the whole of $\mathbf{K}$, $\mathbf{K}'$- spaces. Taking the above into consideration, we obtain after usual calculation the simplified form of Eq. (2.14) in the of homogeneous turbulence as

$$(3.1)\quad \int \psi^{(0),(0)}_{\omega,\,\omega}(\mathbf{K}, -\mathbf{K}, t)\,d\mathbf{K}$$

$$= \int\!\!\int \frac{K^{\mathrm{IV}}_i K^{\mathrm{IV}}_l K^{\mathrm{IV}}_j K^{\mathrm{IV}}_m}{K^{iv^2}} \psi^{(1),(1)}_{i,\,j}(\mathbf{K}, -\mathbf{K}, t)\psi^{(1),(1)}_{m,\,l}(-\mathbf{K}', \mathbf{K}', t)$$

$$+\psi^{(1),(1)}_{i,\,m}(\mathbf{K}', -\mathbf{K}', t)\psi^{(1),(1)}_{j,\,l}(-\mathbf{K}, \mathbf{K}, t)-\psi^{(1),(2)}_{i,\,j}(\mathbf{K}, -\mathbf{K}, t)\psi^{(2),(1)}_{m,\,l}(-\mathbf{K}', \mathbf{K}', t)$$

$$-\psi^{(1),(2)}_{i,\,m}(\mathbf{K}', -\mathbf{K}', t)\psi^{(2),(1)}_{j,\,l}(-\mathbf{K}, \mathbf{K}, t)-\psi^{(2),(1)}_{i,\,j}(\mathbf{K}', -\mathbf{K}', t)\psi^{(1),(2)}_{m,\,l}(-\mathbf{K}', \mathbf{K}', t)$$

$$-\psi^{(2),(1)}_{i,\,m}(\mathbf{K}', -\mathbf{K}, t)\psi^{(1),(2)}_{j,\,l}(-\mathbf{K}, \mathbf{K}, t)+\psi^{(2),(2)}_{i,\,j}(\mathbf{K}', -\mathbf{K}', t)\psi^{(2),(2)}_{m,\,l}(-\mathbf{K}', \mathbf{K}', t)$$

$$+\psi^{(2),(2)}_{i,\,m}(\mathbf{K}', -\mathbf{K}', t)\psi^{(2),(2)}_{j,\,l}(-\mathbf{K}, \mathbf{K}, t)]\,d\mathbf{K}\,d\mathbf{K}' \quad \text{where} \quad K^{\mathrm{IV}} = \mathbf{K}+\mathbf{K}'$$

It is to be noted that $H_2(\mathbf{K}, \mathbf{K}', t)$ appearing in Eq (2.14) has no contribution in the simplified case of homogeneous turbulence as this disappears when requisite integrals are computed.

b) Case of homogeneous and isotropic turbulence:

In the case of homogeneous and isotropic turbulence, the convergences of the type $u_i h'_j$ (**h** being assumed skew) are to vanish identically because of reflexional symmetry (cf. TATSUMI [10]). To obtain the result in a suitable form, we multiply Eq. (2.14) by K^2 and all through follow the same procedure as adopted in deriving Eq. (3.1), and obtain in the case of homogeneous and isotropic turbulence, the equation

$$(3.2)\quad \int K^2\psi^{(0),(0)}_{\omega,\,\omega}(\mathbf{K}', -\mathbf{K}, t)\,d\mathbf{K}$$

$$= \int\!\!\int \frac{K^{\mathrm{IV}}_i K^{\mathrm{IV}}_l K^{\mathrm{IV}}_j K^{\mathrm{IV}}_m}{K^{\mathrm{IV}^2}} [\psi^{(1),(1)}_{i,\,j}(\mathbf{K}, -\mathbf{K}, t)\,\psi^{(1),(1)}_{m,\,l}(-\mathbf{K}', \mathbf{K}', t)$$

$$+\psi^{(1),(1)}_{i,\,m}(\mathbf{K}', -\mathbf{K}', t)\psi^{(1),(1)}_{j,\,l}(-\mathbf{K}, \mathbf{K}, t)+\psi^{(2),(2)}_{i,\,j}(\mathbf{K}, -\mathbf{K}, t)\psi^{(2),(2)}_{m,\,l}(\mathbf{K}, -\mathbf{K}, t)$$

$$+\psi^{(2),(2)}_{i,\,m}(\mathbf{K}', -\mathbf{K}', t)\psi^{(2),(2)}_{j,\,l}(-\mathbf{K}, \mathbf{K}, t)]\,d\mathbf{K}\,d\mathbf{K}'.$$

