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Theories with carrier fields: multiple-interaction nonlocal formulations

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THIS paper examines the consequences of the following hypothesis: One can associate with any physical body a finite number of functions of position and time, called carrier fields, that describe the microscopic properties of the body and are such that appropriately defined operators on the carriers describe the macroscopic properties of the body. The analysis is pursued in terms of two classes of problems. The first class consists of problems with a homogeneous variational principle for the carriers whose Lagrangian function contains bounded linear operators acting on the carriers. The second class replaces the homogeneous variational principle by an inhomogeneous one and the Lagrangian function is allowed to depend on non-linear operators appropriate to the representation of multiple interactions. Governing equations for the carrier fields are obtained for both classes of problems; these governing equations being general integro-differential equations on a Banach space of carrier functions. The equations for the carrier fields are then used in order to obtain equations for the macroscopic properties of the physical bodies under investigation. These results allow the derivation and generalization of the governing equations for theories reported under the general headings of multipolar, micropolar, simple, director, and nonlocal field theories. In addition, the results pose the possibility of unification of these various theories and of obtaining an orderly transition from lattice dynamics to classical continuum mechanics.

Praca niniejsza bada skutki następującej hipotezy: Dowolnemu ciału fizycznemu można przyporządkować skończoną liczbę funkcji położenia i czasu, zwanych nośnikami pól i opisujących mikroskopowe własności ciała. Analizę przeprowadzono na podstawie dwóch klas zagadnień. Pierwsza klasa składa się z zagadnień charakteryzujących się jednorodną zasadą wariacyjną dla nośników, których funkcja Lagrange'a zawiera ograniczone operatory liniowe, działające na nośniki. W klasie drugiej jednorodną zasadę wariacyjną zastąpiono zasadą niejednorodną, przy czym funkcja Lagrange'a może zależeć od operatorów nieliniowych właściwych dla reprezentacji oddziaływań wielokrotnych. Równania podstawowe dla nośników pól otrzymano dla obydwu klas zagadnień. Przyjmują one postać ogólnych równań różniczkowo-całkowych w przestrzeni Banacha funkcji nośników. Równania dla nośników pól wykorzystano następnie do otrzymania równań opisujących makroskopowe własności badanych ciał fizycznych. Otrzymane wyniki pozwalają na wyprowadzenie i uogólnienie równań dla różnych teorii znanych pod ogólnymi nazwami jako teorie multipolarne, mikropolarne, proste, ukierunkowane oraz jako teoria pól nielokalnych. Ponadto wyniki te stwarzają możliwość unifikacji tych różnych teorii i otrzymania prawidłowego przejścia od dynamiki siatek do klasycznej mechaniki kontinuum.

Настоящая работа изучает следствия следующей гипотезы: Произвольному физическому телу можно сопоставить конечное количество функций положения и времени, называемые носителями полей, и описывающие микроскопические свойства тела. Анализ проведен на основе двух классов задач. Первый класс состоит из задач характеризующихся однородным вариационным принципом для носителей, функция Лагранжа которых содержит ограниченные линейные операторы действующие на носители. Во втором классе однородный вариационный принцип заменен неоднородным принципом, причем функция Лагранжа может зависеть от нелинейных операторов свойственных для представления многократных взаимодействий. Основные уравнения для носителей полей получены для обоих классов задач; принимают они вид общих интегро-дифференциальных уравнений в банаховом пространстве функций носителей. Уравнения для носителей полей использованы затем для получения уравнений, описывающих макроскопические свойства исследуемых физических тел. Полученные результаты позволяют вывести и обобщить уравнения для разных теорий известных под общими названиями, как мультиполярные, микрополярные, простые, направленные теории, а также как теория нелокальных полей. Кроме этого эти результаты создают возможность унификации этих разных теорий и получения правильного перехода от динамики решеток к классической механике континуум.

1. Variational considerations and carrier fields

THIS underlying concept that is common to all of the theories considered in this paper can be stated loosely as follows: One can associate with any physical body a finite number of functions of "position" and "time", called carriers, such that all microscopic properties of the body are determined by these carriers. There are several contexts in which we shall use this concept, and indeed, the bare statement of the concept is either tautological or vacuous unless it is closed within a very carefully stated physical domain of discourse.

The first thing we do is to state the necessary analytic preliminaries and common mathematical context. The usual three-dimensional number space is denoted by E_3 and the points of E_3 are labeled by their coordinate values $\{Z^A\}$ relative to a fixed Cartesian coordinate cover (Z) of E_3. Let B denote an open arcwise connected subset of E_3 and let B^* denote the closure of B with respect to the Euclidean topology of E_3. We assume that B^* has a nonzero Euclidean volume measure $\int_{B^*} dV(Z)$ and that ∂B^*, the boundary of B^*, is closed and is a regular two-surface in E_3 with the possible exception of a finite number of edges and vertices. The directed surface measure of ∂B^* is denoted by $\{dS_A(Z)\} \overset{ae}{=} \{N_A(Z^B)dS(Z)\}$. Let T denote a given closed interval $[t_0, t_1]$ of the real line $\mathscr{R}$. We define the point set D^* by $D^* = B^* \times T$ with the naturally induced product topology.

The symbol $\mathscr{D}_1(D^*; N)$ is used to denote the (closed) normed linear space of N-tuples of functions $\{\Phi_\Lambda(X^k, t)\}$, $\Lambda = 1, \dots, N$, of class C^1 on D^* with the norm $\| \ \|$ of uniform convergence for each $\Phi_\Lambda(X^A, t)$ and each first derivative $\partial_A \Phi_\Lambda \equiv \partial \Phi_\Lambda / \partial X^A$, $\partial_t \Phi_\Lambda = \partial \Phi_\Lambda / \partial t$.

We shall also require certain collections of linear operators. Let $W(X^A - Z^A)$ and $h_p(X^A, Z^A)$, $p = 1, \dots, M$ be defined for all $\{X^A\}$ and all $\{Z^A\}$ in B^* and be such that

$$\langle f \rangle_p(X^A, t) = \langle f \rangle_p = \int_{B^*} W(X^A - Z^A) h_p(X^A, Z^A) f(Z^A, t) dV(Z), \tag{1.1}$$

$$\langle f \rangle_p^\dagger(X^A, t) = \langle f \rangle_p^\dagger = \int_{B^*} W(Z^A - X^A) h_p(Z^A, X^A) f(Z^A, t) dV(Z) \tag{1.2}$$

exist (and are finite) for all functions $f(Z^A, t)$ of class C defined on D^*. It will be seen in what follows that it is natural to refer to $\langle \ \rangle_p^\dagger$ as the (variational) dual of $\langle \ \rangle_p$. In general, $\langle \ \rangle_p$ will be used to construct a "descriptor of a macroscopic property" from a carrier function $\Phi_\Lambda(Z^A, t)$:

$$\langle \Phi_\Lambda \rangle_p = \int_{B^*} W(X^A - Z^A) h_p(X^A, Z^A) \Phi_\Lambda(Z^A, t) dV(Z), \tag{1.3}$$

while $\langle\ \rangle_p^+$ will occur in the governing field equations for the quantities $\{\Phi_\Lambda\}$. The function $W(\xi^A)$ will usually vanish outside of some small neighborhood of $\{\xi^A\} = \{0\}$, in which case, $\langle f \rangle_p$ may be viewed as an "averaging operator" with weight function $h_p(\ ,\ )$ over the support of $W(Z^A - X^A)$.

2. Theories with a homogeneous variational principle with linear operators for the carriers

Almost every physical theory on the "microscopic" level possesses a homogeneous([1]) variational principle. For this reason and because the homogeneous variational structure provides a convenient basis for computation and understanding, we shall first examine theories with a homogeneous variational principle that involves linear operators on the carrier fields.

