Generalized continuum theories for directionally reinforced solids

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A THEORY of elasticity with microstructure for fiber reinforced composites with a rectangular array of the fibers is outlined. The theory is based on expansions of the displacements across representative cells. The transition from the actually inhomogeneous composite to a homogeneous continuum is achieved by introducing continuous fields for gross displacements and local deformations. The elastic constants are expressed explicitly in terms of the constants defining the mechanical behavior of the fibers and the matrix material, and the parameters describing the geometrical layout of the composite. A number of specific examples involving the dispersive behavior of time-harmonic waves propagating in directions parallel and normal to the fibers are discussed. These cases are relevant to available data of ultrasonic tests on composite materials. Analytical results are compared with test data, and with some results obtained by the finite element technique.

Naszkicowano teorię sprężystości z mikrostrukturą dla kompozytów wzmocnionych włóknami przy rozmieszczeniu włókien w kształcie prostokątnej tablicy. Teoria bazuje na rozwinięciach przemieszczeń względem reprezentatywnych komórek. Przejście od niejednorodnego do jednorodnego kompozytu osiągnięto dzięki wprowadzeniu ciągłych pól dla dużych przemieszczeń i lokalnych odkształceń. Stałe sprężyste wyrażono jawnie przez stałe określające mechaniczne własności materiału włókien i matrycy oraz parametry opisujące geometryczną strukturę kompozytu. Przedyskutowano kilka odpowiednich przykładów opisujących dyspersyjne zachowanie się fal harmonicznych rozprzestrzeniających się w kierunku równoległym i prostopadłym do włókien. Wybrano takie przypadki, dla których istnieją dostępne dane doświadczalne uzyskane w badaniach ultradźwiękowych, przeprowadzonych na próbkach wykonanych z materiału kompozytowego. Wyniki numeryczne porównano z danymi doświadczalnymi oraz z niektórymi wynikami otrzymanymi metodą elementów skończonych.

Набросана теория упругости с микроструктурой для композитов упрочненных волокнами при распределении волокон в форме прямоугольной таблицы. Эта теория базирует на разложениях перемещений по отношению к представительным ячейкам. Переход от неоднородного к однородному композиту достигнут благодаря введению непрерывных полей для больших перемещений и локальных деформаций. Упругие постоянные выражены явно через постоянные, определяющие механические свойства материала волокон и матрицы, а также параметры, описывающие геометрическую структуру композита. Обсуждено несколько соответствующих примеров, описывающих дисперсионное поведение гармонических волн, распространяющихся в направлении параллельном и перпендикулярном к волокнам. Избраны такие случаи, для которых существуют доступные экспериментальные данные, полученные в ультразвуковых исследованиях, проведенных на образцах, изготовленных из композитного материала. Численные результаты сравнены с экспериментальными данными, а также с некоторыми результатами полученными методом конечных элементов.

1. Introduction

FOR many practical purposes the mechanical response of a directionally reinforced composite can be analyzed adequately on the basis of a theory which accounts for the gross mechanical behavior of the composite material. Gross mechanical behavior is described

by averages of field variables over representative elements. In the simplest theory the averaged stresses are related to averaged strains by means of effective elastic constants. In this theory, which is termed the "effective modulus theory", the mechanical response of the composite medium is equivalent to the response of a homogeneous but generally anisotropic medium whose "effective moduli" are determined in terms of the elastic moduli of the constituents and the parameters describing the geometrical layout of the composite. The computation of effective moduli has been a research topic of long-standing interest.

The effective modulus theory is useful if a pertinent length parameter characterizing the structuring of the composite is sufficiently smaller than a characteristic length of the deformation. Consequently it is conceivable that a number of interesting problems, mainly of a dynamic nature, cannot be analyzed adequately on the basis of the effective modulus theory. It should be noted, for example, that the effective modulus theory cannot account for dispersion of free harmonic waves in an unbounded body, i.e., the dependence of the phase velocity and the group velocity on the wavelength. It has, however, been verified experimentally that dispersion is pronounced if the wavelength is of the same order of magnitude as a characteristing length of the structuring. Wave propagation experiments on composite materials have also revealed higher modes of wave propagation, sometimes called optical modes, in addition to the two lowest modes (which are usually called the acoustical modes). Only the acoustical modes, and then without dispersion, can be described by the effective modulus theory.

The propagation of harmonic waves in a layered composite consisting of alternating layers of two elastic materials can be analyzed rigorously, see e.g. Refs. [1, 2 and 3]. It is also quite simple to construct an effective modulus theory for a laminated medium, as shown in Ref. [4]. Thus, a laminated composite provides a very suitable model to display the limitations of the effective modulus theory. This was done in Ref. [5].

The exact results for a laminated medium presented in Ref. [3] exhibit the different nature of the dispersive behavior for harmonic waves propagating in the direction of the layering, and normal to the layering. For waves propagating along the layering, the layers act as waveguides, and there are no stop-bands, i.e., frequency ranges in which propagating harmonic waves are not possible. For waves propagating normal to the layering, the dynamic interaction between neighboring layers does generate stop-bands, which are very similar to those found in elastic lattices (see e.g. Ref. [6]).

Exact solutions within the context of classical elasticity theory are not available for fiber-reinforced composites. It is, however, to be expected that qualitatively an analogous difference should exist for dispersion of waves propagating in the direction of the fibers, and normal to the fiber-direction. This expectation has been confirmed by experimental results presented in Refs. [7 and 8].

For a laminated medium the restrictions of the effective modulus theory have motivated the formulation of an extension of that theory, see [5]. The extended theory, which is known as the effective stiffness theory, can describe typical dynamic effects due to the structuring. The displacement equations of the effective stiffness theory which were derived in [5] were used to investigate the propagation of plane harmonic waves in the directions parallel to the layering and normal to the layering. The limiting phase velocities at vanish-

ing wavenumbers agreed with the constant phase velocities according to the effective modulus theory, as well as with the limiting phase velocities obtained from an exact treatment. For the lowest modes and for waves propagating parallel to the layering the dispersion curves according to the theory of Ref. [5] agreed over a significant range of wavenumbers with the exact dispersion curves. It was subsequently shown in [9] that the theory can be refined to increase the accuracy. Higher order theories were reviewed in Ref. [10]. For waves propagating normal to the layering, the theories that have been worked out thus far remain, however, unsatisfactory, even with higher order terms. A more accurate representation of the interaction between neighboring layers should produce the desirable improvements.

It is a legitimate question whether the relatively small wavelengths, with frequencies in the megahertz range, at which dispersion effects and higher modes occur, are of practical significance from the point of view of technological applications. In a few examples of pulse propagation that have been worked out, see Refs. [11 and 12], it has been shown that the contribution of the lowest mode often predominates. It has also been shown, however, that the curvature of the phase velocity versus wavenumber curve at zero value of the wavenumber governs the shape of the pulse at larger values of time. On the other hand, there are several other potential sources of dispersion, such as the overall boundaries of the body and inelastic behavior of the constituents, which may produce dispersion predominating that due to the structuring of the composite.

An important motivation for a detailed study of the propagation of harmonic waves in fiber reinforced composites is that effective elastic constants can conveniently be measured by ultrasonic testing techniques. These techniques have the advantage that small specimens can be used, and that good control and reproducibility can be achieved. Typically one measures phase velocity or group velocity for a number of frequencies. The extrapolation to zero frequency then provides the velocities for very long waves, from which the effective elastic constants can be determined. Clearly, these testing procedures require a good understanding of the dynamic behavior of composite materials. The major part of this paper is, therefore, concerned with wave motions that are relevant to ultrasonic testing techniques.

A continuum theory which models more accurately the structuring of a fiber-reinforced composite can, of course, be expected to yield better results for smaller characteristic lengths of deformation. In recent years several attempts have been made towards the development of such more accurate theories. On the basis of kinematical considerations within a typical cell of the directionally reinforced composite, a hierarchy of theories can be developed, which shows a close resemblance to the generalized continuum theories which were apparently first introduced in the literature towards the end of the nineteenth century.

As a research topic in theoretical and applied mechanics, generalized continuum theories enjoyed a renewed interest and a brief period of glory in the fifties and the sixties. Much of this interest was from an abstract theoretical mechanics point of view. Several theories were formulated and several specific problems were solved. The enthusiasm waned, however, when few practical applications could be found. In particular, it turned out to be difficult to relate the multitude of material constants appearing in these generalized

continuum theories to the structuring of real materials, either by theoretical considerations or on the basis of experimental results.

