Archives of Mechanics • Archiwum Mechaniki Stosowanej • 28, 3, pp. 317-324, Warszawa 1976

Scattering of elastic waves by a distribution of inclusions

S. K. DATTA (BOULDER)

THIS paper deals with the propagation of a plane *P*-wave in the presence of a distribution of spheroidal shaped rigid inclusions embedded in a homogeneous isotropic elastic medium. It is assumed that the inclusions are identical in properties and are homogeneously and at random distributed with their symmetry axes parallel to one another. Expressions for the propagation constant $\langle\alpha\rangle$ for propagation along and perpendicular to the axis of symmetry are obtained with an accuracy to $0(c)$, c being the concentration (small).

Praca niniejsza dotyczy rozprzestrzeniania się fali płaskiej (*P*-fali) w jednorodnym i izotropowym ośrodku sprężystym, w którym znajdują się sztywne inkluzje o kształcie kulistym. Zakłada się, że wszystkie inkluzje posiadają identyczne własności i są rozmieszczone równomiernie i jednorodnie, a ich osie symetrii są wzajemnie równoległe. Wyrażenia dla stałej propagacji $\langle\alpha\rangle$ przy propagacji fali wzdłuż osi symetrii i prostopadle do niej otrzymano z dokładnością $0(c)$, gdzie c oznacza koncentrację (małą).

Настоящая работа касается распространения плоской волны (*P*-волны) в однородной и изотропной упругой среде, в которой находятся жесткие включения сферической формы. Предполагается, что все включения обладают идентичными свойствами и распределены равномерно и однородно, а их оси симметрии взимно параллельны. Выражения для постоянной распостранения $\langle\alpha\rangle$, при распространении волны вдоль оси симметрии и церпиендикулярно к ней, получены с точностю 0 (c), где c обозначает концентрацию (малую).

1. Introduction

PROPAGATION of elastic waves in a periodic composite has been the subject of numerous studies in the last few years. Far less attention has been paid to the case of a random array of scatterers. WATERMAN and TRUELL [1] briefly discussed the effect of spherical elastic inclusions. Recently MAL and BOSE [2, 3, 4] and DATTA [5] have considered circular, spherical and elliptical inclusions.

The present study is concerned with the scattering of a plane *P*-wave by a distribution of rigid spheroids. The object is to obtain the averaged propagation constant $\langle\alpha\rangle$ when the wave is moving parallel or perpendicular to the axis of symmetry, assuming that the spheroids have their axes aligned in a particular direction.

The analysis proceeds in two stages. First, the scattered field due to the incidence of a plane *P*-wave on a single spheroid is obtained. For arbitrary angle of incidence this problem cannot be solved exactly. A method of matched asymptotic expansions is used to obtain the displacement field both near and far from the scatterer when the wavelength is larger than its linear dimension. Then, assuming that the spheroid centres are uniformly and homogeneously distributed, a configurational average of the far-field displacement is taken. This leads to the determination of the averaged propagation constant $\langle\alpha\rangle$. $\langle\alpha\rangle$ has been determined with an accuracy to $0(c)$ by neglecting multiple interactions and assuming that the positions of the spheroid centers are uncorrelated.

2. Scattering by a single rigid spheroid

Consider a rigid oblate spheroid centered at the origin of Cartesian coordinates (x, y, z). Oblate spheroidal coordinates are defined as

$$\begin{aligned} &x = \varrho\cos\omega, \quad y = \varrho\sin\omega, \quad z = a\xi\eta \\ &\varrho = a[(\xi^2+1)^{1/2}(1-\eta^2)^{1/2}], \quad \xi \geqslant \xi_0 > 0, \quad -1 \leqslant \eta \leqslant 1; \end{aligned} \tag{2.1}$$

$\xi = \xi_0$ defines the surface of the oblate spheroid. The incident wave is assumed to be

$$\mathbf{u}^{\mathrm{inc}} = u_0 e^{i\varepsilon(\bar{z}\cos\zeta+\bar{y}\sin\zeta)}(\cos\zeta\mathbf{e}_z+\sin\zeta\mathbf{e}_y) \tag{2.2}$$

suppressing the time factor $e^{-i\gamma t}$ on the right-hand side. Here,

$$\varepsilon = \alpha a = \gamma a/c_1, \quad \bar{y} = y/a, \quad \bar{z} = z/a,$$

c_1, c_2 are the compressional and shear wave-speeds in the matrix.

