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Plastic strains in composite media

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THE aim of this paper is derivation of the constitutive equations (in a frame of infinitesimal theory) for a random multiphase medium, each of phases being assumed to be perfectly elastic-plastic. The fields of stress and strains caused by distorsions (in a probabilistic description) are determined and the distorsion field is found on the basis of the energetic theorems of plasticity theory. The parametric representation of the stress-strain curves is obtained in which the solution for a perfectly elastic reference medium is a parameter. The effects of macrononhomogeneity and plastic nonlocality are analyzed.

Celem pracy jest wyprowadzenie związków konstytutywnych (w ramach teorii infinitezymalnej) dla ośrodka losowego wielofazowego, w którym poszczególne fazy są idealnie sprężysto-plastyczne. Wyznacza się pola naprężeń i odkształceń wywołane dystorsjami (w opisie probabilistycznym), pole zaś dystorsji znajduje się na podstawie twierdzeń energetycznych teorii plastyczności. Uzyskuje się parametryczne przedstawienie krzywych naprężenie-odkształcenie, gdzie parametrem jest rozwiązanie dla ośrodka odniesienia idealnie sprężystego. Analizuje się efekty makroniejednorodności i nielokalności plastycznej.

Целью работы является вывод определяющих соотношений (в рамках инфинитезимальной теории) для случайной, многофазной среды, в которой отдельные фазы идеально упруго-пластически. Определяются поля напряжений и деформаций, вызванные дисторсиями (в пробабилистическом описании), поле же дисторсии находится на основе энергетических теорем теории пластичности. Получается параметрическое представление кривых напряжение-деформация, где параметром является решение для среды сравнения — идеально упругой среды. Анализируются эффекты макро-неоднородности и пластической нелокальности.

1. Preliminaries

THE subject of the present paper consists in deriving the constitutive law for a medium composed of perfectly elastic-plastic phases, the internal geometry of the medium being random.

The medium, as usually in the theory of multiphase media, is presupposed to be composed of phases and the phases — composed of grains (and, strictly speaking, of grain boundaries). A grain is defined as a region (i.e. an open and connected set of points) exhibiting homogeneous (constant) mechanical properties including the spatial orientation of tensor quantities. A phase is a (denumerable) set of grains with the same mechanical properties, differing only by their spatial orientation, i.e. by an orthogonal transformation to be performed over the substance of a reference grain. For isotropic phases the definitions of phase and grain coalesce and this is just the case dealt with in the present paper albeit an extention to the anisotropic case will be seen to be straightforward. We assume we know for each phase its tensor of elasticity and its yield criterion, i.e., the equation

of the yield hypersurface in the stress space; in a particular case a phase may be purely elastic. Usual assumptions are made on the continuity of the medium and "smoothed" grain boundaries (which releases us from using generalized functions) the theory being infinitesimal.

The solution is carried out in two steps. First, we determine the fields of stress and strain brought about by a given (known) field of whatever distortions. At the second stage, we assume that the distortions are plastic, i.e., depend on the loading path and determine the actual field of distortion using extremum principles of plasticity. Finally, we come at a parametric description of the macro stress — macro strain relationship, the parameter being supplied by the purely elastic solution to the given boundary value problem for a homogeneous medium.

Define some basic notions and quantities. We introduce a *reference body*, i.e., a body of the same shape, under the same external loads and boundary conditions, which is *homogeneous* and perfectly *elastic*. We assume that the corresponding boundary value problem for this body had been solved and denote the fields of stress and strain obtained by $\boldsymbol{\sigma}^0(\mathbf{x})$ and $\boldsymbol{\epsilon}^0(\mathbf{x})$ (all tensor quantities are in absolute notation, indices being used for numbering phases). The quantities $\boldsymbol{\sigma}^{(0)}(\mathbf{x})$, $\boldsymbol{\epsilon}^{(0)}(\mathbf{x})$ denote fields of stress and strain in an elastic nonhomogeneous (multiphase) body brought about by external loads (body forces and forces at the boundary).

The fields of (whatever) distortions are denoted by $\boldsymbol{\sigma}^d(x)$, $\boldsymbol{\epsilon}^d(\mathbf{x})$. The distortion can be defined in a dual way. Separate an elementary volume element from the medium and consider it under the conditions of the second boundary value problem with zero stress vectors at the boundary; the distortion $\boldsymbol{\epsilon}^d$ is defined as a strain independent of stress

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^d, \quad \boldsymbol{\sigma} = 0. \tag{1.1}$$

