Arch. Mech., 36, 2, pp. 293–296, Warszawa 1984

BRIEF NOTES

On the Prandtl–Reuss equations for a large plastic strain

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IT IS A KNOWN fact [2–4] that utilizing the Zaremba–Jaumann derivative for kinematic hardening at large plastic strains gives unexpected results. For the case of monotonically increasing simple shear strain, an oscillatory shear stress is predicted. Our analysis shows that similar results may be obtained also for isotropic hardening materials. The simple shear and the uniaxial tension for the Zaremba–Jaumann, the Oldroyd and the Cotter–Rivlin stress rates were examined. To obtain more reasonable results, new definitions of the elastic and plastic parts of a total elastic-plastic deformation, which lead to a unique choice of a stress rate in the constitutive equation, were proposed. For small elastic strains, the classical flow rule, with the new plastic strain rate was used. The presented approach led to a new generalization of the Prandtl–Reuss equations, with a new stress rate instead of the Zaremba–Jaumann one.

1. Introduction

FOR A LARGE strain elastic-plastic analysis, generalized Prandtl–Reuss equations are commonly used (see. c.f. [1]). To obtain these equations, the relations of a hypo-elastic material with the Zaremba–Jaumann stress rate instead of Hooke's law are assumed. Utilizing the Zaremba–Jaumann rate for a stress, as well as for a back stress in the Prager kinematic hardening rule, Nagtegaal and de JONG [2] have obtained unexpected results. For monotonically increasing simple shear strain an oscillating shear stress has been predicted. Recently, to obtain more reasonable results, modified Zaremba–Jaumann back stress rates have been proposed [3–4]. Our analysis shows that similar, unexpected results may appear also for the isotropic hardening elastic-plastic model. In this case we may also try to modify the Zaremba–Jaumann stress rate in the Prandtl–Reuss equations. However, it seems that the best way is to reconsider constitutive relations for the elastic and plastic part of deformation. In the paper [5], HILL has shown that any objective time derivative may be used as stress rate and any valid preference must rest on practical consideration alone. Following this idea, the simple shear and uniaxial tension for the Zaremba–Jaumann, the Oldroyd and the Cotter–Rivlin stress rates have been examined in the present paper. It appeared that only the last one gave reasonable results. Regarding this fact, a new definition of an elastic strain rate was introduced. This strain rate satisfied Hooke's law referred to a the actual configuration, instead of hypo-elasticity relations. A plastic strain rate was defined as a difference between the total and the elastic strain rate. Such an approach led to natural generalization of HILL's incremental formulation [6]. The comparison with the conventional formulation, based on $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ decomposition [7], showed that for a small elastic strain the classical flow rule may be used. The new elastic strain rate enabled to present Hooke's law in an adequate differential form, and therefore to derive a new generalization of the Prandtl–Reuss equations.

2. Influence of stress rate choice on the behaviour of elastic-plastic work hardening materials

Let us consider three classes of elastic-plastic materials described by the following constitutive relations:

$$\boldsymbol{\tau}^{\nabla} = \mathcal{L}^{e-p} \cdot \mathbf{d}, \tag{2.1}$$

where $(\ )^{\nabla}$ denotes: the Zaremba–Jaumann derivative for the first class, the Oldroyd for the second and the Cotter–Rivlin for the third.

In Eq. (2.1) $\boldsymbol{\tau}$ is the Kirchhoff stress tensor, $\mathbf{d}$ is the total elastic-plastic strain rate and $\mathcal{L}^{e-p}$ is defined by the matrix of the elastic-plastic moduli described in [1]. The appropriate stress rates are the following:

$$\boldsymbol{\tau}^{\nabla J} = \dot{\boldsymbol{\tau}} - \boldsymbol{\omega}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{\omega}^T = \dot{\boldsymbol{\tau}} + \boldsymbol{\omega}^T\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\omega}, \tag{2.2}$$

$$\boldsymbol{\tau}^{\nabla_0} = \dot{\boldsymbol{\tau}} - \mathbf{L}\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{L}^T, \tag{2.3}$$

$$\boldsymbol{\tau}^{\nabla_R} = \dot{\boldsymbol{\tau}} + \mathbf{L}^T\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{L}, \tag{2.4}$$

where $\mathbf{L}$ is the velocity gradient and $\boldsymbol{\omega}$ — its antisymmetric part.