The scattered displacement field $\mathbf{u}^s$ is to be obtained such that $\mathbf{u}^s$ satisfies the equation

$$\tau^2\nabla\nabla\cdot\mathbf{u}^s-\nabla\wedge\nabla\wedge\mathbf{u}^s = -\nu^2\mathbf{u}^s, \quad \tau^2 = c_1^2/c_2^2 \tag{2.3}$$

and the boundary condition

$$\mathbf{u}^{\mathrm{inc}}+\mathbf{u}^s = \mathbf{U} \quad \text{on} \quad \xi = \xi_0, \tag{2.4}$$

$\mathbf{U}$ being the displacement of the surface of the spheroid. $\mathbf{U}$ is determined from the equation of motion of the spheroid. $\mathbf{u}^s$ also satisfies the radiation condition at infinity.

This boundary value problem for arbitrary ζ cannot be solved by the eigenfunction expansion method. For $\zeta = 0$ the solution can be obtained formally in terms of spheroidal wave functions. However, even then the boundary condition (2.4) leads to a system of an infinite number of equations for the determination of the unknown expansion coefficients. For this reason a method of matched asymptotic expansions was used in [6] to solve the problem for $\zeta = 0$ when the wavelength is larger than a ($\varepsilon < 1$). The case of arbitrary ζ is discussed in [7, 8]. In the present paper attention will be focused on $\zeta = 0$ and $\zeta = \pi/2$. The results pertinent to these two cases will be quoted here without derivation. The interested reader is referred to the papers cited above.

For $\zeta = 0$ the amplitude of oscillation U_z of the spheroid is

$$\frac{1}{u_0}U_z = -aF_z/[M\gamma^2+aF_z^{(t)}]. \tag{2.5}$$

Here, $F_z, F_z^{(t)}$ are functions of ξ_0 and the elastic moduli of the surrounding matrix, M, is the mass of the spheroid. The displacement field away from the scatterer is

$$\mathbf{u}^s = \varepsilon^3\left[\nabla\Phi+\beta^2 r\Psi\mathbf{e}_r+\nabla\left(\frac{\partial}{\partial r}(r\Psi)\right)\right]+0(\varepsilon^4), \tag{2.6}$$

where

$$\begin{aligned} \Phi &= \frac{iV}{4\pi a^3}[(\varrho'/\varrho-1)h_1(\alpha r)P_1(\cos\theta)+a_0h_0(\alpha r)+a_2h_2(\alpha r)P_2(\cos\theta)]\frac{u_0}{\alpha}, \\ \Psi &= \frac{iV}{4\pi a^3}[\tau^2(\varrho'/\varrho-1)h_1(\beta r)P_1(\cos\theta)+\frac{\tau^3}{2}a_2h_2(\beta r)P_2(\cos\theta)]\frac{u_0}{\alpha}. \end{aligned} \tag{2.7$_{1,2}$}$$

The coefficients a_0, a_2 are given by

$$a_0 = \frac{i}{\Delta}\left[\frac{4\sigma-5}{3}(2Q_2^0-\xi_0 Q_2^{0\prime})/(\xi_0^2+1)-\xi_0 Q_2^{0\prime}(\xi_0 Q_1^{0\prime}-Q_1^0)\right],$$

$$a_2 = \frac{i}{\Delta}\frac{4}{3}(1-2\sigma)(2Q_2^0-\xi_0 Q_2^{0\prime})/(\xi_0^2+1),$$

$$\Delta = 2[(2\sigma-1)\xi_0 Q_1^0 Q_2^{0\prime}-\xi_0 Q_1^{0\prime}Q_2^0+(3-4\sigma)Q_1^0 Q_2^0].$$