On the other hand, consider the element under the conditions of the first boundary value problem with zero displacement vectors at the boundary; then we have

$$\boldsymbol{\epsilon} = 0, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^d, \tag{1.2}$$

i.e., the distortion is defined by a distortion stress $\boldsymbol{\sigma}^d$. The relationship between the dual quantities $\boldsymbol{\epsilon}^d$ and $\boldsymbol{\sigma}^d$ is

$$\boldsymbol{\sigma}^d = -C\boldsymbol{\epsilon}^d, \quad \boldsymbol{\epsilon}^d = -S\boldsymbol{\sigma}^d, \tag{1.3}$$

C and S being the tensors of elasticity and elastic compliance (in the sequel, we shall use preferably $\boldsymbol{\sigma}^d$). The quantities $\boldsymbol{\sigma}^{(d)}(\mathbf{x})$ and $\boldsymbol{\epsilon}^{(d)}(\mathbf{x})$ denote fields of stress and strain in the elastic nonhomogeneous body brought about by the given fields $\boldsymbol{\epsilon}^d(\mathbf{x})$ or $\boldsymbol{\sigma}^d(\mathbf{x})$; in other words, these are fields of eigenstress and eigenstrain, respectively. To specify distortions we shall use, for instance, $\boldsymbol{\epsilon}^p$, $\boldsymbol{\sigma}^p$, $\boldsymbol{\epsilon}^{(p)}$, $\boldsymbol{\sigma}^{(p)}$ for plastic distortions, or $\boldsymbol{\epsilon}^t$, $\boldsymbol{\sigma}^t$, $\boldsymbol{\epsilon}^{(t)}$, $\boldsymbol{\sigma}^{(t)}$ for the thermal ones.

2. The auxiliary problem

By the auxiliary problem we mean the first step of solution, described above, i.e., an elastic solution for a given field $\boldsymbol{\sigma}^d(x)$; let us point out, however, that for many problems

(e.g. for thermal distortions) the "auxiliary" problem completes the solution. The solution to this problem falls, in turn, into three partial steps:

(i) we replace the actual inhomogeneous body by a homogeneous one, loaded by a field of body forces of special kind, called polarization field; and find fields of stress and strain generated by polarization;

(ii) we come back to distortions in an inhomogeneous body and find fields of eigenstress and eigenstrain;

(iii) we pass from the deterministic to a stochastic problem performing adequate averaging.

2.1. The polarization problem

Consider the Hooke's law in the following form

$$\boldsymbol{\sigma} = C(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^d) = C_0\boldsymbol{\epsilon} + \boldsymbol{\tau}, \tag{2.1}$$

where

$$\tau = c\boldsymbol{\epsilon} + \boldsymbol{\sigma}^d, \quad c = C - C_0.$$

Here, C_0 is the elastic tensor for the reference body (which, for the time being, may be chosen at will) and $C = C(\mathbf{x})$ the actual elastic tensor at a given point of the multiphase body; τ defines the polarization term including the distortion stress apart of the inhomogeneity expression $c\boldsymbol{\epsilon}$.

Using (2.1) we obtain the following elastic equilibrium equation

$$\nabla \cdot (C_0\boldsymbol{\epsilon}) + \nabla \cdot \boldsymbol{\tau} + P = 0, \tag{2.2}$$

P being an arbitrary external body force. Decompose the solution into two parts, $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^0 + \boldsymbol{\epsilon}'$, the first of which satisfies the equation

$$C_0\nabla \cdot \boldsymbol{\epsilon}^0 + P = 0 \tag{2.3}$$

and the second

$$C_0\nabla \cdot \boldsymbol{\epsilon}' + (\nabla \cdot \boldsymbol{\tau}) = 0. \tag{2.4}$$

The Eq. (2.3), accompanied by adequate boundary conditions, represents a classical problem of infinitesimal elasticity being presupposed to have been solved in advance (expressing the strains by displacements, we obtain the Lamé equation). The Eq. (2.4), with the source term in the form of the divergence of a certain tensor field (resulting in a vector field) together with the zero boundary conditions (stress or displacement vectors disappearing at the boundary) defines the polarization problem.

The solution to (2.4) is given in the form

$$\boldsymbol{\epsilon}'(\mathbf{x}) = \boldsymbol{\Lambda}\boldsymbol{\tau}(\boldsymbol{\xi}), \tag{2.5}$$

the operator $\boldsymbol{\Lambda}$ transforming the polarization field $\boldsymbol{\tau}(\boldsymbol{\xi})$ into the strain field $\boldsymbol{\epsilon}'(\mathbf{x})$ (searched for) where

$$\boldsymbol{\Lambda} = \left(\mathbf{a}\int dv\delta + \bar{\int} dv\boldsymbol{\Lambda}\right) \tag{2.6}$$

or, in Cartesian coordinates,

$$\varepsilon'_{ps}(\mathbf{x}) = a_{psqr}\,\tau_{qr}(\mathbf{x}) + \overline{\int_D} \Lambda_{psqr}(\mathbf{x}, \boldsymbol{\xi})\,\tau_{qr}(\boldsymbol{\xi})\,dv \tag{2.7}$$