We assume that the carriers of the theory are an N-tuple of functions $\{\Phi_\Lambda(Z^A, t)\}$ that belong to $\mathscr{D}_1(D^*; N)$. The physical interpretations that may be attached to these carriers will vary from theory to theory, and, in fact, there may be no simple physical interpretation that can be attached to the carriers in certain instances. In this respect, the remarks of RIVLIN [1] are very much to the point. It is, however, helpful to have something definite in mind, and so we shall think of the carriers $\{\Phi_\Lambda(Z^A, t)\}$ as representing distributions of, say, mass, momentum, energy, etc., on the microscopic level of a material body. With this view in mind, it is natural to introduce operations on the carriers that will lead to the macroscopic quantities which may be associated with the macroscopic properties of material bodies. We accordingly introduce the quantities

$$\langle \Phi_\Lambda \rangle_p(X^A, t) = \int_{B^*} W(X^A - Z^A) h_p(X^A, Z^A) \Phi_\Lambda(Z^A, t) dV(Z), \tag{2.1}$$

where we assume that the functions $W(X^A - Z^A)$ and $h_p(X^A, Z^A)$ are given functions whose choice is dictated by the physics involved([2]).

Physical theories with homogeneous variational principles also involve space and time derivatives, and so we must provide for their occurrence in the theory. As far as time derivatives are concerned, (2.1) gives

$$\langle \partial_t \Phi_\Lambda \rangle_p = \partial_t \langle \Phi_\Lambda \rangle_p \tag{2.2}$$

and hence there is no problem with time derivatives. For space derivatives, however, (2.1) gives

$$\partial_A \langle \Phi_\Lambda \rangle_p \equiv \frac{\partial}{\partial X^A} \langle \Phi_\Lambda \rangle_p(X^B, t) \neq \langle \partial_A \Phi_\Lambda \rangle_p, \tag{2.3}$$

except for *very* special choices of the functions W and h_p. Hence we must make a choice between $\partial_A \langle \Phi_\Lambda \rangle_p$ and $\langle \partial_A \Phi_\Lambda \rangle_p$. It can be shown (see Appendix) that a variational formalism with the choice $\partial_A \langle \Phi_\Lambda \rangle_p$ has a number of deficiencies which are not evidenced

([1]) A variational principle is said to be *homogeneous* if it is of the form $\delta \int L = 0$, while $\delta \int L = \int T^\mu \delta \eta_\mu$ is referred to as an *inhomogeneous* variational principle.

([2]) The interested reader can easily generalize (2.1) and the resulting theory so as to include integrations with respect to t, provided due care is exercised in order to avoid causality problems.

with the choice $\langle \partial_A \Phi_\Lambda \rangle_p$, and hence we elect the latter. This choice is also that which occurs in the nonlocal calculus of variations [2, 3, 4].

The variational theory assumes the existence of a Lagrangian function

$$L = L\big(X^A, t; \Phi_\Lambda(X^A, t), \partial_t \Phi_\Lambda(X^A, t), \partial_B \Phi_\Lambda(X^A, t); \langle \Phi_\Lambda \rangle_1, \partial_t \langle \Phi_\Lambda \rangle_1, \langle \partial_A \Phi_\Lambda \rangle_1; \ldots; \langle \Phi_\Lambda \rangle_M, \partial_t \langle \Phi_\Lambda \rangle_M, \langle \partial_A \Phi_\Lambda \rangle_M\big) \tag{2.4}$$

of class C^2(³) in its $4+5N(1+M)$ arguments. This function defines an action functional by the relation

$$J[\{\Phi_\Lambda\}](L) = \int\limits_T \int\limits_{B^*} L\, dV(X)\, dt. \tag{2.5}$$

We assume that the carrier fields Φ_Λ take given values on part of the boundary of D^*. If $(\partial D^*)_1^\Lambda$ denotes the part of ∂D^* on which Φ^Λ is given, then we write

$$\Phi_\Lambda|_{(\partial D^*)_1^\Lambda} = g_\Lambda,$$

where each g_Λ is a given function on the domain $(\partial D^*)_1^\Lambda$. Since the boundary of D^* consists of $\partial B^* \times T$ and the body at times $t = t_0$ and $t = t_1$, the above given boundary conditions translate into

$$\begin{aligned} \Phi_\Lambda(X^A, t)|_{(\partial B^*)_1^\Lambda} &= u_\Lambda(X^A, t)|_{,(\partial B^*)_1^\Lambda}, \\ \Phi_\Lambda(X^A, t_0)|_{(B^*)_{1,0}^\Lambda} &= v_\Lambda(X^A)|_{(B^*)_{1,0}^\Lambda}, \\ \Phi_\Lambda(X^A, t_1)|_{(B^*)_{1,1}^\Lambda} &= w_\Lambda(X^A)|_{(B^*)_{1,1}^\Lambda}, \end{aligned} \tag{2.6}$$

where u_Λ, v_Λ and w_Λ are given functions of the indicated domains. We define $(\partial D^*)_2^\Lambda$ for each Λ by the requirement $\partial D^* = (\partial D^*)_1^\Lambda \cup (\partial D^*)_2^\Lambda$, $(\partial D^*)_1^\Lambda \cap (\partial D^*)_2^\Lambda = \phi$, so that we have

$$\begin{aligned} \partial B^* = (\partial B^*)_1^\Lambda \cup (\partial B^*)_2^\Lambda, &\quad (\partial B^*)_1^\Lambda \cap (\partial B^*)_2^\Lambda = \phi, \\ B^* = (B^*)_{1,0}^\Lambda \cup (B^*)_{2,0}^\Lambda &= (B^*)_{1,1}^\Lambda \cup (B^*)_{2,1}^\Lambda, \\ (B^*)_{1,0}^\Lambda \cap (B^*)_{2,0}^\Lambda &= (B^*)_{1,2}^\Lambda \cap (B^*)_{2,1}^\Lambda = \phi. \end{aligned}$$

The *field equations and the boundary data* of the theory then obtained from (2.6) and the following requirement (variational principle):

$$J[\{\Phi_\Lambda + \phi_\Lambda\}](L) = J[\{\Phi_\Lambda\}](L) + o(\|\{\phi_\Lambda\}\|) \tag{2.7}$$

must hold for all $\{\phi_\Lambda(X^A, t)\} \in \mathscr{D}_1(D^*; N)$ such that $\phi_\Lambda(X^A; t)|_{(\partial D^*)_1^\Lambda} = 0$, where $o(\|\ \|)$ is with respect to the norm $\|\ \|$ of $\mathscr{D}_1(D^*; N)$. (The more customary statement, where it is also required that $\{\phi_\Lambda(X^A, t)\}|_{(\partial D^*)_2^\Lambda} = 0$, only gives Euler equations, not Euler equations and boundary conditions). Since the Lagrangian function for these theories involves only linear operators of the carriers as arguments, namely $\langle \Phi_\Lambda \rangle_p$, $\partial_t \langle \Phi_\Lambda \rangle_p$, $\langle \partial_A \Phi_\Lambda \rangle_p$, the resulting theories will be referred to as *theories of type one*.

(³) This continuity requirement can be relaxed in many instances without significant changes in the final results. It is assumed here primarily in the interests of simplicity and convenience.

Since the Lagrangian function (2.4) contains the functionals $\langle\Phi_A\rangle_p$, $\partial_t\langle\Phi_A\rangle_p$, $\langle\partial_A\Phi_A\rangle_p$ as arguments, we could use the nonlocal variational calculus [2, 3, 4] to obtain the field equations and boundary conditions([4]). It is illustrative, however, to proceed directly.