In this paper the applicability of generalized continuum theories to the mechanics of directionally reinforced solids with periodic structuring is discussed. A theory of elasticity with microstructure is outlined for fiber reinforced composites with a rectangular array of the fibers. The theory is based on expansions of the displacements across representative cells. The transition from the actually inhomogeneous composite to a homogeneous continuum is achieved by introducing continuous fields for gross displacements and local deformations. The elastic constants of this theory of elasticity with microstructure are expressed explicitly in terms of the constants defining the mechanical behavior of the fibers and the matrix material, and the parameters describing the geometrical layout of the composite. The potential applicability of generalized continuum theories to describe the mechanical behavior of composite materials was earlier discussed by RIVLIN [13, 14] and HERRMANN and ACHENBACH [15].

A number of specific examples involving the dispersive behavior of time-harmonic waves propagating in directions parallel and normal to the fibers are discussed in this paper. The results are compared with some recent experimental results, and with results from other theories, including some obtained by finite element techniques.

Alternative approaches to the one discussed here, have been presented by other authors. Among these we mention mixture theories [16, 17], and variational methods [18, 19].

2. A homogeneous continuum model

In this section the basic ideas for the construction of generalized continuum theories for fiber-reinforced composites are presented with reference to a specific model. The equations governing a general state of deformation, according to this particular model, are lengthly. These equations have already been presented in some detail in Ref. [20] and they are, therefore, not listed once again in this paper.

We consider a fiber-reinforced composite consisting of uni-directional fibers embedded in a matrix material. It is assumed that the fibers are cylindrical rods of radius a arranged in rectangular arrays. The distances between the center lines of the fibers are d_2 and d_3, in the x_2- and x_3-directions, respectively, as shown in Fig. 1. Each fiber is identified by two indices: the first index identifies the row and the second index identifies the column in which the fiber is located. The position of the center line of fiber (k, l) is defined by $x_2 = x_2^l$ and $x_3 = x_3^k$. The elastic constants of the high-modulus reinforcing fibers and the low-modulus matrix material are denoted by λ_f, μ_f and λ_m, μ_m, respectively.

To describe the displacement field the fiber-reinforced medium is divided into strips by the planes of structural symmetry of the composite, see Fig. 1. Each strip is of width d_2 and of height d_3, and each strip contains one fiber. We focus attention on the strip which contains fiber (k, l). An element of unit length of this strip is labeled cell (k, l). Next, we define a system of local cylindrical coordinates r, θ, x_1, as well as a system of local Cartesian coordinates $x_1, \bar{x}_2, \bar{x}_3$, see Fig. 2. Now, provided that the characteristic length of the deformation is sufficiently larger than either d_2 or d_3, the displacements
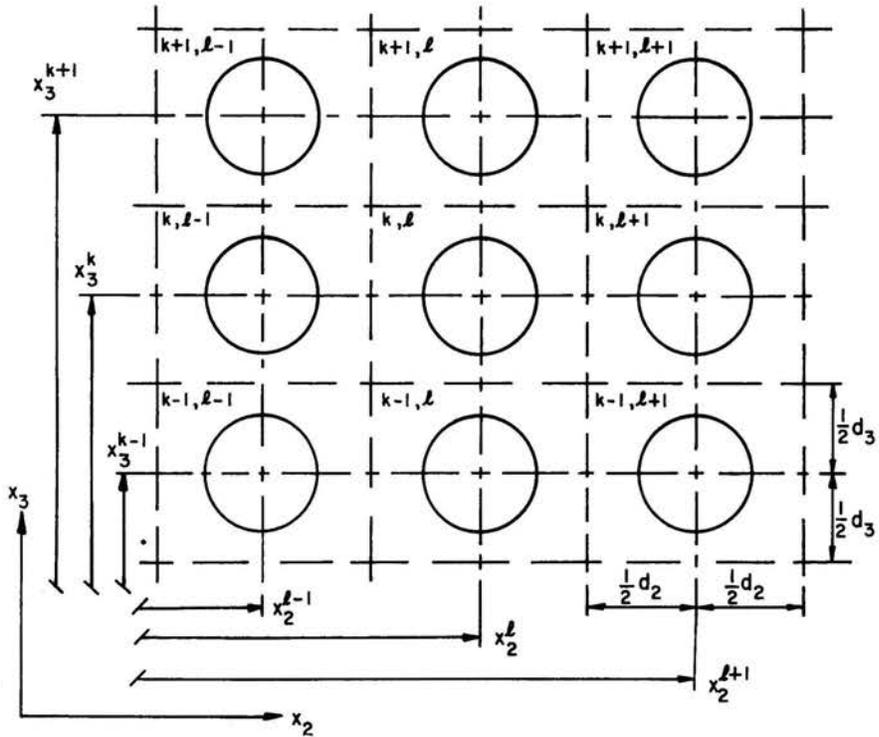
FIG. 1. Fiber-reinforced composite.
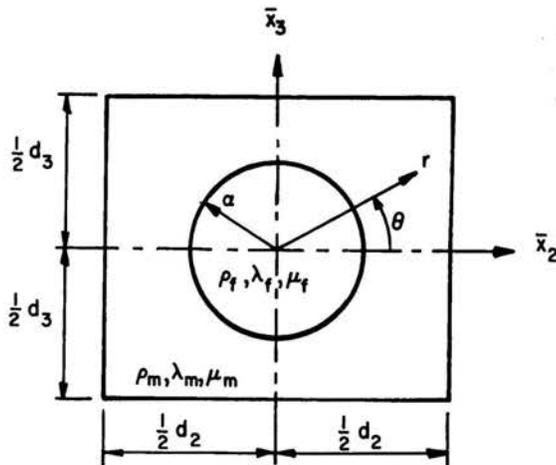
FIG. 2. Cell (k, l) of the fiber-reinforced composite.

in the cell can be approximated by expansions in terms of quantities which are defined at the center line of the fiber, which is also the center line of the cell. These expansions are analogous to the expansions used in rod theories. For the particular model considered here, we consider linear expansions of the forms (in indicial notation $i = 1, 2, 3$):

in fiber (k, l), $(r < a)$:

$$(2.1) \qquad\qquad u_i^{f(k,l)} = \bar{u}_i^{(k,l)} + r\cos\theta\psi_{3i}^{f(k,l)} + r\sin\theta\psi_{3i}^{f(k,l)};$$

in the matrix material of cell (k, l), $(r > a)$:

$$(2.2) \qquad u_i^{m(k,l)} = \bar{u}_i^{(k,l)} + a\cos\theta\psi_{2i}^{f(k,l)} + a\sin\theta\psi_{3i}^{f(k,l)} + (r-a)\cos\theta\psi_{2i}^{m(k,l)} + (r-a)\sin\theta\psi_{3i}^{m(k,l)}.$$

Thus, the displacement in the matrix is expressed as the displacement at the fiber-matrix interface plus additional terms which increase linearly with the distance from the interface. By expressing $u_i^{m(k,l)}$ in the form (2.2) the displacement satisfies the condition of continuity at the fiber-matrix interface. Equation (2.2) can also be written in the form

$$(2.3) \qquad u_i^{m(k,l)} = \bar{u}_i^{(k,l)} + a\cos\theta(\psi_{2i}^{f(k,l)} - \psi_{2i}^{m(k,l)}) + a\sin\theta(\psi_{3i}^{f(k,l)} - \psi_{3i}^{m(k,l)}) + \bar{x}_2\psi_{2i}^{m(k,l)} + \bar{x}_3\psi_{3i}^{m(k,l)}.$$

The field quantities and their dependence on the coordinates are summarized as

> gross displacements $\qquad \bar{u}_i^{(k,l)}(x_1, x_2^l, x_3^k, t)$;
>
> local fiber deformations $\quad \psi_{2i}^{f(k,l)}(x_1, x_2^l, x_3^k, t), \qquad \psi_{3i}^{f(k,l)}(x_1, x_2^l, x_3^k, t)$;
>
> local matrix deformations $\psi_{3i}^{m(k,l)}(x_1, x_2^l, x_3^k, t), \qquad \psi_{3i}^{m(k,l)}(x_1, x_2^l, x_3^k, t)$.