For $\zeta = \pi/2$ on the other hand

$$\frac{1}{u_0}U_y = -aF_y/[M\gamma^2+F_y^{(t)}] \tag{2.8}$$

and

$$\Phi = \frac{iV}{4\pi a^3}[(\varrho'/\varrho-1)h_1(\alpha r)P_1^1(\cos\theta)\sin\omega+\bar{a}_0 h_0(\alpha r)$$

$$+\bar{a}_2 h_2(\alpha r)P_2(\cos\theta)+\bar{a}_4 h_2(\alpha r)P_2^2(\cos\theta)\cos 2\omega]\frac{u_0}{\alpha}, \tag{2.9$_{1,2}$}$$

$$\Psi = \frac{iV}{4\pi a^3}\left[(\varrho'/\varrho-1)\tau^2 h_1(\beta r)P_1^1(\cos\theta)\sin\omega\right.$$

$$\left.+\frac{\tau^3}{2}\bar{a}_2 h_2(\beta r)P_2(\cos\theta)+\frac{\tau^3}{2}\bar{a}_4 h_2(\beta r)P_2^2(\cos\theta)\cos 2\omega\right]\frac{u_0}{\alpha}$$

with

$$\bar{a}_0 = \frac{i}{\xi_0}\left[B_1\left\{\sqrt{\xi_0^2+1}Q_1^{1\prime}-(3-4\sigma)\frac{\xi_0}{\sqrt{\xi_0^2+1}}Q_1^1-\frac{2}{3}\frac{4\sigma-5}{(\xi_0^2+1)^{1/2}}\right\}\right.$$

$$\left.+C_1\left\{\xi_0 Q_1^{0\prime}-(3-4\sigma)Q_1^0+\frac{1}{3(\xi_0^2+1)^{1/2}}\right\}+\frac{\xi_0}{\sqrt{\xi_0^2+1}}\right],$$

$$\bar{a}_2 = \frac{i}{\xi_0}\frac{4(1-2\sigma)}{3\sqrt{\xi_0^2+1}}\left(B_1+C_1/\sqrt{\xi_0^2+1}\right),$$

$$\bar{a}_4 = \frac{i}{\xi_0}\frac{2(1-2\sigma)}{3\sqrt{\xi_0^2+1}}B_1,$$

$$B_1 = \frac{2Q_2^2\xi_0-(\xi_0^2+1)Q_2^{2\prime}}{\Delta_1},$$

$$\Delta_1 = (4\sigma-2)(\xi_0^2+1)Q_1^1 Q_2^{2\prime}-2(\xi_0^2+1)Q_1^{1\prime}Q_2^2+2(3-4\sigma)\xi_0 Q_1^1 Q_2^2,$$

$$C_1 = -\frac{1}{2\Delta}[B_1\{2(\xi_0^2+1)Q_1^{1\prime}Q_2^0-2(3-4\sigma)\xi_0 Q_1^1 Q_2^0$$

$$-2(2\sigma-1)Q_1^1 Q_2^{0\prime}(\xi_0^2+1)\}+2\xi_0 Q_2^0-(\xi_0^2+1)Q_2^{0\prime}].$$

Here, $Q_n^{m\prime} = dQ_n^m/d\xi|_{\xi=\xi_0}$, $h_n(z)$ is the Bessel function of the third kind, ϱ' and ϱ are the densities of the inclusion and the matrix, respectively, and σ is Poisson's ratio.

3. Effect of a distribution of inclusions

If multiple scattering effects are neglected, then the scattered field at any point will simply be the sum of the fields given by (2.6) when referred to the centers of the inclusions. It must be remembered that (2.6) may be interpreted as the scattered field everywhere due to one inclusion which, to this order of approximation, is reduced to a point. This approach was taken in [5] and will be followed here as well.