For the simple shear case the velocity field is as follows:

$$v_1 = \dot{\gamma} x_2, \quad v_2 = 0, \quad v_3 = 0, \tag{2.5}$$

where

$$\gamma = \operatorname{tg}\varphi \tag{2.6}$$

and φ denotes a shear angle.

The relation (2.1) is reduced to four equations for τ_{12}, τ_{11}, τ_{22} and τ_{33}. These equations may be solved step by step using the perturbation technique [8], when the nonlinear parts with respect to γ are perturbed. The solutions are assumed to have the form

$$\tau_{ij} = \tau_{ij}^{(0)} + \varepsilon\tau_{ij}^{(1)} + \varepsilon^2\tau_{ij}^{(2)} + \ldots. \tag{2.7}$$

For the Zaremba–Jaumann stress rate an instability of the $\tau_{12}-\gamma$ curve is observed. The first step of calculations gives the solution for the hypo-elastic material analysed by TRUESDELL [9]. The Oldroyd stress rate gives a monotonically increasing $\tau_{12}-\gamma$ curve and a tension in the shear direction as a higher order effect. For the Cotter–Rivlin stress rate, the relation $\tau_{12}-\gamma$ is the same as the previous one, but as a higher order effect; the compression between planes parallel to the shear direction appears. Considering the known fact that metal specimens elongate during torsion, the result obtained for the Cotter–Rivlin stress rate seems to be reasonable.

In a similar way we may examine the above stress rates for the uniaxial tension case. Then it will appear that each of them is admissible.

3. New definitions of the elastic and plastic part of total deformation and its consequences

Let us define the measure of elastic strain as a tensor

$$\hat{\mathbf{e}}^e \equiv (\mathcal{L}^e)^{-1}\boldsymbol{\tau}, \tag{3.1}$$

where $\mathcal{L}^e$ is the tensor of elastic moduli, and $\boldsymbol{\tau}$ is the Kirchhoff stress tensor. For purely elastically deformed material, the tensor $\hat{\mathbf{e}}^e$ becomes the Almansi strain tensor and Eq. (3.1) is Hooke's law referred to a current configuration. The measure of elastic strain rate is defined as follows

$$\hat{\mathbf{d}}^e \equiv (\overset{-1}{\mathbf{F}})^T \dot{\overline{\mathbf{F}^T \hat{\mathbf{e}} \mathbf{F}}} \overset{-1}{\mathbf{F}}, \tag{3.2}$$

where $\mathbf{F}$ is the gradient of the total elastic-plastic deformation and $\dot{\overline{(\ )}}$ denotes the material time derivative. For purely elastic deformation, the tensor $\hat{\mathbf{d}}^e$ becomes the Cotter–Rivlin rate of the Almansi strain tensor.

The measure of the plastic strain rate is assumed in the form

$$\hat{\mathbf{d}}^p \equiv \mathbf{d} - \hat{\mathbf{d}}^e, \tag{3.3}$$

where the tensor $\mathbf{d}$ is the strain rate of the total deformation. For the above definition of the plastic strain rate, we propose the classical flow rule

$$\hat{\mathbf{d}}^p = \lambda \boldsymbol{\tau}', \tag{3.4}$$

where $\boldsymbol{\tau}'$ denotes a deviatoric part of $\boldsymbol{\tau}$.

The commonly used definitions of elastic and plastic strain rate base on the concept of the multiplicative decomposition [6]

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p, \tag{3.5}$$

and are defined as follows:

$$\mathbf{d}^e \equiv (\overset{-1}{\mathbf{F}^e})^T \dot{\overline{(\mathbf{F}^e)^T \mathbf{e}^e \mathbf{F}^e}} \overset{-1}{\mathbf{F}^e}, \tag{3.6}$$

$$\mathbf{d}^p \equiv (\overset{-1}{\mathbf{F}^p})^T \dot{\overline{(\mathbf{F}^p)^T \mathbf{e}^p \mathbf{F}^p}} \overset{-1}{\mathbf{F}^p} \tag{3.7}$$

where

$$\mathbf{e}^\alpha \equiv \frac{1}{2} [\mathbf{I} - (\overset{-1}{\mathbf{F}^\alpha})^T \overset{-1}{\mathbf{F}^\alpha}], \qquad \alpha = e, p. \tag{3.8}$$