Note that within the actual fiber-reinforced composite the gross displacements and the local deformations are defined at discrete values of x_2 and x_3, but they are continuous functions of x_1 and t.

The displacements should be continuous at the interfaces between cell (k, l) and the neighboring cells. It is, however, not possible to require point by point continuity. What can be done is to impose the condition that the average displacement is continuous at the interfaces of the cells. Thus, at the interface between cells (k, l) and $(k, l+1)$ we require

$$(2.4) \qquad \int_{-\frac{1}{2}d_3}^{\frac{1}{2}d_3} \{[u_i^{m(k,l+1)}]_{x_2 = -\frac{1}{2}d_2} - [u_i^{m(k,l)}]_{x_2 = \frac{1}{2}d_2}\} d\bar{x}_3 = 0.$$

Substituting Eq. (2.3) into (2.4) we obtain upon working out the integrals

$$(2.5) \qquad \bar{u}_i^{(k,l+1)} - \bar{u}_i^{(k,l)} - \frac{ad_2}{d_3}\ln[\zeta + (1+\zeta^2)^{1/2}](\psi_{2i}^{f(k,l+1)} - \psi_{2i}^{m(k,l+1)}$$

$$+ \psi_{2i}^{f(k,l)} - \psi_{2i}^{m(k,l)}) - \frac{1}{2}d_2(\psi_{2i}^{m(k,l+1)} + \psi_{2i}^{m(k,l)}) = 0.$$

In Eq. (2.5) the ratio ζ is defined as

$$(2.6) \qquad\qquad\qquad \zeta = \frac{d_3}{d_2}.$$

The displacement expansions (2.1) and (2.3) can be used to compute the corresponding strains. Substituting Eq. (2.1) into the expression for the components of the small strain tensor,

$$(2.7) \qquad\qquad\qquad \varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j),$$

in which

(2.8) $$\partial_j u_i = \partial u_i / \partial x_j,$$

and where the differentiation in the x_2- and x_3-directions should be with respect to the local coordinates $\bar{x}_2$ and $\bar{x}_3$, we find

(2.9) $$\varepsilon_{\gamma\gamma}^{f(k,l)} = \psi_{\gamma\gamma}^{f(k,l)} \quad \text{(no summation)},$$

(2.10) $$\varepsilon_{23}^{f(k,l)} = \frac{1}{2}\,(\psi_{23}^{f(k,l)} + \psi_{32}^{f(k,l)}),$$

(2.11) $$\varepsilon_{11}^{f(k,l)} = \partial_1 \bar{u}_1^{(k,l)} + \bar{x}_2\,\partial_1 \psi_{21}^{f(k,l)} + \bar{x}_3\,\partial_1 \psi_{31}^{f(k,l)},$$

(2.12) $$\varepsilon_{1\gamma}^{f(k,l)} = \varepsilon_{\gamma 1}^{f(k,l)} = \frac{1}{2}\,(\partial_1 \bar{u}_{\gamma}^{(k,l)} + \bar{x}_2\,\partial_1 \psi_{2\gamma}^{f(k,l)} + \bar{x}_3\,\partial_1 \psi_{3\gamma}^{f(k,l)} + \psi_{\gamma 1}^{f(k,l)}).$$

In Eqs. (2.9)–(2.12) the Greek index γ can assume the values 2 or 3 only. An analogous set of strains can be obtained from the displacement distributions in the matrix material. These strains are listed in Ref. [20].

In an isotropic linearly elastic body the strain energy density can be written as

(2.13) $$W = \frac{1}{2}\,(\lambda + 2\mu)\,(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2) + \lambda(\varepsilon_{11}\varepsilon_{22} + \varepsilon_{11}\varepsilon_{33} + \varepsilon_{22}\varepsilon_{33}) + 2\mu(\varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{13}^2),$$

where λ and μ are Lamé's elastic constants, and ε_{ij} are the components of the strain tensor.

Substitution of the strains (2.9)–(2.12) into W, and integration over A_f, where

(2.14) $$A_f = \pi a^2$$

is the cross-sectional area of the fiber, yields the strain energy $W^{f(k,l)}$ stored in the fiber element of cell (k, l). Substitution of the expressions for the strains in the matrix material over A_m, where

(2.15) $$A_m = d_2 d_3 - \pi a^2,$$

yields the strain energy $W^{m(k,l)}$ stored in the matrix material of the cell (k, l). The total strain energy averaged over the volume of cell (k, l) is

(2.16) $$W_{\text{ave}}^{(k,l)} = \frac{1}{d_2 d_3}\,(W^{f(k,l)} + W^{m(k,l)}).$$

The displacement expansions (2.1)–(2.3) can also be used to compute particle velocities. For the kinetic energy stored in the fiber element of cell (k, l) we have

(2.17) $$T^{f(k,l)} = \frac{1}{2}\,\varrho_f \int\!\!\int_{A_f} \sum_{i=1}^{3} (\dot{u}_i^{f(k,l)})^2 dA_f.$$

By employing Eq. (2.1) we find

(2.18) $$T^{f(k,l)} = \frac{1}{2}\,\varrho_f \sum_{i=1}^{3} \{A_f(\dot{\bar{u}}_i^{(k,l)})^2 + I_3^f(\dot{\psi}_{2i}^{f(k,l)})^2 + I_2^f(\dot{\psi}_{3i}^{f(k,l)})^2\},$$

where

(2.19)
$$I_2^f = \int\int_{A_f} \bar{x}_3^2 dA_f,$$

(2.20)
$$I_3^f = \int\int_{A_f} \bar{x}_2^2 dA_f.$$

The kinetic energy stored in the matrix material of cell (k, l), $T^{m(k,l)}$, can be computed in the same manner. The total kinetic energy stored in cell (k, l) is the sum of Eq. (2.18) and $T^{m(k,l)}$. The average over the volume of the cell is

(2.21)
$$T_{\text{ave}}^{(k,l)} = \frac{1}{d_2 d_3} (T^{f(k,l)} + T^{m(k,l)}).$$

The displacement distribution in the fiber-reinforced composite is now described by the field variables $\bar{u}_i^{(k,l)}$, $\psi_{2i}^{f(k,l)}$, $\psi_{3i}^{f(k,l)}$, $\psi_{2i}^{m(k,l)}$ and $\psi_{3i}^{m(k,l)}$. These variables are defined only on discrete lines $x_2 = x_2^l$ and $x_3 = x_3^k$. To obtain a continuum model we now introduce fields that are continuous in x_2 and x_3, and whose values at $x_2 = x_2^l$ and $x_3 = x_3^k$ coincide with the values of the actual field variables at the center lines of the cells. The step is indicated by writing $\bar{u}_i(x_j, t)$ rather than $u_i^{(k,l)}(x_1, x_2^l, x_3^k, t)$, etc. In this manner five continuous fields are introduced:

gross displacements $\bar{u}_i(x_j, t)$,
local deformations $\psi_{2i}^f(x_j, t)$ and $\psi_{3i}^f(x_j, t)$,
                                $\psi_{2i}^m(x_j, t)$ and $\psi_{3i}^m(x_j, t)$.

As a direct implication of the foregoing step we can also state a strain energy density $W(x_i, t)$ and a kinetic energy density $T(x_i, t)$, which are continuous functions of x_i and t, and whose values at $x_2 = x_2^k$ and $x_3 = x_3^l$ agree with $W_{\text{ave}}^{(k,l)}$ and $T_{\text{ave}}^{(k,l)}$. The explicit expressions for $W(x_i, t)$ and $T(x_i, t)$ are stated in Ref. [20].