The first thing to do is to evaluate $\langle f+u\rangle_p$. When (2.1) is used, we have $\langle f+u\rangle_p = \langle f\rangle_p+\langle u\rangle_p$. It then follows from (2.4) and (2.5) that (2.7) is satisfied if, and only if,

$$(2.8)\qquad 0=\int\limits_T\int\limits_{B^*}\Bigg\{\frac{\partial L}{\partial\Phi_A}\phi_A+\frac{\partial L}{\partial(\partial_A\Phi_A)}\partial_A\phi_A+\frac{\partial L}{\partial(\partial_t\Phi_A)}\partial_t\phi_A + \sum_{p=1}^{M}\left(\frac{\partial L}{\partial\langle\Phi_A\rangle_p}\langle\phi_A\rangle_p+\frac{\partial L}{\partial(\partial_t\langle\Phi_A\rangle_p)}\partial_t\langle\phi_A\rangle_p+\frac{\partial L}{\partial\langle\partial_A\Phi_A\rangle_p}\langle\partial_A\phi_A\rangle_p\right)\Bigg\}dV(X)dt$$

holds for all $\{\phi_A\}\in\mathscr{D}_1(D^*;N)$ which vanish on $(\partial D^*)^A_1$. Now, we have

$$\int\limits_{B^*}U^p(X^A,t)\langle f\rangle_p(X^A,t)dV(X) = \int\limits_{B^*}\int\limits_{B^*}U^p(X^A,t)W(X^A-Z^A)h_p(X^A,Z^A)f(Z^A,t)dV(Z)dV(X).$$

Since the domains of the Z-wise and X-wise integrations are the same, we can interchange their orders by standard theorems in analysis so as to obtain

$$(2.9)\qquad \int\limits_{B^*}U^p(X^A,t)\langle f\rangle_p(X^A,t)dV(X) = \int\limits_{B^*}\int\limits_{B^*}U^p(Z^A,t)W(Z^A-X^A)h_p(Z^A,X^A)f(X^A,t)dV(Z)dV(X) = \int\limits_{B^*}\langle U^p\rangle_p^\dagger(X^A,t)f(X^A,t)dV(X),$$

where $\langle\ \rangle_p^\dagger$ is the variational dual of $\langle\ \rangle_p$ that is defined by (1.2). When this result is used to rewrite (2.8), we obtain the requirement that

$$(2.10)\qquad 0=\int\limits_T\int\limits_{B^*}\Bigg\{\left(\frac{\partial L}{\partial\Phi_A}+\sum_{p=1}^{M}\left\langle\frac{\partial L}{\partial\langle\Phi_A\rangle_p}\right\rangle_p^\dagger\right)\phi_A(X^A,t) + \left(\frac{\partial L}{\partial(\partial_t\Phi_A)}+\sum_{p=1}^{M}\left\langle\frac{\partial L}{\partial(\partial_t\langle\Phi_A\rangle_p)}\right\rangle\right)\partial_t\phi_A(X^A t,) + \left(\frac{\partial L}{\partial(\partial_A\Phi_A)}+\sum_{p=1}^{M}\left\langle\frac{\partial L}{\partial\langle\partial_A\Phi_A\rangle_p}\right\rangle_p^\dagger\right)\partial_A\phi_A(X^A,t)\Bigg\}dV(X)\,dt$$

must hold for all $\{\phi_A\}\in\mathscr{D}_1(D^*;N)$ which vanish on $(\partial D^*)^A_1$.

([4]) This is accomplished by the trivial modification in the definition of $\langle\ \ \rangle_p$:

$$\langle f\rangle_p\equiv\int\limits_T\delta(t-\tau)\int\limits_{B^*}W(X^A-Z^A)h_p(X^A,Z^A)f(Z,\tau)dV(Z)d\tau.$$

Although it is not essential, since discontinuities may be accounted for by introducing the appropriate supports for the discontinuities (this leads to the "jump conditions" on the field variables), we shall assume that all of the terms in (2.10) are sufficiently continuous that we may perform the indicated integrations by parts. We then have that

$$(2.11)\quad 0 = \int\limits_T \int\limits_{B^*} \Bigg\{ \frac{\partial L}{\partial \Phi_A} + \sum_{p=1}^{M} \Big\langle \frac{\partial L}{\partial \langle \Phi_A \rangle_p} \Big\rangle_p^{\dagger} - \partial_A \Bigg(\frac{\partial L}{\partial(\partial_A \Phi_A)} + \sum_{p=1}^{M} \Big\langle \frac{\partial L}{\partial \langle \partial_A \Phi_A \rangle_p} \Big\rangle_p^{\dagger} \Bigg)$$

$$- \partial_t \Bigg(\frac{\partial L}{\partial(\partial_t \Phi_A)} + \sum_{p=1}^{M} \Big\langle \frac{\partial L}{\partial(\partial_t \langle \Phi_A \rangle_p)} \Big\rangle_p^{\dagger} \Bigg) \Bigg\} \phi_A(X^A, t)\, dV(X)\, dt$$

$$+ \int\limits_T \int\limits_{\partial B^*} N_A(X) \Bigg(\frac{\partial L}{\partial(\partial_A \Phi_A)} + \sum_{p=1}^{M} \Big\langle \frac{\partial L}{\partial \langle \partial_A \Phi_A \rangle_p} \Big\rangle_p^{\dagger} \Bigg) \phi_A(X^A, t)\, dS(X)\, dt$$

$$+ \int\limits_{B^*} \Bigg\{ \Bigg(\frac{\partial L}{\partial(\partial_t \Phi_A)} + \sum_{p=1}^{M} \Big\langle \frac{\partial L}{\partial(\partial_t \langle \Phi_A \rangle_p)} \Big\rangle_p^{\dagger} \Bigg) \phi_A(X^A, t) \Bigg\} \Bigg|_{t_0}^{t_1} dV(X)$$

must hold for all $\{\phi_A\} \in \mathscr{D}_1(D^*; N)$ which vanish on $(\partial D^*)_1^A$. Now, a subset of all $\{\phi_A(X^A, t)\}$ belonging to $\mathscr{D}_1(D^*; N)$ consists of those $\{\phi_A(X^A, t)\}$ which are such that

$$\{\phi_A(X^A, t)\}|_{\partial B^*} = 0, \quad \{\phi_A(X^A, t_0)\} = 0, \quad \{\phi_A(X^A, t_1)\} = 0.$$

When this set of elements of $\mathscr{D}_1(D^*; N)$ is used in (2.11), the fundamental lemma of the calculus of variations shows that (2.11) can hold if, and only if, $\{\Phi_A(X^A, t)\}$ satisfies the N Euler-Lagrange equations

$$(2.12)\quad 0 = \frac{\partial L}{\partial \Phi_A} - \partial_A \Bigg(\frac{\partial L}{\partial(\partial_A \Phi_A)} \Bigg) - \partial_t \Bigg(\frac{\partial L}{\partial(\partial_t \Phi_A)} \Bigg)$$

$$+ \sum_{p=1}^{M} \Bigg\{ \Big\langle \frac{\partial L}{\partial \langle \Phi_A \rangle_p} \Big\rangle_p^{\dagger} - \partial_A \Big\langle \frac{\partial L}{\partial \langle \partial_A \Phi_A \rangle_p} \Big\rangle_p^{\dagger} - \partial_t \Bigg(\Big\langle \frac{\partial L}{\partial(\partial_t \langle \Phi_A \rangle_p)} \Big\rangle_p^{\dagger} \Bigg) \Bigg\}, \qquad A = 1, \ldots, N$$

at all points interior to $B^* \times T$. When (2.12) is then used to simplify (2.11), we have the requirement that

$$(2.13)\quad 0 = \int\limits_T \int\limits_{\partial B^*} N_A(X^B) \Bigg(\frac{\partial L}{\partial(\partial_A \Phi_A)} + \sum_{p=1}^{M} \Big\langle \frac{\partial L}{\partial \langle \partial_A \Phi_A \rangle_p} \Big\rangle_p^{\dagger} \Bigg) \phi_A(X^A, t)\, dS(X)\, dt$$