3.1. Axial incidence ($\zeta=0$)

Suppose the plane wave is propagating in the direction of the axes of the spheroids. Let $\boldsymbol{\rho}_i(\xi_i, \eta_i, \zeta_i)$ be the coordinate of the center of the i-th spheroid. Consider a large volume V containing N inclusions. Assuming that the distribution of spheroid centers is random and statistically uniform, the probability density of a scatterer at $\boldsymbol{\rho}_i$ in V is

$$P(\boldsymbol{\rho}_i) = N/V = n, \tag{3.1}$$

where n is the number density of the scatterers. Furthermore, if the scatterer positions are sufficiently removed from one another and if all physical positions of the second scatterer are equally probable when the first scatterer is at $\boldsymbol{\rho}_i$, then,

$$\begin{aligned} P(\boldsymbol{\rho}_j|\boldsymbol{\rho}_i) &= n, \quad |\boldsymbol{\rho}_j-\boldsymbol{\rho}_i| > 2L, \\ &= 0, \quad |\boldsymbol{\rho}_j-\boldsymbol{\rho}_i| \leqslant 2L. \end{aligned} \tag{3.2}$$

Here, $2L$ is the major axis of the spheroid. The configurational average of $f(\mathbf{r}|\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_N)$ will be denoted by

$$\langle f(\mathbf{r})\rangle = n\int f(\mathbf{r}|\boldsymbol{\rho})d\boldsymbol{\rho} \tag{3.3$_1$}$$

and the partial average with one scatterer fixed at $\boldsymbol{\rho}_i$ by

$$\begin{aligned} \langle f(r|\boldsymbol{\rho}_i)\rangle &= n\int f(\mathbf{r}|\boldsymbol{\rho};\boldsymbol{\rho}_i)d\boldsymbol{\rho}, \\ &|\boldsymbol{\rho}-\boldsymbol{\rho}_i| \geqslant 2L. \end{aligned} \tag{3.3$_2$}$$

It may be noted that to the lowest order of approximation (point scatterers) L may be made to go to zero.

Now, using the notation of Ref. [1] the exciting field acting on the i-th scatterer will be denoted by $\mathbf{u}^E(r|\boldsymbol{\rho}_i; \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_N)$. Taking the average one then obtains

$$\langle \mathbf{u}^E(\mathbf{r}|\boldsymbol{\rho}_i)\rangle = \mathbf{u}^{\mathrm{inc}} + n\int T(\boldsymbol{\rho})\langle \mathbf{u}^E(r|\boldsymbol{\rho})\rangle d\boldsymbol{\rho}, \tag{3.4}$$

where $T(\boldsymbol{\rho})$ is the scattering operator which, acting on the exciting field of the scatterer at $\boldsymbol{\rho}$, gives the scattered field at $\mathbf{r}$. Thus,

$$\mathbf{u}^s(\mathbf{r}|\boldsymbol{\rho}_j; \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_N) \equiv T(\boldsymbol{\rho}_j)\mathbf{u}^E(\mathbf{r}|\boldsymbol{\rho}_j; \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_N).$$

In writing (3.4) use has been made of the assumption of uncorrelated scatterer positions.

Equation (3.4) is an integral equation to be solved for $\langle \mathbf{u}^E(\mathbf{r}|\boldsymbol{\rho})\rangle$. To find this solution assume that the composite occupies the space $z > 0$ and that the solution is of the form

$$\langle \mathbf{u}^E(\mathbf{r}|\boldsymbol{\rho})\rangle = \bar{u}_0 e^{i\langle\alpha\rangle z}\mathbf{e}_z. \tag{3.5}$$

Using (3.5) together with (2.6) and (2.7) Eq. (3.4) becomes

$$\bar{u}_0 e^{i\langle\alpha\rangle\zeta_i}\mathbf{e}_z = u_0 e^{i\alpha\zeta_i}\mathbf{e}_z + \frac{ic\alpha^3}{4\pi\alpha}\bar{u}_0 \int \sum_{n=0}^{2} e^{i\langle\alpha\rangle\zeta}[\mathscr{A}_n \mathbf{L}_n^{(3)} + \mathscr{C}_n \mathbf{N}_n^{(3)}]\,d\boldsymbol{\rho}, \tag{3.6}$$