It remains to examine what happens to the continuity conditions in the transition from the system of variables defined in discrete planes to the system of continuous variables. Considering $\bar{u}_i$, etc., as continuous functions of x_2 and x_3, Eq. (2.5) is a difference relation of the form

(2.22) $\Delta_2 \bar{u}_i - \dfrac{ad_2}{d_3} \ln[\zeta + (1+\zeta^2)^{1/2}](2\psi_{2i}^f + \Delta_2 \psi_{2i}^f - 2\psi_{2i}^m - \Delta_2 \psi_{2i}^m)$

$$- \frac{1}{2} d_2 (2\psi_{2i}^m + \Delta_2 \psi_{2i}^m) = 0.$$

In Eq. (2.22) the field variables are considered at $x_2 = x_2^l$, $x_3 = x_3^k$. The difference $\Delta_2 \bar{u}_i$ is defined as

(2.23)
$$\Delta_2 \bar{u}_i = \bar{u}_i\big|_{x_2 = x_2^{l+1}} - \bar{u}_i\big|_{x_2 = x_2^l},$$

with analogous definitions for $\Delta_2 \psi_{3i}^f$ and $\Delta_2 \psi_{3i}^m$. Noting that $x_2^{l+1} = x_2^l + d_2$, we see that in the limit $d_3 \to 0$, $d_2 \to 0$, but keeping $\zeta = d_3/d_2$ and a/d_3 constant, the difference relation (2.22) can be replaced by the differential relation

(2.24)
$$S_{2i} = \partial_2 \bar{u}_i - \frac{2a}{d_3} \ln[\zeta + (1+\zeta^2)^{1/2}] (\psi_{2i}^f - \psi_{2i}^m) - \psi_{2i}^m = 0.$$

Similarly we obtain for displacement continuity between cells (k, l) and $(k+1, l)$

$$(2.25) \qquad S_{3i} = \partial_3 \bar{u}_i - \frac{2a}{d_2} \ln \left[\frac{1 + (1 + \zeta^2)^{1/2}}{\zeta} \right] (\psi_{3i}^f - \psi_{3i}^m) - \psi_{3i}^m = 0.$$

It is now assumed that (2.24) and (2.25) are also valid for finite values of d_2 and d_3. The continuity conditions in the system of discrete cells have thus been turned into constraint conditions between the continuous field variables.

At this stage we have constructed a strain energy density as an expression in terms of local deformations and the gradients of the local deformations and the gross displacements. A kinetic energy density has been obtained in terms of the first-order time derivatives of the gross displacements and the local deformations. Considering a fixed regular region V of the medium, the displacement equations of motion can then be obtained by invoking Hamilton's principle for independent variations of the dependent field quantities in V and in a specified time interval $t_0 \leqslant t \leqslant t_1$. For the region V, Hamilton's principle states that

$$(2.26) \qquad \delta \int_{t_0}^{t_1} \int_V (T - W) \, dt \, dV + \int_{t_0}^{t_1} \delta W_1 \, dt = 0,$$

where δW_1 is the variation of the work done by external forces and dV is the scalar volume element. Here we are interested only in the displacement equations of motion and we restrict the admissible variations to ones that vanish identically on the bounding surface of V. In the absence of body forceds the variational problem then reduces to finding the Euler equations for

$$(2.27) \qquad \delta \int_{t_0}^{t} \int_V F \, dt \, dV = 0,$$

where the functional F is defined as

$$(2.28) \qquad F = T - W.$$

An elegant and convenient method of taking the continuity conditions (2.24) and (2.25) into account is to introduce them as subsidiary conditions through the use of Lagrangian multipliers. The variational problem may then be redefined by using the functional

$$(2.29) \qquad F = T - W - \sum_{i=1}^{3} (\Gamma_{2i} S_{2i} + \Gamma_{3i} S_{3i}),$$

in Eq. (2.27), where the Lagrangian multipliers Γ_{2i} and Γ_{3i} are functions of x_j and t. Since the functional F as given by Eq. (2.29) depends only on the dependent field variables and their first-order derivatives, the system of Euler equations may be written as

$$(2.30) \qquad \sum_{r=1}^{4} \frac{\partial}{\partial q_r} \left[\frac{\partial F}{\partial (\partial f_s / \partial q_r)} \right] - \frac{\partial F}{\partial f_s} = 0.$$

In Eq. (2.30), f_s represents the dependent variables $\bar{u}_i$, $\psi_{\gamma i}^f$, $\psi_{\gamma i}^m$ and $\Gamma_{\gamma i}$, and q_r are the spatial variables x_i and time t. A system of 21 governing equations follows from the Euler equation (2.30) and from the constraint conditions (2.24) and (2.25).

3. Particular examples: theory and experiment

The model described in the previous section can represent the effects of the discrete structuring of a fiber-reinforced composite on the mechanical field variables. It may be expected that the model is superior to the effective modulus theory, especially for dynamic problems.

One check on the accuracy of the model is a comparison with experimental results of the phase velocity at various frequencies, for specific harmonic wave motions. This comparison shows that linear expansions within a cell give good results, over a substantial range of frequencies, for transverse waves propagating in the direction of the fibers. The analytical results are valid over a smaller range of frequencies for longitudinal waves propagating in the direction of the fibers, and over a still smaller range for longitudinal waves propagating normal to the fibers.

In this section it is shown that the homogeneous continuum model can be improved by using more accurate displacement distributions, and by improving the representation of the interaction between neighboring cells. We will consider a number of special cases for which experimental results are available.

3.1. Transverse waves propagating in the direction of the fibers

For a number of frequencies, measurements of what is thought to be the group velocity, have been presented by TAUCHERT and GUZELSU [7].

A simple theory for transverse waves can be based on the following assumed displacement representations in cell (k, l), see Fig. 2:

in fiber (k, l), $(r < a)$:

$$(3.1) \qquad u_1^{f(k,l)} = \bar{x}_2 \psi_{21}^{f(k,l)},$$

$$(3.2) \qquad u_2^{f(k,l)} = \bar{u}_2^{(k,l)};$$

in the matrix material of cell (k, l)

$$(3.3) \qquad u_2^{m(k,l)} = \bar{u}_2^{(k,l)};$$

$$\bar{x}_2 \geqslant 0, \; |\bar{x}_3| \leqslant a: u_1^{m(k,l)} = (a^2 - \bar{x}_3^2)^{1/2} \psi_{21}^{f(k,l)} + [\bar{x}_2 - (a^2 - \bar{x}_3^2)^{1/2}] \psi_{21}^{m(k,l)},$$

$$(3.4) \qquad a \leqslant |\bar{x}_3| \leqslant \frac{1}{2} d_3: u_1^{m(k,l)} = \bar{x}_2 \psi_{21}^{m(k,l)},$$

$$\bar{x}_2 \leqslant 0, \; |\bar{x}_3| \leqslant a: u_1^{m(k,l)} = -(a^2 - \bar{x}_3^2)^{1/2} \psi_{21}^{m(k,l)} + [\bar{x}_2 + (a^2 - \bar{x}_3^2)^{1/2}] \psi_{21}^{m(k,l)}.$$

The corresponding strains are

$$(3.5)_{1,2} \qquad \begin{aligned} \varepsilon_{12}^{f(k,l)} &= \frac{1}{2} (\partial_1 \bar{u}_2^{(k,l)} + \psi_{21}^{f(k,l)}), \\ \varepsilon_{11}^{f(k,l)} &= \bar{x}_2 \partial_1 \psi_{21}^{f(k,l)}; \end{aligned}$$

$$(3.6)_{1,2} \qquad \begin{aligned} \varepsilon_{12}^{m(k,l)} &= \frac{1}{2} (\partial_1 \bar{u}_2^{(k,l)} + \psi_{21}^{m(k,l)}), \\ \varepsilon_{11}^{m(k,l)} &= \bar{x}_2 \partial_1 \psi_{21}^{m(k,l)} \end{aligned}$$

or

(3.6)$_3$
$$\varepsilon_{11}^{m(k,l)} = \pm (a^2 - \bar{x}_3^2)^{1/2} \partial_1 \psi_{21}^{f(k,l)} + [\bar{x}_2 \pm (a^2 - \bar{x}_3^2)^{1/2}] \partial_1 \psi_{21}^{m(k,l)}.$$

The strains $\varepsilon_{13}^{m(k,l)}$ are neglected.