$$+ \int\limits_{B^*} \Bigg\{ \Bigg(\frac{\partial L}{\partial(\partial_t \Phi_A)} + \sum_{p=1}^{M} \Big\langle \frac{\partial L}{\partial(\partial_t \langle \Phi_A \rangle_p)} \Big\rangle_p^{\dagger} \Bigg) \phi_A(X^A, t) \Bigg\} \Bigg|_{t_0}^{t_1} dV(X)$$

must hold for all $\{\phi_A\} \in \mathscr{D}_1(D^*; N)$ which vanish on $(\partial D^*)_1^A$. If we now consider those elements of $\mathscr{D}_1(D^*; N)$ which satisfy the requirement $\{\phi_A(X^A, t)\}|_{\partial B^*} = 0$, then (2.13) reduces to

$$\int\limits_{B^*} \Bigg\{ \Bigg(\frac{\partial L}{(\partial_t \Phi_A)} + \sum_{p=1}^{M} \Big\langle \frac{\partial L}{\partial(\partial_t \langle \Phi_A \rangle_p)} \Big\rangle_p^{\dagger} \Bigg) \phi_A(X^A, t) \Bigg\}_{t_0}^{t_1} dV(X) = 0.$$

Since ϕ_A vanishes on $(\partial D^*)_1^A$, it vanishes on $(B^*)_{1,0}^A$ and $(B^*)_{1,1}^A$. Hence the fundamental lemma of the calculus of variations yields the initial and terminal conditions

$$
(2.14)\qquad
\begin{aligned}
0 &= \left(\frac{\partial L}{\partial(\partial_t \Phi_A)} + \sum_{p=1}^{M} \left\langle \frac{\partial L}{\partial(\partial_t \langle \Phi_A \rangle_p)} \right\rangle_p^{\dagger}\right)\Bigg|_{t=t_0} \quad \text{on } (B^*)_{2,0}^A,\\
0 &= \left(\frac{\partial L}{\partial(\partial_t \Phi_A)} + \sum_{p=1}^{M} \left\langle \frac{\partial L}{\partial(\partial_t \langle \Phi_A \rangle_p)} \right\rangle_p^{\dagger}\right)\Bigg|_{t=t_1} \quad \text{on } (B^*)_{2,1}^A.
\end{aligned}
$$

The conditions on $(B^*)_{1,0}^A$ and $(B^*)_{1,1}^A$ are given by (2.6). Finally, taking $\{\phi_A(X^A, t)\} = \{\phi_A(X^A)\}$ and applying the fundamental lemma of the calculus of variations on the three-manifold $\partial B^* \times T$, we obtain the boundary data

$$
(2.15)\qquad 0 = N_A(X^B)\left(\frac{\partial L}{\partial(\partial_A \Phi_A)} + \sum_{p=1}^{M} \left\langle \frac{\partial L}{\partial \langle \partial_A \Phi_A \rangle_p} \right\rangle_p^{\dagger}\right)\Bigg|_{(\partial B^*)_2^A}.
$$

The boundary data on $(\partial B^*)_1^A$ is given by (2.6). The Euler-Lagrange equations, the initial data, the terminal data and the boundary data are thus given by (2.12), (2.14), (2.15) and (2.6), respectively.

The reader should carefully note that the Euler-Lagrange equations and data we have just obtained *are equations for the determination of the carrier fields* $\{\Phi_A(X^A, t)\}$. This point is of fundamental importance in later sections, since this formulation determines the macroscopic quantities $\langle \Phi_A \rangle_p$,

$$
\langle \partial\ \Phi_A \rangle_p, \quad \partial_t \langle \Phi_A \rangle_p, \quad \frac{\partial L}{\partial \langle \Phi_A \rangle_p}, \quad \frac{\partial L}{\partial \langle \partial_A \Phi_A \rangle_p}, \quad \frac{\partial L}{\partial(\partial_t \langle \Phi_A \rangle_p)}
$$

only, after we solve the *integro-differential* systems of equations (2.12), (2.14), (2.15) and (2.6) for the carriers $\{\Phi_A\}$.

It is quite easily seen that the Euler-Lagrange equations (2.12) may be interpreted as local equations of balance. In fact, integrating (2.12) over an arbitrary part P of B^*, we obtain the local balance laws

$$
\begin{aligned}
(2.16)\qquad \frac{d}{dt}\int_P \left\{\frac{\partial L}{\partial(\partial_t \Phi_A)} + \sum_{p=1}^{M} \left\langle \frac{\partial L}{\partial(\partial_t \langle \Phi_A \rangle_p)} \right\rangle_p^{\dagger}\right\} dV(X)
&= \int_{\dot P} \left\{\frac{\partial L}{\partial \Phi_A} + \sum_{p=1}^{M} \left\langle \frac{\partial L}{\partial \langle \Phi_A \rangle_p} \right\rangle_p^{\dagger}\right\} dV(X)\\
&\quad - \int_{\partial P} \left\{\frac{\partial L}{\partial(\partial_A \Phi_A)} + \sum_{p=1}^{M} \left\langle \frac{\partial L}{\partial \langle \partial_A \Phi_A \rangle_p} \right\rangle_p^{\dagger}\right\} N_A\, dS(X).
\end{aligned}
$$

However, if P is the whole body B^*, then (2.15) holds and we have

$$(2.17)\quad \frac{d}{dt}\int_{B^*}\left\{\frac{\partial L}{\partial(\partial_t\Phi_A)}+\sum_{p=1}^{M}\left\langle\frac{\partial L}{\partial(\partial_t\langle\Phi_A\rangle_p)}\right\rangle_p^{\dagger}\right\}dV(X)$$

$$=\int_{B^*}\left\{\frac{\partial L}{\partial\Phi_A}+\sum_{p=1}^{M}\left\langle\frac{\partial L}{\partial\langle\Phi_A\rangle_p}\right\rangle_p^{\dagger}\right\}dV(X)$$

$$-\int_{(\partial B^*)_1^A}\left\{\frac{\partial L}{\partial(\partial_A\Phi_A)}+\sum_{p=1}^{M}\left\langle\frac{\partial L}{\partial\langle\partial_A\Phi_A\rangle_p}\right\rangle_p^{\dagger}\right\}N_A\,dS(X).$$

If $(\partial B^*)_1^A=\phi$, then the global law of balance (2.17) is quite different from the local law (2.12). As an example, suppose that $M=N=1$, $\Phi_1(X^A,t)=\Phi$, and

$$2L=\varrho_0\left(\frac{\partial\Phi}{\partial t}\right)^2-\alpha^{AB}(X^C)\frac{\partial\Phi}{\partial X^A}\frac{\partial\Phi}{\partial X^B}+\beta^{AB}(X^C)\frac{\partial\Phi}{\partial X^A}\left\langle\frac{\partial\Phi}{\partial X^B}\right\rangle_1,$$

then (2.12) gives the field equation

$$\varrho_0\frac{\partial^2\Phi}{\partial t^2}=\frac{1}{2}\frac{\partial}{\partial X^A}\left\{(\alpha^{AB}+\alpha^{BA})\frac{\partial\Phi}{\partial X^B}-\beta^{AB}\left\langle\frac{\partial\Phi}{\partial X^B}\right\rangle_1-\left\langle\beta^{BA}\frac{\partial\Phi}{\partial X^B}\right\rangle_1^{\dagger}\right\}.$$

Thus, when (1.1) and (1.2) are used with

$$W(X^A-Z^A)h_1(X^A,Z^A)=K(X^A,Z^A),$$

we have

$$\varrho_0\frac{\partial^2\Phi}{\partial t^2}=\frac{1}{2}\frac{\partial}{\partial X^A}\left\{(\alpha^{AB}+\alpha^{BA})\frac{\partial\Phi}{\partial X^B}\right\}$$

$$-\frac{1}{2}\frac{\partial}{\partial X^A}\int_{B}\{\beta^{AB}(X)K(X,Z)+\beta^{BA}(Z)K(Z,X)\}\frac{\partial\Phi(Z,t)}{\partial Z^B}dV(Z).$$

3. Theories with non-linear macroscopic operators of multiple interactions and an inhomogeneous variational principle

The theories developed in the previous section were based upon the assumption that the macroscopic independent variables of the Lagrangian function obtain from linear operators

$$\langle f\rangle_p=\int_{B^*}W(X^A-Z^A)h_p(X^A;Z^A)f(Z^A)dV(Z)$$

acting on the carrier fields and their derivatives: they are of type one. Although theories based upon this assumption provide a structure for describing a large class of physical phenomena, they are not adequate in all instances. We therefore consider theories of type two which are significantly more general.