(summation convention!), where $\overline{\int_D}$ denotes the principal value of the singular integral extended over the whole body (domain D) and $\mathbf{a}$ is a constant fourth-order tensor; $\mathbf{x}$ and $\boldsymbol{\xi}$ are the given and the current point, respectively. Here we have

$$\Lambda_{psqr} = G_{q(p,s)r}, \qquad a_{psqr} = -\lim_{r\to 0} \oint_S G_{q(p,s)}\, n_r\, dS,$$

where $G_{pq}(\mathbf{x}, \boldsymbol{\xi})$ is the Green tensor of the Lamé problem for the reference body, S the surface of a small sphere including the point $\mathbf{x}$, having a radius r and the outward unit normal $\mathbf{n}$ (for the details of derivation cf. [1]). For example, for an infinite isotropic medium we obtain (cf. [1])

$$\Lambda_{psqr} = \frac{\lambda_0+\mu_0}{8\pi\mu_0(\lambda_0+2\mu_0)r^3}\left[\frac{\mu_0}{\lambda_0+\mu_0}(\delta_{sq}\delta_{pr}+\delta_{pq}\delta_{sr}) - \delta_{qr}\delta_{ps}\right.$$
$$+3\left(-\frac{\mu_0}{\lambda_0+\mu_0}\delta_{sq}n_pn_r+\delta_{qr}n_pn_s+\delta_{ps}n_qn_r+\delta_{pr}n_sn_q+\delta_{sr}n_pn_q\right.$$
$$\left.\left.-\frac{\mu_0}{\lambda_0+\mu_0}\delta_{pq}n_sn_r\right)-15n_pn_sn_qn_r\right],$$

$$a_{psqr} = \frac{1}{6\mu_0}(\delta_{pq}\delta_{sr}+\delta_{pr}\delta_{sq}) - \frac{\lambda_0+\mu_0}{15\mu_0(\lambda_0+2\mu_0)}(\delta_{pq}\delta_{sr}+\delta_{qs}\delta_{pr}+\delta_{sp}\delta_{qr}),$$

where

$$n_p = \frac{x_p-\xi_p}{r},$$

$$r = |\mathbf{x}-\boldsymbol{\xi}| = \sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2},$$

δ_{pq} are Kronecker symbols and λ_0, μ_0 are the Lamé constants for the reference medium.

Let us remark that the solution (2.7) is the result of the one out of two possible ways of approach, since we also may not use the polarization representation (2.1). In that second case one obtains an integro-differential operator instead of $\boldsymbol{\Lambda}$, which is perhaps a little more convenient in the problem of finding the macro-tensor of elasticity, while the present approach lends itself better to the calculations of macro-fields of stress and strains and seems to be indispensable in the plastic case, $\boldsymbol{\sigma}\Rightarrow\boldsymbol{\epsilon}$ relations being no more one-to-one and therefore calling for a parametric representation.

2.2. The distortion problem

The solution (2.5) would be sufficient for a homogeneous body, i.e., when the field τ does not contain the strain-depending term $c\boldsymbol{\epsilon}$ [cf. (2.1)]. In the general case, however, we must search for operators transforming the known fields $\boldsymbol{\sigma}^d(\mathbf{x})$, $\boldsymbol{\epsilon}^0(\mathbf{x})$ into the fields

of stress and strain taking account of the elastic inhomogeneity. To this end, substitute (2.5) and (2.1) in the fundamental solutions of (2.3) and (2.4):

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^0 + \boldsymbol{\epsilon}' = \boldsymbol{\epsilon}^0 + \boldsymbol{\Lambda}(\boldsymbol{\sigma}^d + c\boldsymbol{\epsilon}) = (\boldsymbol{\Lambda}c)\boldsymbol{\epsilon} + (\boldsymbol{\epsilon}^0 + \boldsymbol{\Lambda}\boldsymbol{\sigma}^d). \tag{2.8}$$

The solution to this integral equation [with regard to $\boldsymbol{\epsilon}(\mathbf{x})$] may be represented in the form of a Neumann-type series

$$\boldsymbol{\epsilon} = [J + (\boldsymbol{\Lambda}c) + (\boldsymbol{\Lambda}c)^2 + \ldots]\,(\boldsymbol{\epsilon}^0 + \boldsymbol{\Lambda}\boldsymbol{\sigma}^d) \tag{2.9}$$

(for details of this representation cf. [1]) and analogically for $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} = C[J + (\boldsymbol{\Lambda}c) + (\boldsymbol{\Lambda}c)f + \ldots]\,(\boldsymbol{\epsilon}^0 + \boldsymbol{\Lambda}\boldsymbol{\sigma}^d), \tag{2.10}$$

where J is the identity operator.