Next, we will consider the conditions at the interface between cell (k, l) and cell $(k+1, l)$. Since periodicity with respect to x_2 implies $\psi_{21}^{f(k,l)} = \psi_{21}^{f(k+1,l)}$, and $\psi_{21}^{m(k,l)} = \psi_{21}^{m(k+1,l)}$, we obtain by virtue of Eq. (2.4):

(3.7)
$$\psi_{21}^{m(k,l)} = -\frac{\eta}{1-\eta} \psi_{21}^{f(k,l)},$$

where η is the volume density of the fibers, i.e.,

(3.8)
$$\eta = \frac{A_f}{A_f + A_m}.$$

The strains given by Eqs. (3.5)$_{1,2}$ and (3.6)$_{1-3}$ are now substituted in Eq. (2.13), and the resulting expressions are integrated over the appropriate regions of cell (k, l). The computation of the total strain energy averaged over the volume of cell (k, l), as defined by Eq. (2.16), and the subsequent transition to the continuum model, as described following Eq. (2.21), then yield the following strain energy density

(3.9)
$$W = \frac{1}{2} a_1 (\partial_1 \bar{u}_2)^2 + a_2 (\partial_1 \bar{u}_2) \psi_{21}^f + \frac{1}{2} a_3 (\psi_{21}^f)^2 + \frac{1}{2} a_4 (\partial_1 \psi_{21}^f)^2.$$

Here we have used the relation between ψ_{21}^m and ψ_{21}^f given by Eq. (3.7). The constants are:

(3.10)$_{1-4}$
$$a_1 = \eta \mu^f + (1-\eta) \mu^m,$$
$$a_2 = (\mu^f - \mu^m) \eta,$$
$$a_3 = \eta \mu^f + \frac{\eta^2}{1-\eta} \mu^m,$$
$$a_4 = 0.25 \eta (\lambda_f + 2\mu_f) a^2 + (\lambda_m + 2\mu_m) C,$$

where

(3.10)$_5$
$$C = \left(\frac{4}{3} \frac{a^3}{d_3} - \frac{3}{4} \eta a^2 \right) \left(1 + \frac{\eta}{1-\eta} \right)^2 - 2 \left(\frac{1}{8} \eta d_2^2 - \frac{3}{8} \eta a^2 \right)$$
$$\times \left(1 + \frac{\eta}{1-\eta} \right) \frac{\eta}{1-\eta} + \left(\frac{1}{12} d_2^2 - \frac{1}{4} \eta a^2 \right) \left(\frac{\eta}{1-\eta} \right)^2.$$

The kinetic energy density is obtained as

(3.11)
$$T = \frac{1}{2} \bar{\varrho} (\dot{u}_2)^2 + \frac{1}{2} b (\dot{\psi}_{21}^f)^2,$$

where

(3.12)
$$\bar{\varrho} = \eta \varrho_f + (1-\eta) \varrho_m, \quad b = 0.25 \eta \varrho_f a^2 + \varrho_m C,$$

and C is defined by Eq. (3.10)$_5$.

Application of the Euler-Poisson equation (2.30), where $F = T - W$ (the interface conditions have already been taken into account via Eq. (3.7)), yields

$$(3.13) \qquad \bar{\varrho}\ddot{\bar{u}}_2 - a_1\,\partial_1\,\partial_1\bar{u}_2 - a_2\,\partial_1\,\psi_{21}^f = 0,$$

$$(3.14) \qquad b\ddot{\psi}_{21}^f - a_4\,\partial_1\,\partial_1\,\psi_{21}^f + a_2\,\partial_1\bar{u}_2 + a_3\,\psi_{21}^f = 0.$$

We will consider harmonic waves of the forms

$$(3.15) \qquad (\bar{u}_2,\,\psi_{21}^f) = (U_2,\,\Psi_{21}^f)e^{ik(x_1-ct)},$$

where $k = 2\pi/\Lambda$ is the wavenumber, Λ being the wavelength, and c is the phase velocity. Substitution of Eqs. (3.15) into (3.13) and (3.14) yields

$$(3.16) \qquad \bar{\varrho}c^2 = a_1 - \frac{a_2^2}{(a_4-bc^2)k^2+a_3}.$$

This is a quadratic equation for the phase velocity.

We will simplify the computation of c, by observing that for $k \to \infty$, we have $\bar{\varrho}c^2 = a_1$. Since this upper limit is reached quickly, we substitute this result in the denominator, to obtain the explicit but approximate result:

$$(3.17) \qquad \bar{\varrho}c^2 = a_1 - \frac{a_2^2}{(a_4-ba_1/\varrho)k^2+a_3}.$$

The group velocity c_g, is related to the phase velocity by

$$(3.18) \qquad c_g = c + k\frac{dc}{dk}.$$

Table 1. Mechanical and geometric parameters of the boron-epoxy composite [7]

Mechanical parameters	Boron	Epoxy (PR-279 resin)
Young's modulus in fiber direction, 10^6 psi, (E)	55.0	0.73
Mass density, $10^{-6}\dfrac{\text{lb sec}^2}{\text{in}^4}$, ($\varrho$)	251	118
Poisson's ratio, (ν) (estimated)	0.2	0.4
Geometric parameters		
fiber radius, in, (a)	0.002	
volume density, (η)	0.54	
fiber radius/fiber spacing, (a/d)	0.41	

Numerical results are presented for a boron-epoxy composite, for which experimental results were presented by TAUCHERT and GUZELSU [7]. The mechanical and geometrical parameters are summarized in Table 1. We use the same system of units as in Ref. [7]. The values of the relevant ratios are

$$\gamma = \mu_f/\mu_m = 88.1, \qquad \theta = \varrho_f/\varrho_m = 2.13,$$

while

$$(c_T)_m = \left(\frac{\mu_m}{\varrho_m}\right)^{1/2} = 0.0469 \frac{\text{in}}{\mu\sec}.$$
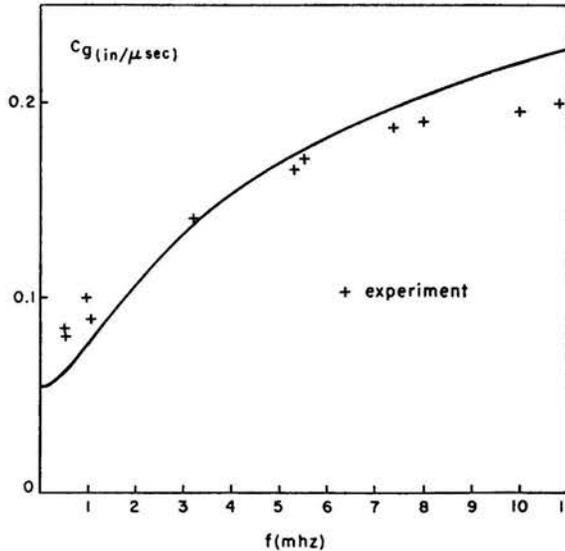
FIG. 3. Analytical and experimental results for transverse waves propagating in the direction of the fibers; see Table 1 for mechanical and geometric parameters.

The results computed from Eqs. (3.17) and (3.18) are plotted in Fig. 3. Experimental results from [7], Fig. 6, are also plotted in Fig. 3. It is noted that there are some deviations at small and relatively large frequencies, but the agreement is not altogether unsatisfactory.

3.2. Longitudinal waves propagating normal to the fibers

Measurements of the phase velocity for various frequencies have been presented by, among others, SUTHERLAND and LINGLE [8].

To construct a useful model for this more difficult case, it is necessary to include quadratic terms in the expansions inside cell (k, l) and to provide an accurate representation of the interaction between neighboring cells.

Let us start off with expressions for the strains which are consistent with longitudinal motions in the x_2-direction. We consider

$$(3.19)_{1,2} \qquad \varepsilon_{22}^{f(k,l)} = \psi_{22}^{f(k,l)}, \qquad \varepsilon_{33}^{f(k,l)} = \psi_{33}^{f(k,l)};$$

$$(3.20)_{2,1} \qquad \varepsilon_{22}^{m(k,l)} = \psi_{22}^{m(k,l)} + 2\frac{\bar{x}_2}{d_2}\varphi_{22}, \qquad \varepsilon_{33}^{m(k,l)} = \psi_{33}^{m(k,l)}.$$