where

$$\begin{gathered}
\mathscr{A}_0 = \frac{\langle\alpha\rangle}{\alpha} a_0, \quad \mathscr{A}_1 = \left(\frac{\varrho'}{\varrho}\frac{\langle\alpha\rangle^2}{\alpha^2} - 1\right),\\
\mathscr{A}_2 = \frac{\langle\alpha\rangle}{\alpha} a_2, \quad \mathscr{C}_1 = \tau^2 \mathscr{A}_1, \quad \mathscr{C}_2 = \frac{\tau^3}{2}\mathscr{A}_2,\\
\mathbf{L}_n^{(3)} = \mathbf{e}_R \frac{\partial}{\partial R} h_n(\alpha R) P_n(\cos\Theta) + e_\Theta \frac{1}{R} h_n(\alpha R) \frac{\partial}{\partial\Theta} P_n,\\
\mathbf{N}_n^{(3)} = \mathbf{e}_R \frac{(n+1)}{R} h_n(\beta R) P_n + \mathbf{e}_\Theta \frac{1}{R}\frac{\partial}{\partial R}\big(R h_n(\beta R)\big)\frac{\partial P_n}{\partial\Theta}.
\end{gathered} \tag{3.7}$$

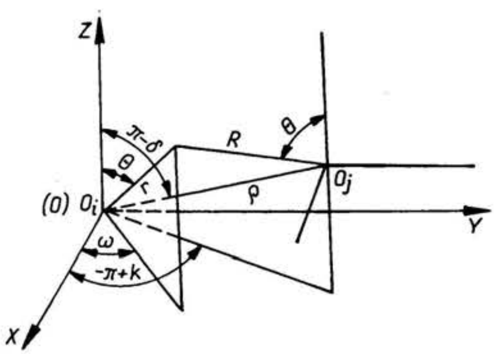

FIG. 1.

The coordinates (r, θ), (R, Θ) and (ϱ, δ) are shown in Fig. 1. The integral in (3.6) can be evaluated by using the translational addition theorems for spherical vector wave functions [9]. This is done in the Appendix. Equation (3.6) then gives

$$\frac{\langle\alpha\rangle^2}{\alpha^2} = \frac{1 + c\varrho'/\varrho}{1 + c(1 + ia_2 - ia_0)} \tag{3.8$_1$}$$

which may also be written as

$$\frac{\langle c_1\rangle^2}{c_1^2} = \frac{1 + c(1 + ia_2 - ia_0)}{1 + c\varrho'/\varrho}. \tag{3.8$_2$}$$

Here, $c = nV$.

This then gives the relationship between longitudinal wave speed in the z-direction in the composite and that in the matrix.

In the limit when the spheroid tends to a sphere, equation $(3.8)_2$ takes the form

$$\frac{\langle c_1\rangle^2}{c_1^2} = \frac{1 + c\left(1 + \dfrac{3(3-5\sigma)}{4-5\sigma}\right)}{1 + c\varrho'/\varrho} \tag{3.9}$$

and when the spheroid tends to a disc it is found that

$$\frac{\langle c_1\rangle^2}{c_1^2} = \frac{1}{1+nM/\varrho}. \tag{3.10}$$

The above derivation applies for an oblate spheroid. However, the results for a prolate spheroid are obtained by letting $a \to ia$, $\xi_0 \to -i\xi_0$.

Table 1. Values of $a^3\xi_0(\xi_0^2+1)\,(ia_2-ia_0)/A^3$ ($=\lambda$)

ξ_0	0	0.6	∞	$-1.2i$	$-i$
λ	0	1.59384	1.71429	1.58297	0

Table 1 lists the values of λ, which is defined as $a^3\xi_0(\xi_0^2+1)\,(ia_2-ia_0)/A^3$, for $\sigma = 1/3$, A being the mean of the semi-major and minor axes of a spheroid.