Let each of the functions

$$g_a^p(X^A, \Phi_A(X^A, t), \partial_B\Phi_A(X^A, t), \partial_t\Phi_A(X^A, t); Z_1^A, \Phi_A(Z_1^A, t), \partial_B\Phi_A(Z_1^A, T), \partial_t\Phi_A(Z_1^A, t); \dots; Z_p^A, \Phi_A(Z_p^A, t), \partial_B\Phi_A(Z_p^A, t), \partial_t\Phi_A(Z_p^A, t)),$$
$$a = 1, \dots, Q, \quad p = 1, \dots, R$$

be of class C^2 in each of its indicated arguments. For each (X^A, t) in D^*, we define the functionals k_a^p by the relations

$$(3.1) \qquad k_a^p(X^A, t; \{\Phi_A\}) = \int\int\limits_{B^*}\int g_a^p dV(Z_1) \dots dV(Z_p).$$

These functionals correspond with the quantities $\langle\ \rangle_p$ of the last section.

Let $L(X^A, t; \Phi_A(X^A, t); \partial_B\Phi_A(X^A, t); \partial_t\Phi_A(X^A, t); k_a^p)$ be a function of class C^2 in all of its arguments and define the functional J_R by

$$(3.2) \qquad J_R[\{\Phi_A\}](L) = \int\limits_T \int\limits_{B^*} L dt dV(X).$$

In addition, we postulate the existence of given functions $\{q^A(X^A, t)\}$ on $B^* \times T$, $\{T^A(X^A, t)\}$ on $(\partial B^*)_2^A \times T$, $\{\mathscr{I}^A(X^A)\}$ on $(B^*)_{2.0}^A$ and $\{\mathscr{T}^A(X^A)\}$ on $(B^*)_{2.1}^A$. (Since the data (2.6) are assumed, it would be inconsistent to assume that T^A is given on $(\partial B^*)_1^A$, etc.). These functions are used to construct the linear functional

$$(3.3) \qquad j[\{\Phi_A\}] = \int\limits_T \int\limits_{B^*} q^A(X^A, t)\phi_A(X^A, t) dV(X) dt + \int\limits_T \int\limits_{(\partial B^*)_2^A} T^A(X^A, t)\phi_A(X^A, t) dS(X) dt + \int\limits_{(B^*)_{2.1}^A} \mathscr{T}^A(X^A)\phi_A(X^A, t_1) dV(X) - \int\limits_{(B^*)_{2,0}} \mathscr{I}^A(X^A)\phi_A(X^A, t_0) dV(X).$$

The field equations and the data for the theory now obtained from the inhomogeneous variational statement require that

$$(3.4) \qquad J[\{\Phi_A + \phi_A\}](L) = J[\{\Phi_A\}](L) + j[\{\phi_A\}] + o(\|\{\phi_A\}\|)$$

shall hold for all $\{\phi_A(X^A, t)\} \in \mathscr{D}_1(D^*; N)$ that vanish on $(\partial D^*)_1^A$.

The generality of the above formulation is such that there is little purpose in proceeding directly. Accordingly, we shall make use of the results of the nonlocal variational calculus given in [2, 3, 4, 5]. Define the operators $\mathscr{H}_r$ by the relations

$$(3.5) \qquad \mathscr{H}_r(U(X^A, Z_1^A, \dots, Z_{r-1}^A, Z_r^A, Z_{r+1}^A, \dots, Z_p^A)) = U(Z_r^A, Z_1^A, \dots, Z_{r-1}^A, X^A, Z_{r+1}^A, \dots, Z_p^A).$$

These operators are then used to construct the *extended Lagrangian* function $\mathscr{L}$ by

$$(3.6) \qquad \mathscr{L} = L + \sum_{a=1}^{Q} \sum_{p=1}^{R} \sum_{r=1}^{p} \int\int\limits_{B^*}\int \mathscr{H}_r\left(\frac{\partial L}{\partial k_a^p} g_a^p\right) dV(Z_1) \dots dV(Z_p).$$

It may then be shown that the condition (3.4) is satisfied for all $\{\phi_A(X^A, t)\} \in \mathscr{D}_1(D^*; N)$ that vanish on $(\partial D^*)^A_1$ if, and only if, the carrier fields $\{\Phi_A(X^A, t)\}$ satisfy the N nonlocal Euler equations

$$(3.7) \qquad q^A = \frac{\partial \mathscr{L}}{\partial \Phi_A(X^B, t)} - \frac{\partial}{\partial X^A}\left(\frac{\partial \mathscr{L}}{\partial(\partial_A \Phi_A(X^B, t))}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial(\partial_t \Phi_A(X^B, t))}\right)$$

at all points interior to $D^* = B^* \times T$, the data (2.6), and

$$(3.8) \qquad T^A = N_A(X^B)\left\{\frac{\partial \mathscr{L}}{\partial(\partial_A \Phi_A(X^B, t))}\right\}$$

on $(\partial B^*)^A_2$,

$$(3.9) \qquad \mathscr{I}^A = \left\{\frac{\partial \mathscr{L}}{\partial(\partial_t \Phi_A(X^A, t))}\right\}\Bigg|_{t=t_0}$$

on $(B^*)^A_{2,0}$ and,

$$(3.10) \qquad \mathscr{T}^A = \left\{\frac{\partial \mathscr{L}}{\partial(\partial_t \Phi_A(X^A, t))}\right\}\Bigg|_{t=t_1}$$

on $(B^*)^A_{2,1}$.

These field equations and boundary-initial data may be viewed to advantage in terms of Lagrangian functions that represent multiple interactions. The reason for this is that we may take for a typical Lagrangian function as one which assumes the form

$$\frac{1}{2}a\left(\frac{\partial \Phi}{\partial t}\right)^2 + \int_B G_1(X^A, Z^A_1)\Phi(Z^A_1)dV(Z_1)$$

$$+ \iint_{B^*} G_2(X^A, Z^A_1, Z^A_2)\Phi(Z^A_1)\Phi(Z^A_2)dV(Z_1)dV(Z_2) + \ldots$$

$$+ \iiint_{B^*} G_3(X^A, Z^A_1, Z^A_2, Z^A_3)\Phi(Z^A_1)\Phi(Z^A_2)\Phi(Z^A_3)dV(Z_1)dV(Z_2)dV(Z_3) + \ldots$$

and interpret the double integral as a two-point interaction, the triple integral as a three-point interaction, etc.