Corresponding displacements in the fiber of cell (k, l) are

$$(3.21)_{1,2} \qquad u_2^{f(k,l)} = \bar{u}_2(x_2, t) + \bar{x}_2\psi_{22}^{f(k,l)}, \qquad u_3^{f(k,l)} = \bar{x}_3\psi_{33}^{f(k,l)},$$

while in the matrix material we have

$$\bar{x}_2 \geqslant 0, \; |\bar{x}_3| \leqslant a:$$

$$(3.22)_1 \qquad u_2^{m(k,l)} = \bar{u}_2 + (a^2 - \bar{x}_3^2)^{1/2}\psi_{22}^{f(k,l)} + [\bar{x}_2 - (a^2 - \bar{x}_3^2)^{1/2}]\psi_{22}^{m(k,l)} + [\bar{x}_2^2 - (a^2 - \bar{x}_3^2)]\frac{\varphi_{22}}{d_2};$$

$$\bar{x}_2 \leqslant 0, \; |\bar{x}_2| \leqslant a:$$

$$(3.22)_2 \qquad u_2^{m(k,l)} = \bar{u}_2 - (a^2 - \bar{x}_3^2)^{1/2}\psi_{22}^{f(k,l)} + [\bar{x}_2 + (a^2 - \bar{x}_3^2)^{1/2}]\psi_{22}^{m(k,l)} + [\bar{x}_2^2 - (a^2 - \bar{x}_3^2)]\frac{\varphi_{22}}{d_2};$$

$$a \leqslant |x_3| \leqslant \frac{1}{2}d_3:$$

$$(3.22)_3 \qquad\qquad\qquad u_2^{m(k,l)} = \bar{u}_2 + \bar{x}_2 \psi_{22}^{m(k,l)} + \bar{x}_2^2 \frac{\varphi_{22}}{d_2};$$

$$\bar{x}_3 \geqslant 0, \; |\bar{x}_2| < a:$$

$$(3.22)_4 \qquad\qquad\qquad u_3^{m(k,l)} = (a^2 - \bar{x}_2^2)^{1/2}\psi_{33}^{f(k,l)} + [\bar{x}_3 - (a^2 - \bar{x}_2^2)^{1/2}]\psi_{33}^{m(k,l)};$$

$$\bar{x}_3 \leqslant 0, \; |\bar{x}_2| < a:$$

$$(3.22)_5 \qquad\qquad\qquad u_3^{m(k,l)} = -(a^2 - \bar{x}_2^2)^{1/2}\psi_{33}^{f(k,l)} + [\bar{x}_3 + (a^2 - \bar{x}_2^2)^{1/2}]\psi_{33}^{m(k,l)};$$

$$a \leqslant |\bar{x}_2| \leqslant \frac{1}{2}d_2:$$

$$(3.22)_6 \qquad\qquad\qquad u_3^{m(k,l)} = \bar{x}_3 \psi_{33}^m.$$

These displacement distributions also give rise to shear strains $\varepsilon_{23}^{m(k,l)}$, which are easily computed. It is evident that $\varepsilon_{23}^{m(k,l)}$ assumes different values in different regions of the matrix material of cell (k, l), and that the expressions for $\varepsilon_{23}^{m(k, l)}$ depend on the local co-ordinates. The averages, $\bar{\varepsilon}_{23}^{m(k,l)}$, over these regions are, however, easily computed. For example, for the region $a \leqslant \bar{x}_3 \leqslant \frac{1}{2}d_3$, $-a \leqslant \bar{x}_2 \leqslant 0$ we find

$$(3.23) \qquad\qquad\qquad \bar{\varepsilon}_{23}^{m(k,l)} = \frac{1}{2}(\psi_{33}^{f(k,l)} - \psi_{33}^{m(k,l)}),$$

while for $0 \leqslant \bar{x}_2 \leqslant a$, $(a^2 - \bar{x}_2^2)^{1/2} < \bar{x}_3 < a$, we have

$$(3.24) \qquad\qquad\qquad \bar{\varepsilon}_{23}^{m(k,l)} = \frac{1}{4-\pi}(-\psi_{22}^{f(k,l)} + \psi_{22}^{m(k,l)} - \psi_{33}^{f(k\,l)} + \psi_{33}^{m(k,l)}).$$

These expressions can be further simplified, since $\psi_{22}^{f(k,l)}$ and $\psi_{33}^{f(k,l)}$ may be neglected as compared to $\psi_{22}^{m(k,l)}$ and $\psi_{33}^{m(k,l)}$, respectively.

Just as discussed in Sect. 2, and exemplified in the first part of the present section, the strains in the discrete cells lead us to the construction of a strain energy density. We find

$$(3.25) \qquad W = \frac{1}{2}a_{22}^f(\psi_{22}^f)^2 + \frac{1}{2}a_{33}^f(\psi_{33}^f)^2 + a_{23}^f \psi_{22}^f \psi_{33}^f + \frac{1}{2}a_{22}^m(\psi_{22}^m)^2$$

$$+ \frac{1}{2}a_{33}^m(\psi_{33}^m)^2 + a_{23}^m \psi_{22}^m \psi_{33}^m + \frac{1}{2}b_{22}^m(\varphi_{22}^2),$$

where

$$a_{22}^f = a_{33}^f = \eta(\lambda_f + 2\mu_f),$$

$$a_{23}^f = \eta\lambda_f,$$

$$a_{22}^m = (1-\eta)\,(\lambda_m + 2\mu_m) + \frac{2}{\pi}\frac{1}{4-\pi}\eta\mu_m + \left(\frac{d_2}{a}-2\right)\frac{1}{\pi}\,\eta\mu_m,$$

(3.26)

$$a_{33}^m = (1-\eta)\,(\lambda_m + 2\mu_m) + \frac{2}{\pi}\frac{1}{4-\pi}\eta\mu_m + \left(\frac{d_3}{a}-2\right)\frac{1}{\pi}\,\eta\mu_m,$$

$$a_{23}^m = (1-\eta)\,\lambda_m + \frac{4}{\pi}\frac{1}{4-\pi}\eta\mu_m,$$

$$b_{22}^m = \left(\frac{1}{3}-\eta\,\frac{a^2}{d_2^2}\right)(\lambda_m + 2\mu_m).$$

In these expressions, η is defined by Eq. (3.8).

In the computation of the kinetic energy we only take into account the gross displacements, and we find

(3.27)
$$T = \frac{1}{2}\,\bar{\varrho}\,(\dot{\bar{u}}_2)^2,$$

where $\bar{\varrho}$ is defined by Eq. (3.12)$_1$.

The conditions at the interfaces of neighboring cells require careful consideration. Substituting the displacement expressions in Eq. (2.14) we find

(3.28)
$$\Delta_2\bar{u}_2 - p_3 a\psi_{22}^f - \frac{1}{2}p_3 a\Delta_2\psi_{22}^f - (d_2 - p_3 a)\psi_{22}^m - \frac{1}{2}(d_2 - p_3 a)\Delta_2\psi_{22}^m$$
$$+ d_2\left(\frac{1}{4} - \frac{4}{3}\frac{a^3}{d_2^2 d_3}\right)\Delta_2\varphi_{22} = 0,$$

where

(3.29)$_{1,2}$
$$p_2 = \frac{\pi a}{d_2},\qquad p_3 = \frac{\pi a}{d_3}.$$

In a similar manner we find

(3.30)
$$S_3 = -p_2 a\psi_{33}^f - (d_3 - p_2 a)\psi_{33}^m = 0.$$

To place Eq. (3.28) within the context of the transition to the continuum model, we introduce Taylor expansions for the differences $\Delta_2 u$, etc. Defining the operator $P[\ ]$ as

(3.31)
$$P[\ ] = \sum_{n=1}^{\infty}\frac{1}{n!}\,(d_2)^n\,\frac{\partial^n}{\partial x_2^n},$$

we find that Eq. (3.28) can be replaced by

(3.32)
$$S_2 = P[\bar{u}_2] - p_3 a\psi_{22}^f - \frac{1}{2}p_3 a P[\psi_{22}^f] - (d_2 - p_3 a)\psi_{22}^m$$
$$- \frac{1}{2}(d_2 - p_3 a)P[\psi_{22}^m] + d_2\left(\frac{1}{4} - \frac{4}{3}\frac{a^3}{d_2^2 d_3}\right)P[\varphi_{22}] = 0.$$

2*

The functional to be used in Hamilton's principle is now defined as

(3.33) $$F = T - W - \lambda_2 S_2 - \lambda_3 S_3,$$

where λ_2 and λ_3 are Lagrangian multipliers. To obtain the appropriate Euler-Poisson equation which follows from Eq. (2.27) with F defined by Eq. (3.33), we employ a well-known result which states that the Euler-Poisson equation for the integral

$$I = \int F(x, y_1, y_1', \dots y_1^{(n_1)}, y_2 \dots y_2^{(n_2)}, \dots, y_m \dots y_m^{(n_m)}) dx$$

is given by

(3.34) $$\sum_{k=0}^{n_i} (-1)^k \frac{d^k}{dx^k} F_{y_i^{(k)}}' = 0.$$

The system of equations resulting from the application of (3.34) to (2.27) and (3.33) is

$$-\bar{\varrho} \ddot{u}_2 - Q[\lambda_2] = 0,$$

$$-a_{22}^f \psi_{22}^f - a_{23}^f \psi_{33}^f + p_3 a \lambda_2 + \frac{1}{2} p_3 a Q[\lambda_2] = 0,$$

$$-a_{22}^m \psi_{22}^m - a_{23}^m \psi_{33}^m + (d_2 - p_3 a) \lambda_2 + \frac{1}{2} (d_2 - p_3 a) Q[\lambda_2] = 0,$$