3.2. Incidence normal to the axis of symmetry ($\zeta = \pi/2$)

For propagation perpendicular to the axis of symmetry, equations (3.6) and (3.7) will be modified to, considering the half-space $y > 0$,

$$\bar{u}_0 e^{i\langle\alpha\rangle\eta l}\mathbf{e}_y = u_0 e^{i\alpha\eta l}\mathbf{e}_y + \frac{ic\varepsilon^3}{4\pi a^3\alpha}\bar{u}_0 \int \sum_{n=0}^{2}{}' \sum_{m=-n}^{n}{}' [\mathscr{A}_{mn}\mathbf{L}_{mn}^{(2)} + \mathscr{C}_{mn}\mathbf{N}_{mn}^{(3)}] \times e^{i\langle\alpha\rangle\eta} d\boldsymbol{\rho}, \tag{3.11}$$

where

$$\mathscr{A}_{00} = \frac{\langle\alpha\rangle}{\alpha}\bar{a}_0, \quad \mathscr{A}_{11} = -\frac{i}{2}(\varrho'/\varrho - \langle\alpha\rangle^2/\alpha^2) = \frac{1}{2}\mathscr{A}_{-11},$$

$$\mathscr{A}_{02} = \bar{a}_2\frac{\langle\alpha\rangle}{\alpha}, \quad \mathscr{A}_{22} = \frac{1}{2}\bar{a}_4\frac{\langle\alpha\rangle}{\alpha}\frac{1}{24}\mathscr{A}_{-22},$$

$$\mathscr{C}_{m1} = \tau^2\mathscr{A}_{m1}, \quad \mathscr{C}_{m2} = \frac{\tau^3}{2}\mathscr{A}_{m2}, \tag{3.12}$$

$$\mathbf{L}_{mn}^{(3)} = \left[\mathbf{e}_R\frac{\partial}{\partial R}h_nP_n^m + \mathbf{e}_\Theta\frac{1}{R}h_n(\alpha R)\frac{\partial P_n^m}{\partial\Theta} + e_\Omega\frac{im}{R\sin\Theta}h_nP_n^m\right]e^{im\Omega},$$

$$\mathbf{N}_{mn}^{(3)} = \left[\mathbf{e}_R\frac{n(n+1)}{R}h_nP_n^m + \mathbf{e}_\Theta\frac{1}{R}\frac{\partial}{\partial R}h_n(\beta R)\frac{\partial}{\partial\Theta}P_n^m + e_\Omega\frac{im}{R\sin\Theta}\frac{\partial}{\partial R}(Rh_n)P_n^m\right]e^{im\Omega}.$$

Note that all other $\mathscr{A}_{mn}$'s and $\mathscr{C}_{mn}$'s are zero.

For the purpose of integrating, a transformation of coordinates is made so that Θ is now measured from the y-axis and Ω from the z-axis. The Eq. (3.11) still has the same form with the modified coefficients

$$\mathscr{A}_{02} = \frac{\langle\alpha\rangle}{\alpha}\left(-3\bar{a}_4 - \frac{1}{2}\bar{a}_2\right),$$

$$\mathscr{A}_{22} = \frac{1}{2}\frac{\langle\alpha\rangle}{\alpha}\left(\frac{1}{4}\bar{a}_2 - \frac{1}{2}\bar{a}_4\right) = \frac{1}{24}\mathscr{A}_{-22}.$$

Other coefficients remain the same.

The evaluation of the integral in (3.11) is then straightforward. The final result is

$$\frac{\langle c_1\rangle^2}{c_1^2} = \frac{1+c\left[1-\frac{1}{2}(i\bar{a}_2+2i\bar{a}_0+6i\bar{a}_4)\right]}{1+c\varrho'/\varrho}. \tag{3.13}$$

In the limit when $\xi_0 \to 0$ (3.13) reduces to

$$\frac{\langle c_1\rangle^2}{c_1^2} = \frac{1+16\pi a^3 n\,\frac{(1-2\sigma)(13-16\sigma)}{(3-4\sigma)(7-8\sigma)}}{1+nM/\varrho}. \tag{3.14}$$

Table 2 lists the values of $\bar{\lambda}$, which is defined as $a^3\xi_0(\xi_0^2+1)\left[-\frac{i}{2}(2\bar{a}_0+\bar{a}_2+6\bar{a}_4)\right]/A^3$.