We note in passing that these equations reproduce the results obtained in Sect. 2 under the identification of k^1_a with $\langle \Phi^A \rangle_p, \ldots$. In fact, under this identification, (3.6) yields

$$(3.11) \qquad \mathscr{L} = L + \left\langle \frac{\partial L}{\partial \langle \Phi_A \rangle_p} \right\rangle^\dagger_p + \left\langle \frac{\partial L}{\partial \langle \partial_A \Phi_A \rangle_p} \right\rangle^\dagger_p + \left\langle \frac{\partial L}{\partial(\partial_t \langle \Phi_A \rangle_p)} \right\rangle^\dagger_p,$$

and (3.7) reproduces the field equations (2.12) for $q^A = 0$ in (3.7). Thus, if we substitute (3.11) into (3.7)–(3.10), we obtain the governing equations and data for the case with linear operators governed by the inhomogeneous variational principle (3.4). There is thus no need to redo the cases considered in Sect. 2 for inhomogeneous variational principles.

4. Macroscopic formulations

The formulations given in the previous sections may be referred to as *microscopic*, since the field equations and data are for the determination of the (microscopic) carrier fields $\{\Phi_\Lambda(X^A, t)\}$ rather than macroscopic quantities such as the quantities k^p_a, $\langle\ \rangle_p$ or $\langle\ \rangle^\dagger_p$. Although such formulations give a complete system of governing equations, there is an undesirable feature of the microscopic formulation when we attempt to use it in posing and solving macroscopic problems. This comes about because the equations of the microscopic formulation must be solved for the carrier fields $\{\Phi_\Lambda(X^A, t)\}$ and then these fields must be used to determine the macroscopic quantities. If the carrier fields are indeed physically measurable quantities, then the above procedure is an acceptable one although the equations which must be solved are a somewhat formidable array of integro-differential equations. On the other hand, if the carrier fields are not physically measurable quantities (wave functions, etc.), then we would certainly prefer to have governing equations for macroscopic systems that involve the physical macroscopic variables directly; in fact, this is the preferable situation whenever we wish to give a description of a macroscopic system.

There are three distinct avenues by which we can obtain governing equations for macroscopic systems from the results given above. The first of these is to assume that the carrier fields $\{\Phi_\Lambda(X^A, t)\}$ are actually macroscopic variables such as mass, displacement, magnetization, etc. The Euler-Lagrange field equations and the boundary and initial conditions obtained above are then the governing *nonlocal* macroscopic equations of the theory. Although this alternative is a familiar one, since it is equivalent to a formulation using a classical variational principle (the only difference from classical theories being the fact that the Lagrangian function L can now depend on integrals of the dependent variables), we lose a certain degree of generality since we no longer have quantities which can represent the underlying microscopic properties of material bodies.

The second alternative arises from the assumption that the basic quantities of the macroscopic theory are taken to be a $4NM$-tuple of functions $\{U_{\Lambda p}(X^A, t),\ U_{A\Lambda p}(X^A, t)\}$, $\Lambda = 1, \ldots, N$, $p = 1, \ldots, M$, $A = 1, 2, 3$. For the theories considered in Sect. 2, the carrier fields $\{\Phi^\Lambda(X^A, t)\}$ are now considered to be arbitrary to within the constraints

$$\langle\Phi_\Lambda\rangle_p = U_{\Lambda p}, \quad \langle\partial_A\Phi_\Lambda\rangle_p = U_{A\Lambda p}(^5). \tag{4.1}$$

Further, since the macroscopic theory is to involve only the macroscopic variables $\{U_{\Lambda p}, U_{A\Lambda p}\}$, we assume that the Lagrangian function has the form

$$L = L(X^A, t; U_{\Lambda p}, U_{A\Lambda p}, \partial_t U_{\Lambda p}) \tag{4.2}$$

and that $(\partial D^*)^A_1 = \phi$. When (4.1) and (4.2) are used in conjunction with the microscopic equations (3.7)–(3.10) with the appropriate modifications for the special nature of the g'_as (i.e. $g_a = g'_a$), the assumption of underlying carrier fields with an inhomogeneous variational principle yields the following system of *macroscopic equations of the first kind*:

---

(5) Since $\langle\partial_t\Phi_\Lambda\rangle_p = \partial_t\langle\Phi_\Lambda\rangle_p$, we have $\langle\partial_t\Phi_\Lambda\rangle_p = \partial_t\langle\Phi_\Lambda\rangle_p = \partial_t U_{\Lambda p}$.

$$q^{A} = \sum_{p=1}^{M}\left\{\left\langle\frac{\partial L}{\partial U_{Ap}}\right\rangle_{p}^{\dagger} - \partial_{A}\left\langle\frac{\partial L}{\partial U_{AAp}}\right\rangle_{p}^{\dagger} - \partial_{t}\left(\left\langle\frac{\partial L}{\partial(\partial_{t}U_{Ap})}\right\rangle_{p}^{\dagger}\right)\right\}, \tag{4.3}$$

$$T^{A} = N_{A}(X^{B})\sum_{p=1}^{M}\left\langle\frac{\partial L}{\partial U_{AAp}}\right\rangle_{p}^{\dagger}\bigg|_{\partial B^{*}} \quad \text{on } (\partial B^{*})_{2}^{A}, \tag{4.4}$$

$$\mathcal{I}^{A}(X^{A}) = \sum_{p=1}^{M}\left\langle\frac{\partial L}{\partial(\partial_{t}U_{Ap})}\right\rangle_{p}^{\dagger}\bigg|_{t=t_{0}} \quad \text{on } (B^{*})_{2,0}^{A}, \tag{4.5}$$

$$\mathcal{T}^{A}(X^{A}) = \sum_{p=1}^{M}\left\langle\frac{\partial L}{\partial(\partial_{t}U_{Ap})}\right\rangle_{p}^{\dagger}\bigg|_{t=t_{1}} \quad \text{on } (B^{*})_{2,1}^{A}. \tag{4.6}$$

Macroscopic theories of the first kind thus preserve the existence of a definite Lagrangian function L and the functions W and h_p which define the operators $\langle\ \rangle_p^{\dagger}$. Furthermore, the macroscopic Euler-Lagrange equations (4.3) are of the second order with respect to timewise differentiation and so the data (4.5) and (4.6) are well posed.

Macroscopic formulations of the second kind can be obtained under quite different circumstances. For formulations of this kind, the Lagrangian function, the operators k_a^p and the carriers $\{\Phi_A\}$ are *all* assumed to be arbitrary to within the constraints

$$\begin{aligned} R^{A}(X^{B}, t) &= \frac{\partial\mathcal{L}}{\partial(\Phi_{A}(X^{A}, t))}, \\ S^{AA}(X^{B}, t) &= \frac{\partial\mathcal{L}}{\partial(\partial_{A}\Phi_{A}(X^{B}, t))}, \\ P^{A}(X^{B}, t) &= \frac{\partial\mathcal{L}}{\partial(\partial_{t}\Phi_{A}(X^{B}, t))}, \end{aligned} \tag{4.7}$$

where $\{R^A, S^{AA}, P^A\}$ is a $6N$-tuple of functions of $\{X^A, t\}$ that is identified with the macroscopic variables of the theory. With these assumptions, the requirement of consistency with the microscopic equations (3.7)–(3.10) yields the following system of *macroscopic equations of the second kind*:

$$q^{A} = \{R^{A} - \partial_{A}S^{AA} - \partial_{t}P^{A}\}, \tag{4.8}$$

$$T^{A} = N_{A}(X^{B})S^{AA}|_{\partial B^{*}}, \quad \text{on } (\partial B^{*})_{2}^{A}, \tag{4.9}$$

$$\mathcal{I}^{A}(X^{A}) = P^{A}|_{t=t_{0}}, \quad \text{on } (B^{*})_{2,0}^{A}, \tag{4.10}$$

$$\mathcal{T}^{A}(X^{A}) = P^{A}|_{t=t_{1}}, \quad \text{on } (B^{*})_{2,1}^{A}. \tag{4.11}$$

Macroscopic theories of the second kind do not preserve the notion of a Lagrangian function and the field equation (4.8) are only of the first order with respect to differentiation with respect to time. Satisfaction of both the initial conditions (4.10) and the terminal conditions (4.11) will not, in general, be possible. Accordingly, we must view the functions $\mathcal{T}^{A}(X^{A})$ occurring in the terminal condition (4.11) as undetermined until after the fact.