(3.35) $$-a_{33}^f \psi_{33}^f - a_{23}^f \psi_{22}^f + p_2 a \lambda_3 = 0,$$

$$-a_{33}^m \psi_{33}^m - a_{23}^m \psi_{22}^m + (d_3 - p_2 a) \lambda_2 = 0,$$

$$-b_{22}^m \varphi_{22} - \frac{1}{4} d_2 Q[\lambda_2] = 0.$$

In these equations the operator $Q[\ ]$ is defined as

(3.36) $$Q[\ ] = \sum_{n=1}^{\infty} \frac{1}{n!} (d_2)^n (-1)^n \frac{\partial^n}{\partial x_2^n}.$$

We will again consider expressions for the field variables representing harmonic waves, in this case propagating in the x_2-direction:

$$(\bar{u}_2, \lambda_2, \lambda_3, \varphi) = (U_2, \Lambda_2, \Lambda_3, \Phi) e^{ik(x_2 - ct)},$$

$$(\psi_{22}^f, \psi_{33}^f, \psi_{22}^m, \psi_{33}^m) = (\Psi_{22}^f, \Psi_{33}^f, \Psi_{22}^m, \Psi_{33}^m) e^{ik(x_2 - ct)}.$$

Substitution of these expressions into Eqs. (3.35) and (3.30), (3.32) yields a system of eight homogeneous equations for the eight amplitudes: $U_2 \dots \Psi_{33}^m$. The condition that the determinant must vanish yields an explicit expression for the frequency in terms of the wavenumber. We find

(3.37) $$\Omega^2 = \frac{F(d_2 k)}{\eta \theta + 1 - \eta},$$

where $\theta = \varrho_f / \varrho_m$, and the dimensionless frequency Ω is defined as

(3.38) $$\Omega^2 = \frac{\omega^2 d_2^2}{\mu_m / \varrho_m}$$

and η is defined by Eq. (3.8). The function $F(\ \ )$, which is a function of the dimensionless wavenumber $d_2 k$, is defined as

(3.39)
$$F(d_2 k) = \frac{1 - \cos(d_2 k)}{[1 + \cos(d_2 k)] M + [(1 - \cos(d_2 k)] N},$$

where

$$M = \frac{0.25\eta[\eta D - (1 - \eta) B] + 0.25(1 - \eta)[(1 - \eta) A - \eta C]}{AD - BC},$$

$$N = \frac{0.0625(\lambda_m + 2\mu_m)}{[0.333 - \eta a^2 / d_2^2] \mu_m}$$

and

$$A = \left(a_{22}^f - \frac{(d_3 - p_2 a)^2}{p_2 a} \frac{(a_{23}^f)^2}{E} \right) \frac{1}{\mu_m}, \quad B = C = (d_3 - p_2 a) \frac{a_{23}^m a_{23}^f}{\mu_m E},$$

$$D = \left(a_{22}^m - \frac{p_2 a (a_{23}^m)^2}{E} \right) \frac{1}{\mu_m}, \quad E = \frac{(d_3 - p_2 a)^2}{p_2 a} a_{33}^f + p_2 a a_{33}^m.$$

The function $F(d_2 k)$ given by Eq. (3.39) implies a typical feature of wave propagation normal to the direction of the fibers, namely a maximum for Ω, with a corresponding stop band, and a value $d_2 k = \pi$, i.e. Λ = wavelength = $2d_2$, at which the phase velocity vanishes.

Table 2. Mechanical and geometric parameters of the tungsten-aluminum composite [8]

Mechanical parameters	Tungsten	Aluminum
Longitudinal modulus, dyne/cm², $(\lambda + 2\mu)$	5.15×10^{12}	
Shear modulus, dyne/cm², (μ)		2.65×10^{11}
Poisson's ratio, (ν)	0.28	0.34
Density, gm/cm³	19.19	2.44 (22.1%)
		2.7 (2.2%)
Geometric parameters	2.2%	22.1%
volume density, (η)	0.022	0.221
fiber radius, mm, (a)	0.127	0.127
fiber radius/fiber spacing		
a/d_2	0.098	0.20
a/d_3	0.071	0.353

Experimental results for a tungsten-aluminum composite are presented in Ref. [8]. The mechanical and geometrical parameters of this composite are summarized in Table 2. We use the same system of units as in Ref. [8]. The phase velocity $c = \omega/k$ was computed from Eq. (3.37), and the results are compared to the experimentally obtained values in Fig. 4. Satisfactory agreement between theory and experiment was obtained.
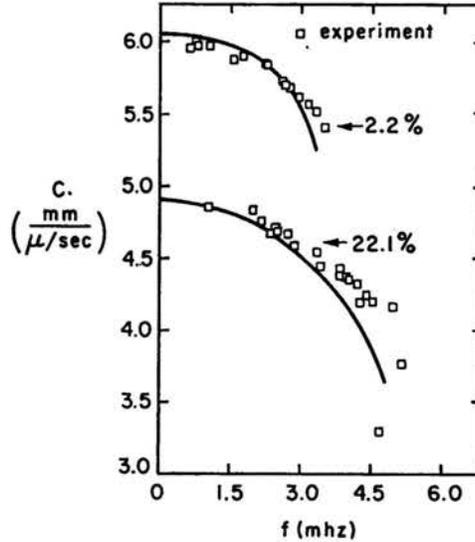
FIG. 4. Analytical and experimental results for longitudinal waves propagating normal to the fibers; see Table 2 for mechanical and geometric parameters.

3.3. Longitudinal waves propagating along the fibers

For longitudinal motions in the x_1-direction, the displacement distributions are symmetric with respect to the planes of structural symmetry of the fiber-reinforced composite. The case $d_2 = d_3 = d$, for which experimental information is available, has the additional simplifying feature that the dependence of the field variables on $\bar{x}_2$ is just the same as the dependence on $\bar{x}_3$.

Consistent with the foregoing observations we consider the following displacement distributions in the fiber of cell (k, l).

$$(3.40)_{1-3} \quad u_1^{f(k,l)} = \bar{u}_1(x_1, t), \quad u_2^{f(k,l)} = \bar{x}_2 \psi^{f(k,l)}(x_1, t), \quad u_3^{f(k,l)} = \bar{x}_3 \psi^{f(k,l)}(x_1, t).$$

The corresponding strains are

$$\varepsilon_{11}^{f(k,l)} = \partial_1 \bar{u}_1, \quad \varepsilon_{22}^{f(k,l)} = \varepsilon_{33}^{f(k,l)} = \psi^{f(k,l)},$$

$$(3.41)$$

$$\varepsilon_{12}^{f(k,l)} = \frac{1}{2} \bar{x}_2 \partial_1 \psi^{f(k,l)}, \quad \varepsilon_{13}^{f(k,l)} = \frac{1}{2} \bar{x}_3 \partial_1 \psi^{f(k,l)}.$$

For the displacements in the matrix material we choose

$$(3.42) \qquad u_1^{m(k,l)} = \bar{u}_1(x_1, t) + d \sin\left(\frac{\pi}{2} \frac{r-a}{b-a}\right) \varphi(x_1, t),$$

where b is a radius such that

$$(3.43) \qquad\qquad\qquad \pi b^2 = d^2.$$

Note that the displacements in the x_1-direction are continuous at $r = a$, and that the slope vanishes at $r = b$. The latter approximates the condition of displacement symmetry at the boundaries $\bar{x}_2 = \pm \frac{1}{2} d$ and $\bar{x}_3 = \pm \frac{1}{2} d$ of cell (k, l). The displacements

of the matrix material in the $\bar{x}_2$- and $\bar{x}_3$-directions are neglected. The strains in the matrix material follow as:

$$\varepsilon_{11}^{m(k,l)} = \partial_1\bar{u}_1 + d\sin\left(\frac{\pi}{2}\frac{r-a}{b-a}\right)\partial_1\varphi,$$