Table 2. Values of $a^3\xi_0(\xi_0^2+1)(\bar{a}_2+2\bar{a}_0+6\bar{a}_4)/2iA^3$ $(=\bar{\lambda})$

ξ_0	0	0.6	∞	$-1.2i$	0
λ	4.24615	2.76123	1.71429	0.33664	0

Appendix

To evaluate the integral in (3.6) the vector wave functions $\mathbf{L}_n^{(3)}$ and $\mathbf{N}_n^{(3)}$ are expanded about the point $\mathbf{r} = \boldsymbol{\rho}_i$. It is convenient then to change the origin to this point (see Fig. 1). The required expansions are then (see [4] and [9])

$$\mathbf{L}_n^{(3)} = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} A_{\mu\nu}^{on}\mathbf{L}_{\mu\nu}^{(1)}, \tag{A.1}$$

$$\mathbf{N}_n^{(3)} = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} [C_{\mu\nu}^{on}M_{\mu\nu}^{(1)}+B_{\mu\nu}^{on}\mathbf{N}_{\mu\nu}^{(1)}] \tag{A.2}$$

with

$$\begin{aligned} A_{\mu\nu}^{on} &= \sum_p (-1)^\mu i^{\nu+p-n}(2\nu+1)a(0,n|-\mu,\nu|p)h_p(\alpha\varrho)P_p^{-\mu}(\cos\delta)e^{-i\mu k},\\ B_{\mu\nu}^{on} &= \sum_p (-1)^\mu i^{\nu+p-n}a(n,\nu,p)a(0,n|-\mu,\nu|p)h_p(\beta\varrho)P_p^{-\mu}(\cos\delta)e^{-i\mu k},\\ C_{\mu\nu}^{on} &= \sum_p (-1)^\mu i^{\nu+p-n}b(n,\nu,p)a(0,n|-\mu,\nu|p,p-1)h_p(\beta\varrho)P_p^{-\mu}(\cos\delta)e^{-i\mu k}. \end{aligned} \tag{A.3}$$

The coefficients appearing in (A.3) are given in the references cited above. Since the integral is symmetric in k it will be zero except for $\mu = 0$. Also, the spherical Bessel function $j_\nu(\alpha r)$ appearing in $\mathbf{L}_{\mu\nu}^{(1)}$, etc., can be expanded in powers of αr and, keeping only the lowest order terms, one obtains (setting $\mu = 0$)

$$\mathbf{L}_n^{(3)} = \frac{\alpha}{3}\mathbf{e}_z A_{01}^{on}, \qquad \mathbf{N}_n^{(3)} = \frac{2\beta}{3}\mathbf{e}_z B_{01}^{on}. \tag{A.4}$$

Substituting these in the integrand, the integral is found to be

$$\text{(A.5)}\qquad \frac{2\pi\alpha}{\alpha^3}e_z\Bigg[(\mathscr{A}_0-\mathscr{A}_2)\left\{\frac{-e^{i\alpha\zeta i}}{\langle\alpha\rangle/\alpha-1}+\frac{2\langle\alpha\rangle/\alpha}{(\langle\alpha\rangle/\alpha)^2-1}e^{i<\alpha>\zeta i}\right\} -i\mathscr{A}_1\left\{\frac{e^{i\alpha\zeta i}}{\langle\alpha\rangle/\alpha-1}+\frac{2}{(\langle\alpha\rangle/\alpha)^2-1}e^{i<\alpha>\zeta i}\right\}\Bigg].$$

Using (A.5) in (3.6) one obtains

$$\text{(A.6)}\qquad 1 = -\frac{c}{1-\langle\alpha\rangle^2/\alpha^2}\left[\varrho'/\varrho+\frac{\langle\alpha\rangle^2}{\alpha^2}(ia_0-ia_2-1)\right]$$

which leads to (3.8).

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DEPARTMENT OF MECHANICAL ENGINEERING
UNIVERSITY OF COLORADO, BOULDER, COLORADO 80302.