It is of interest to note that both (4.3) and (4.8) lead to equations of balance:

$$\text{(4.12)}\qquad \frac{\partial}{\partial t}\int_{B^*}\left\langle\frac{\partial L}{\partial(\partial_t U_{\Lambda p})}\right\rangle_p^{\dagger} dV(X) = \int_{B^*}\left\{\left\langle\frac{\partial L}{\partial U_{\Lambda p}}\right\rangle_p^{\dagger} - q^{\Lambda}\right\} dV(X) - \int_{\partial B^*}\left\langle\frac{\partial L}{\partial U_{\Lambda A p}}\right\rangle_p^{\dagger} N_A(X)\,dS(X),$$

$$\text{(4.13)}\qquad \frac{\partial}{\partial t}\int_{B^*} P^{\Lambda} dV(X) = \int_{B^*}\{R^{\Lambda} - q^{\Lambda}\}\, dV(X) - \int_{\partial B^*} S^{A\Lambda} N_A(X)\, dS(X).$$

When the boundary conditions (4.4) and (4.9) are used, we then obtain

$$\text{(4.14)}\qquad \frac{\partial}{\partial t}\int_{B^*}\left\langle\frac{\partial L}{\partial(\partial_t U_{\Lambda p})}\right\rangle_p^{\dagger} dV(X) = \int_{B^*}\left\{\left\langle\frac{\partial L}{\partial U_{\Lambda p}}\right\rangle_p^{\dagger} - q^{\Lambda}\right\} dV(X) - \int_{\partial B^*} T^{\Lambda} dS(X),$$

$$\text{(4.15)}\qquad \frac{\partial}{\partial t}\int_{B^*} P^{\Lambda} dV(X) = \int_{B^*}\{R^{\Lambda} - q^{\Lambda}\}\, dV(X) - \int_{\partial B^*} T^{\Lambda} dS(X)\ (^6).$$

We note a particular application of the macroscopic theory of the second kind which has appeared in the recent literature. Let $W(Z^A - X^A) = \overline{W}(\|Z - X\|) = \overline{W}(\xi)$, where $\overline{W}(\xi)$ is zero for $\xi > \mu$ and $\overline{W}(\xi) > 0$ for $0 \leqslant \xi < \mu$, then $W(Z^A, X^A)$ may be viewed as a function that localizes the contribution of $f(Z)$ about $f(X)$ in

$$\text{(4.16)}\qquad \langle f\rangle_p^{\dagger} = \int_{B^*} W(Z^A - X^A)\, h_p(Z^A, X^A) f(Z^A)\, dV(\xi).$$

We now take $p = 1$ and suppress the p occurrence in all terms. Further, if we expand $h(Z^A, X^A) = h(Z^A - X^A)$ in a power series in $(Z^A - X^A)$, we then have

$$\text{(4.17)}\qquad \langle f\rangle^{\dagger} = \sum_{i=0}^{\infty} h_{B_1\ldots B_i}(X^A)\langle f\rangle^{B_1\ldots B_i}$$

with

$$\text{(4.18)}\qquad h_{B_1\ldots B_i}(X^A) = \left.\frac{\partial^i h(Y^A)}{\partial Y^{B_1}\ldots\partial Y^{B_i}}\right|_{Y^A = X^A},$$

$$\text{(4.19)}\qquad \langle f\rangle^0 = \int_{B^*} W(Z^A - X^A) f(Z^A)\, dV(Z),$$

$$\text{(4.20)}\qquad \langle f\rangle^{B_1 = B_i} = \int_{B^*} W(Z^A - X^A)(Z^{B_1} - X^{B_1})\ldots(Z^{B_i} - X^{B_i}) f(Z^A)\, dV(Z).$$

When (4.19), (4.20) are used in (4.8)–(4.10), we have

$$\text{(4.21)}\qquad q^A = \sum_{i=0}^{\infty} h_{B_1\ldots B_i}(X)\{\langle R^{\Lambda}\rangle^{B_1\ldots B_i} - \partial_A\langle S^{A\Lambda}\rangle^{B_1\ldots B_i} - \partial_t(\langle P^{\Lambda}\rangle^{B_1\ldots B_i})\} - \sum_{i=0}^{\infty}\partial_A h_{B_1\ldots B_i}\langle S^{A\Lambda}\rangle^{B_1\ldots B_i},$$

(6) The quantities $\{q^{\Lambda}, T^{\Lambda}\}$ that appear in (4.12) and (4.14) are not necessarily the same as the quantities $\{q^{\Lambda}, T^{\Lambda}\}$ in (4.13) and (4.15).

$$\mathscr{T}^{\Lambda} = \sum_{i=0}^{\infty} N_A(X) h_{B_1\ldots B_i}(X) \langle S^{A\Lambda}\rangle^{B_1\ldots B_i}|_{\partial B^*}, \tag{4.22}$$

$$\mathscr{I}^{\Lambda} = \sum_{i=0}^{\infty} h^{B_1\ldots B_i} \langle P^{\Lambda}\rangle^{B_1\ldots B_i}|_{t=t_0}, \quad \Lambda = 1, \ldots, N. \tag{4.23}$$

The Eqs. (4.21) are identical in form to what ERINGEN [6] refers to as the "master balance equation". Further, since the function $h(X^A, Z^A)$ is arbitrary (i.e. the explicit form of the operator $\langle\ \rangle$ need not be assumed in macroscopic theories of the second kind), we can allow $h(Z^A, X^A)$ to span a space of testing functions with respect to $\{Z^A\}$ for each fixed $\{X^A\}$. Under these conditions (4.21) through (4.23) can hold if, and only if, the coefficient of each $h_{B_1\ldots B_i}$ vanishes separately. Thus, with

$$\begin{aligned} q_{\Lambda} &= \sum_{i=0}^{\infty} h_{B_1\ldots B_i} q^{\Lambda B_1\ldots B_i}, \\ \mathscr{T}^{\Lambda} &= \sum_{i=0}^{\infty} h_{B_1\ldots B_i} \mathscr{T}^{\Lambda B_1\ldots B_i}, \\ \mathscr{I}^{\Lambda} &= \sum_{i=0}^{\infty} h_{B_1\ldots B_i} \mathscr{I}^{\Lambda B_1\ldots B_i}, \end{aligned} \tag{4.24}$$

and the corresponding"distributional" structure that is accordingly attached to q^{Λ}, $\mathscr{T}^{\Lambda}$, $\mathscr{I}^{\Lambda}$, we have

$$q^{\Lambda 0} = \langle R^{\Lambda}\rangle^0 - \partial_A\langle S^{A\Lambda}\rangle^0 - \partial_t(\langle P^{\Lambda}\rangle^0), \ldots, \tag{4.25}$$

$$q^{\Lambda B_1\ldots B_i} = \langle R^{\Lambda}\rangle^{B_1\ldots B_i} + S\langle S^{B_1\Lambda}\rangle^{B_2\ldots B_i} - \partial_A\langle S^{A\Lambda}\rangle^{B_1\ldots B_i} - \partial_t(\langle P^{\Lambda}\rangle^{B_1\ldots B_i}), \tag{4.26}$$

$$\mathscr{T}^{\Lambda 0} = N_A\langle S^{A\Lambda}\rangle^0|_{\partial B^*}, \ldots, \tag{4.27}$$

$$\mathscr{T}^{\Lambda B_1\ldots B_i} = N_A\langle S^{A\Lambda}\rangle^{B_1\ldots B_i}|_{\partial B^*}, \tag{4.28}$$