Let us now introduce the isotropic forms for the spectrum tensors

$$\psi^{(0),(0)}_{\omega,\,\omega}(\mathbf{K}, -\mathbf{K}, t), \quad \psi^{(1),(1)}_{i,\,j}(\mathbf{K}, -\mathbf{K}, t) \quad \text{and} \quad \psi^{(2),(2)}_{i,\,j}(K, K', t),$$

respectively, as

$$(3.3)\qquad \psi^{(0),(0)}_{\omega,\,\omega}(\mathbf{K}, -\mathbf{K}, t) = \frac{1}{4\pi K^2} E^{(0)}(k, t),$$

$$(3.4)\qquad \psi^{(1),(1)}_{i,\,j}(\mathbf{K}, -\mathbf{K}, t) = \frac{E^{(1)}(k, t)}{4\pi K^2}\left\{\delta_{ij} - \frac{K_i K_j}{K^2}\right\},$$

and

$$(3.5)\qquad \psi^{(2),(2)}_{i,\,j}(\mathbf{K}, -\mathbf{K}, t) = \frac{E^{(2)}(k, t)}{4\pi K^2}\left\{\delta_{ij} - \frac{K_i K_j}{K^2}\right\},$$

where $E^{(0)}(k, t)$ is the spectrum for the total pressure; $E^{(1)}(k, t)$, $E^{(2)}(k, t)$ are respectively the kinetic energy spectrum and magnetic energy spectrum.

Substituting Eqs. (3.3)–(3.5) and expressions similar to Eq. (3.4) and (3.5) for the second rank tensors appearing in Eq. (3.2), we obtain the reduced version of Eq. (3.2) as

$$(3.6)\qquad \int_0^\infty k^2E^{(0)}(k,t)dk = \int_0^\infty\int_0^\infty\int_{-1}^{1} \frac{Q^2}{16k^{\mathrm{IV}2}k^2k'^2}[E^{(1)}(k,t)E^{(1)}(k',t)+$$

$$+E^{(2)}(k,t)E^{(2)}(k't)]dkdk'd\mu,$$

where the space integral $d\mathbf{K}$ has been replaced by $4\pi k^2dk$ and the multiple integral $d\mathbf{K}d\mathbf{K}$ has been replaced by $4\pi k^2 dk\cdot 2\pi k'^2dk'd\mu$, μ being the cosine of the angle between $\mathbf{K}$ and $\mathbf{K}'$, and Q has the symmetric form given by

$$(3.7)\qquad Q = k^4+k'^4+k^{\mathrm{IV}4}-2k^2k'^2-2k'^2k^{\mathrm{IV}2}-2k^2jk^{\mathrm{IV}2}.$$

Now taking

$$(3.8)\qquad \overline{(\mathrm{grad}\bar{\omega})^2} = \int_0^\infty k^2E^{(0)}(k,t)dk$$

into account and effecting integration on the right hand side of Eq. (3.6) with respect to μ, the reduced version of Eq. (3.6) is obtained as

$$\overline{(\mathrm{grad}\bar{\omega})^2} = \int_0^\infty\int_0^\infty \{E^{(1)}(k)E^{(1)}(k')+E^{(2)}(k)E^{(2)}(k')\}kk'\phi\left(\frac{k}{k'}\right)dkdk',$$

where

$$\phi(s) = \phi\left(\frac{1}{s}\right) = -\frac{1}{8}(s^3-s^{-3})+\frac{11}{24}(s+s^{-1})+\frac{1}{16}(s-s^{-1})^4\ln\frac{1+s}{|1-s|}.$$

Expression (3.8) for $\overline{(\mathrm{grad}\bar{\omega})^2}$ thus obtained may be considered as a generalization of Heisenberg's expression for $\overline{(\mathrm{grad}p)^2}$ e.g.

$$(3.9)\qquad \overline{(\mathrm{grad}\,p)^2} = p^2\int_0^\infty\int_0^\infty E^{(1)}(k)E^{(1)}(k')kk'\phi\left(\frac{k}{k'}\right)dkdk'$$

to the case of homogeneous and isotropic MHD turbulence. Estimation of $\overline{(\mathrm{grad}\bar{\omega})^2}$ might be desirable for analyzing certain aspects of fluctuations of total pressure associated with the homogeneous and isotropic MHD turbulence; however, a comprehensive study will be possible when much experimental information regarding measurements of fluctuations of total pressure is available.

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INSTITUTE FOR MECHANICS,
TECHNICAL UNIVERSITY OF NORWAY, TRONDHEIM, NORWAY.

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