A comparison of the newly defined strain rates with the classical ones gives us

$$\hat{\mathbf{d}}^e = (\overset{-1}{\mathbf{F}})^T \dot{\overline{\mathbf{F}^T e^e \mathbf{F}}} \overset{-1}{\mathbf{F}}, \tag{3.9}$$

$$\hat{\mathbf{d}}^p = (\overset{-1}{\mathbf{F}^e})^T \mathbf{d}^p \overset{-1}{\mathbf{F}^e}. \tag{3.10}$$

From Eq. (3.10) it yields that for small elastic strains the proposed flow rule (3.4) is equivalent to the classical one:

$$\mathbf{d}^p = \lambda \boldsymbol{\tau}'. \tag{3.11}$$

With regard to Eq. (3.2) Hooke's law

$$\boldsymbol{\tau} = \mathscr{L}^e \cdot \hat{\mathbf{e}}^e, \tag{3.12}$$

may be presented in the following differential form:

$$\boldsymbol{\tau}^{\nabla H} = \mathscr{L}^e \cdot \mathbf{d}, \tag{3.13}$$

where

$$\boldsymbol{\tau}^{\nabla H} \equiv \mathscr{L}^e [(\mathscr{L}^e)^{-1} \cdot \boldsymbol{\tau}]^{\nabla R}, \tag{3.14}$$

is a new stress rate measure. This measure is an objective one and depends on the elastic moduli tensor. For the isotropic $\mathscr{L}^e$ tensor one can write

$$\boldsymbol{\tau}^{\nabla H} = \dot{\boldsymbol{\tau}} + \mathbf{L}^T \boldsymbol{\tau} + \boldsymbol{\tau}\, \mathbf{L} + \frac{2\nu}{1-2\nu} \mathbf{I} \left[\operatorname{tr}(\mathbf{d}\boldsymbol{\tau}) - \frac{\nu}{1+\nu} \operatorname{tr} \mathbf{d} \operatorname{tr} \boldsymbol{\tau} \right] - \frac{2\nu}{1+\nu} \mathbf{d} \operatorname{tr} \boldsymbol{\tau}. \tag{3.15}$$

For $\nu = 0$, $\boldsymbol{\tau}^{\nabla H}$ is equal the Cotter–Rivlin stress rate $\boldsymbol{\tau}^{\nabla R}$. If σ_y is the yield stress and $\bar{\varepsilon}^p$ describes a length of the plastic strain path, the assumed Huber–Mises yield condition has the form

$$\varphi(\tau_{ij}) = \frac{1}{2}\sigma'_{ij}\sigma'_{ij} - \frac{1}{2}\sigma_y^2(\bar{\varepsilon}^p) = 0. \tag{3.16}$$

Then

$$\dot{\varphi} = \frac{\partial\varphi}{\partial\tau_{ij}}\dot{\tau}_{ij} + \frac{\partial\varphi}{\partial\bar{\varepsilon}^p}h(\bar{\varepsilon}^p)\dot{\bar{\varepsilon}}^p = 0. \tag{3.17}$$

The above expression and Eq. (3.15) enable us to obtain a new generalization of the Prandtl–Reuss equations:

$$\boldsymbol{\tau}^{\nabla H} = \hat{\mathscr{L}}^{e-p}\cdot\mathbf{d}, \tag{3.18}$$

where

$$\hat{\mathscr{L}}^{e-p}_{ijkl} \equiv \mathscr{L}^{e-p}_{ijkl} + \frac{3\sigma'_{ij}\varkappa_{kl}\dfrac{E}{1+\nu}}{2\sigma_y^2(\bar{\varepsilon}^p)\left[\dfrac{2}{3}h(\bar{\varepsilon}^p) + \dfrac{E}{1+\nu}\right]}, \tag{3.19}$$

$$\varkappa_{kl} \equiv 2\left(\sigma'_{kp}\sigma'_{pl} - \frac{\nu}{1+\nu}\sigma_{pp}\sigma'_{kl}\right), \tag{3.20}$$

and all the remaining denotations are the same as used in [1]. An examination of Eq. (3.18) for the simple shear case gives us a monotonically increasing $\tau_{12}-\gamma$ curve, and the compression in the x_1, x_2, x_3 directions as a higher order effect. Also the uniaxial tension test leads to results consistent with the assumed work hardening relation.

Concluding one can say that the presented considerations give a simple and consistent description of elastic-plastic behaviour at finite strain.

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Received January **7**, **1983.**