$$\mathscr{I}^{\Lambda 0} = \langle P^{\Lambda}\rangle^0|_{t=t_0}, \ldots, \tag{4.29}$$

$$\mathscr{I}^{B_1\ldots B_i} = \langle P^{\Lambda}\rangle^{B_1\ldots B_i}|_{t=t_0}, \tag{4.30}$$

where S denotes complete symmetrization with respect to the B-indices. The $S\langle S^{B_1\Lambda}\rangle$ come from the fact that

$$\partial_A h_{B_1\ldots B_i} = \frac{\partial}{\partial X^A}\left(\frac{\partial^i h(Y^A)}{\partial Y^{B_1}\ldots\partial Y^{B_i}}\bigg|_{Y^A=X^A}\right) = h_{B_1\ldots B_i A}$$

and that $h_{B_1\ldots B_i}$ are completely symmetric in the B's. The Eqs. (4.25), (4.26) are identical with those given by ERINGEN [6]. *However, the constitutive relations and the state variables that must be used to augment the system* (4.25)–(4.26) *cannot be the classical ones, since the carriers are now quite arbitrary, and in fact there will be no simple quantity such as "displacements" and "displacement gradients" which could possibly serve as state variables.* This follows from the fact that "averages" of displacement gradients are no longer gradients of macroscopic displacements; i.e.,

$\langle\partial_A\Phi_{\Lambda}\rangle_p \neq \partial_A\langle\Phi_{\Lambda}\rangle_p$ so that $\partial_B\langle\partial_A\Phi_{\Lambda}\rangle_p \neq \partial_A\langle\partial_B\Phi_{\Lambda}\rangle$ in general.

There is another obvious alternative which does not require such an arbitrary structure for $h_p(X^A, X^A)$: we simply take $h_p(Z^A; X^A)$ to be the set of functions $\{1 = h_0, Z^{A_1} - X^{A_1} =$

$= h_{A_1}, (Z^{A_1}-X^{A_1})(Z^{A_2}-X^{A_2}) = h_{A_1A_2}, \ldots, (Z^{A_1}-X^{A_1})(Z^{A_2}-X^{A_2})\ldots(Z^{A_r}-X^{A_r}) = h_{A_r\ldots A_r}\}$. In this case, however, (4.8) yields only N equations of the form

$$(4.31) \qquad q^A = \sum_i \{\langle R^{AB_1\ldots B_i}\rangle^{B_1\ldots B_i} - \partial_A\langle S^{AAB_1\ldots B_i}\rangle^{B_1\ldots B_i} - \partial_t(\langle P^{AB_1\ldots B_i}\rangle^{B_1\ldots B_i})\}$$

rather than separate equations for each choice of the B indices as is given in (4.25), (4.26). Setting each collection of terms with the same B indices equal to given functions which are such that (4.31) is satisfied would be sufficient, but is not necessary.

Particular note should be made of the results in [7], in which it is shown that the theoretical structure obtained for linearized, nonlocal, isotropic, elastic solids can be used to obtain an exact fit of the acoustical and optical branches of elastic shear waves within one Brillouin zone. These results provide a simple continuum model of periodic one-dimensional lattices and their associated dynamics, and suggests the breadth and generality of nonlocal theories. The results are also highly suggestive that nonlocal theories may provide theoretical constructs whereby an orderly transition from lattice dynamics to classical elasticity may be effected. This is further substantiated by the results established in [8] concerning nonlocal formulations of the Rayleigh surface wave problem and the correlations between the theory and known experimental dispersion relations for such surface wave phenomena.

Appendix

This appendix examines variational theories in which spatial derivatives occur in the form $\partial_A\langle\Phi_A\rangle_p$ rather than in the form $\langle\partial_A\Phi\rangle_p$. We accordingly assume the existence of a Lagrangian function

$$(A.1) \qquad L = L(X^A, t; \Phi_A(X^A, t), \dot{\Phi}_A(X^A, t), \partial_B\Phi_A(X^A, t); \langle\Phi_A\rangle_1, \partial_t\langle\Phi_A\rangle_1, \partial_B\langle\Phi_A\rangle_1; \ldots; \langle\Phi_A\rangle_M, \partial_t\langle\Phi_A\rangle_M, \partial_B\langle\Phi_A\rangle_M,$$

of class C^2 in its $4+5N(1+M)$ arguments which defines the action functional

$$(A.2) \qquad J[\{\Phi_A\}](L) = \int_T\int_{B^*} L\,dV(X)\,dt.$$

Then, for simplicity, field equations and the boundary data are obtained from the requirement that $J[\{\Phi_A+\phi_A\}](L) = J[\{\Phi_A\}](L)+o(\|\{\phi_A\}\|_1)$ must hold for all $\{\phi_A(X^A, t)\} \in \in \mathscr{D}_1(D^*; N)$. On introducing the notation

$$\{e|L\}_\xi = \frac{\partial L}{\partial\xi} - \partial_A\left(\frac{\partial L}{\partial(\partial_A\xi)}\right) - \partial_t\left(\frac{\partial L}{\partial(\partial_t\xi)}\right)$$

for the local Euler-Lagrange derivative [2] and using the same arguments as used in Sects. 2 and 3, we obtain the field equations

$$(A.3) \qquad 0 = \{e|L\}_{\Phi A} + \sum_{p=1}^{M}\langle\{e|L\}_{\langle\Phi_A\rangle_p}\rangle_p^\dagger + \sum_{p=1}^{M}\int_{\partial B^*}\frac{\partial L}{\partial(\partial_A\langle\Phi_A\rangle_p)}(Z^E)W(Z^E-X^E)h_p(Z^E-X^E)N_A(Z^E)\,dS(Z),$$

the boundary conditions

$$0 = \frac{\partial L}{\partial(\partial_A \Phi_\Lambda)}\bigg|_{\partial B^*} N_A(X^B). \tag{A.4}$$

The reader should carefully note that there are no boundary conditions involving $\frac{\partial L}{\partial\langle\partial_A \Phi_\Lambda\rangle_p}$; instead, these terms now occur in the field equations (A.3). If terms like $\frac{\partial L}{\partial(\partial_A\langle\Phi_\Lambda\rangle_p)}$ or integrals of such terms are to be identified with the macroscopic properties of a body, then field equations of the form of (A.3) with boundary conditions (A.4) would usually be unacceptable. It is for this reason that we based the previous analysis on Lagrangian functions whose general form is given by (2.4).

References

1. R. S. RIVLIN, In "Instituto Nacionale di Alta Mathematica, Symposia Mathematica", **1**, 358, 1968.
2. D. G. B. EDELEN, *Nonlocal Variations and Local Invariance of Fields*, American Elsevier, New York 1969.
3. D. G. B. EDELEN, Int. J. Engng. Sci., **7**, 269, 1969; **7**, 287, 1969; **7**, 373, 1969; **7**, 391, 1969; **7**, 401, 1969; **7**, 677, 1969; **7**, 843, 1969; **8**, 517, 1970; **9**, 741, 1971; **9**, 801, 1971; **9**, 815, 1971; **9**, 912, 1971; **11**, 1109, 1973.
4. V. P. BHATKAR, Int. J. Engng. Sci., **9**, 871, 1971.
5. D. G. B. EDELEN, *Nonlocal variational mechanics — IX. Lagrangians with multiple interactions*, Int. J. Engng. Sci., **13**, 861, 1975.
6. A. C. ERINGEN, Int. J. Engng. Sci., **8**, 819, 1970.
7. A. C. ERINGEN, Letters Appl. Engng. Sci., **1**, 129, 1973.
8. A. C. ERINGEN, Letters Appl. Engng. Sci., **1**, 11, 1973.

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