$(3.45)_{1-3}$
$$\varepsilon_{12}^{m(k,l)} = \frac{\pi}{4}\frac{d}{b-a}\cos\left(\frac{\pi}{2}\frac{r-a}{b-a}\right)\frac{\bar{x}_2}{r}\varphi,$$

$$\varepsilon_{13}^{m(k,l)} = \frac{\pi}{4}\frac{d}{b-a}\cos\left(\frac{\pi}{2}\frac{r-a}{b-a}\right)\frac{\bar{x}_3}{r}\varphi.$$

The usual steps lead to the following strain energy density

(3.46)
$$W = \frac{1}{2}a_1(\partial_1\bar{u}_1)^2 + \frac{1}{2}a_2(\psi^f)^2 + a_3\partial_1\bar{u}_1\psi^f$$

$$+ \frac{1}{2}a_4(\partial_1\psi^f)^2 + a_5\partial_1\bar{u}_1\partial_1\varphi + \frac{1}{2}a_6(\partial_1\varphi)^2 + \frac{1}{2}a_7\varphi^2,$$

where

$$a_1 = \eta(\lambda_f+2\mu_f)+(1-\eta)(\lambda_m+2\mu_m), \quad a_2 = 2\eta(\lambda_f+2\mu_f)+2\eta\lambda_f,$$

$$a_3 = 2\eta\lambda_f, \quad a_4 = \frac{1}{2}\eta a^2\mu_f,$$

(3.47)
$$a_5 = \left[\frac{8}{\pi^2}(1-\eta^{1/2})^2 + \frac{4}{\pi}\eta^{1/2}(1-\eta^{1/2})\right]d(\lambda_m+2\mu_m),$$

$$a_6 = \left[\frac{8}{\pi^2}(1-\eta^{1/2})^2\left(\frac{\pi^2}{16}+\frac{1}{4}\right) + \eta^{1/2}(1-\eta^{1/2})\right]d^2(\lambda_m+2\mu_m),$$

$$a_7 = \left[\frac{\pi}{2}\left(\frac{\pi^2}{4}-1\right) + \frac{\pi^3}{4}\eta^{1/2}(1-\eta^{1/2})^{-1}\right]\mu_m.$$

For the computation of a kinetic energy density we consider the following particle velocities

$(3.48)_{1,2}$
$$\dot{\bar{u}}_1^{f(k,l)} = \dot{\bar{u}}_1, \quad \dot{\bar{u}}_1^{m(k,l)} = \dot{\bar{u}}_1 + d\sin\left(\frac{\pi}{2}\frac{r-a}{b-a}\right)\dot{\varphi}.$$

We find

(3.49)
$$T = \frac{1}{2}\bar{\varrho}\dot{\bar{u}}_1^2 + b_1\dot{\bar{u}}_1\dot{\varphi} + \frac{1}{2}b_2\dot{\varphi}^2,$$

where $\bar{\varrho}$ is defined by Eq. $(3.12)_1$, and

$(3.50)_{1,2}$
$$b_1 = \left[\frac{8}{\pi^2}(1-\eta^{1/2})^2 + \frac{4}{\pi}\eta^{1/2}(1-\eta^{1/2})\right]d\varrho_m,$$

$$b_2 = \left[\frac{8}{\pi^2}(1-\eta^{1/2})^2\left(\frac{\pi^2}{16}+\frac{1}{4}\right) + \eta^{1/2}(1-\eta^{1/2})\right]d\varrho_m.$$

A straightforward application of Eq. (2.30) to $F = T-W$ yields the following set of governing equations

(3.51)
$$\bar{\varrho}\ddot{\bar{u}}_1 + b_1\ddot{\varphi} - a_1\partial_1\partial_1\bar{u}_1 - a_3\partial_1\psi^f - a_5\partial_1\partial_1\varphi = 0,$$

(3.52)
$$a_2\psi^f + a_3\partial_1\bar{u}_1 - a_4\partial_1\partial_1\psi^f = 0,$$

(3.53)
$$b_1\ddot{\bar{u}}_1 + b_2\ddot{\varphi} - a_5\partial_1\partial_1\bar{u}_1 - a_6\partial_1\partial_1\varphi + a_7\varphi = 0.$$

Substituting harmonic wave solutions of the forms

$$(\bar{u}_1, \psi^f, \varphi) = (U, \Psi^f, \Phi)e^{ik(x_1 - ct)}$$

in Eqs. (3.51)–(3.53) yields a relation between the phase velocity and the wavenumber as

(3.54)
$$\bar{\varrho}c^2 - a_1 + \frac{a_3^2}{a_2 + a_4 k^2} - \frac{(b_1 c^2 - a_5)^2 k^2}{b_2 c^2 k^2 - a_6 k^2 - a_7} = 0.$$

This is a quadratic equation for c^2, which can easily be solved.

Table 3. Mechanical and geometric parameters of a fiber-reinforced composite of silica fibers and polystyrene matrix material [21]

Mechanical parameters	Silica	Polystyrene
Shear modulus, dyne/cm², (μ)	3.12×10^{11}	0.1323×10^{11}
Poisson's ratio, (v)	0.17	0.353
Density, gm/cm³	2.2	1.056
Geometric parameters		
fiber radius, cm, (a)	0.051	
fiber spacing, cm, (d)	0.236	
volume density $= \pi a^2/d^2$, (η)	0.147	

In a recent article, Ref. [21], the finite element method was employed to investigate the dispersive characteristics of a fiber-reinforced composite, for longitudinal motions propagating in the direction of the fibers. The computations were carried out for a composite whose mechanical and geometric parameters are summarized in Table 3. The results are shown in Fig. 5 by the solid line. The circles indicate experimental results
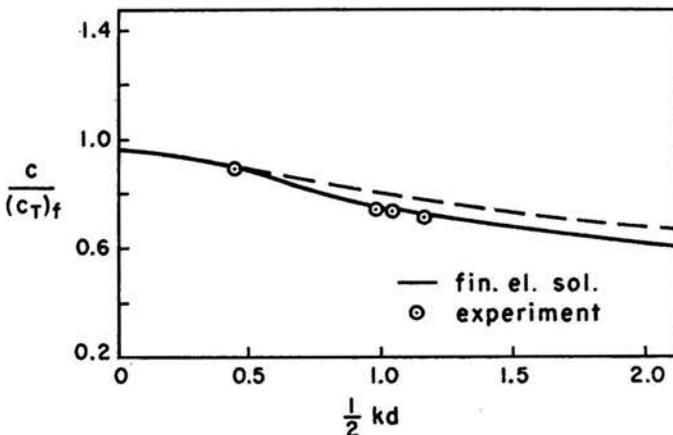
FIG. 5. Analytical, experimental, and numerical results for longitudinal waves propagating in the direction of the fibers; see Table 3 for mechanical and geometric results.

presented in Ref. [21]. For this composite the dimensionless phase velocity $c/(\mu_f/\varrho_f)^{1/2}$ was computed from Eq. (3.54) versus $\frac{1}{2}kd$, and the results have also been plotted in Fig. 5, by the dashed line.

4. Concluding remarks

In this paper we have outlined a procedure to construct a generalized continuum theory for fiber-reinforced composites. For certain special wave motions, which are relevant to available data of ultrasonic tests on composite materials, equations governing the mechanical behavior were presented in detail, and analytical and experimental results were compared.

Within the framework of the theory presented here, the mechanical parameters of the constituents, and the geometric parameters describing the structuring of the composite, enter into coefficients in the set of governing partial differential equations. Thus the governing equations can be determined if relevant information on the constituents and the structuring of the composite is available. No unknown correction factors or other fudge devices, introduced for curve fitting purposes, enter in the theory presented here.

It should be realized, of course, that the mechanical behavior of the constituents is not always known to the accuracy desired by theoreticians. In fact, due to manufacturing processes, the mechanical properties of the constituents may be somewhat different when part of a composite, as compared to the solitary state. In this light it seems to this writer that it is hardly necessary to require agreement on three digits accurate with "exact results". The agreement with experimental results presented here, which is in the five to ten percent range, is very satisfactory.

Acknowledgement

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