This yields following operators for partial influences of $\boldsymbol{\epsilon}^0$ and $\boldsymbol{\sigma}^d$ (which superpose),

$$\begin{aligned} \boldsymbol{\epsilon}^{(d)} &= A\boldsymbol{\sigma}^d, \quad & \boldsymbol{\epsilon}^{(0)} &= B\boldsymbol{\epsilon}^0, \\ \boldsymbol{\sigma}^{(d)} &= A'\boldsymbol{\sigma}^d, \quad & \boldsymbol{\sigma}^{(0)} &= B'\boldsymbol{\epsilon}^0, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} A &= [J + (\boldsymbol{\Lambda}c) + \ldots]\boldsymbol{\Lambda}, \\ B &= [J + (\boldsymbol{\Lambda}c) + \ldots], \\ A' &= C[J + (\boldsymbol{\Lambda}c) + \ldots]\boldsymbol{\Lambda}, \\ B' &= C[J + (\boldsymbol{\Lambda}c) + \ldots]. \end{aligned}$$

Point out that the operators B and B' yield the solution to the problem of an inhomogeneous body under external loads, while A and A' do the same for known distortion fields.

It is convenient for further (probabilistic) calculations to split the powers of the operator $\boldsymbol{\Lambda}c$ into $\boldsymbol{\Lambda}$- and c-terms, e.g. for B,

$$B = J + \boldsymbol{\Lambda}_{01}c_1 + \boldsymbol{\Lambda}_{012}c_1c_2 + \ldots \tag{2.12}$$

where

$$\boldsymbol{\Lambda}_{01\ldots n} = \boldsymbol{\Lambda}_{01} \otimes \boldsymbol{\Lambda}_{12} \otimes \ldots \otimes \boldsymbol{\Lambda}_{n-1,n},$$

$$c_1c_2 \ldots c_n = c_1 \otimes c_2 \otimes \ldots \otimes c_n$$

and indices denote consecutive points; for example, using Cartesian coordinates we have

$$\boldsymbol{\Lambda}_{012}c_1c_2 = (\boldsymbol{\Lambda}_{01} \otimes \boldsymbol{\Lambda}_{12})\,(c_1 \otimes c_2) \Rightarrow \boldsymbol{\Lambda}_{ijkl}(\mathbf{x}_0, \mathbf{x}_1)\boldsymbol{\Lambda}_{mnpq}(\mathbf{x}_1, \mathbf{x}_2)c_{klmn}(\mathbf{x}_1)c_{pqrs}(\mathbf{x}_2),$$

where e.g. (cf. (2.7))

$$\boldsymbol{\Lambda}_{ijkl} = a_{ijkl}\delta(\mathbf{x}_0 - \mathbf{x}_1) + \int_D d^3\mathbf{x}_1 \boldsymbol{\Lambda}_{ijkl}(\mathbf{x}_0, \mathbf{x}_1).$$

Tensor indices in $\boldsymbol{\Lambda}_{012}$ are seen to overlap and sum over the indices in the c_1c_2 term except for the first two (i, j) and the last two (r, s), so that always a fourth-order tensor is obtained. Analogically, we obtain for A, A', B'

$$\begin{aligned} A &= \boldsymbol{\Lambda}_{01} + \boldsymbol{\Lambda}_{012}c_1 + \boldsymbol{\Lambda}_{0123}c_1c_2 + \ldots, \\ B' &= C_0 + \boldsymbol{\Lambda}_{01}C_0c_1 + \boldsymbol{\Lambda}_{012}C_0c_1c_2 + \ldots, \\ A' &= \boldsymbol{\Lambda}_{01}C_0 + \boldsymbol{\Lambda}_{012}C_0c_1 + \boldsymbol{\Lambda}_{0123}C_0c_1c_2 + \ldots. \end{aligned} \tag{2.13}$$

2.3. The stochastic problem

The solutions (2.11) are deterministic, while for a random medium all elements are stochastic except for the field $\boldsymbol{\epsilon}^0(\mathbf{x})$. In particular, we have stochastic operators A, B, A', B' if the multiphase body exhibits random fields of $C(\mathbf{x})$, and stochastic fields $\sigma^d(x)$ when the distortions depend on random phases. The fundamental characteristics of random quantities are their mean values, obtained by averaging over ensemble (denoted by the symbol $\langle\ \rangle$, while mean values are denoted by an upperscore). When averaging is performed over a given phase only, we come at conditional averages ($\langle\ \rangle_{K_0}$ denotes averaging over the phase K_0) where K_L, $L = 0, 1, 2, \ldots$ is the K_L-th phase; note that we cannot simply number phases by $1, 2, \ldots$, since some (dyadic) quantities will depend on the order of phase indices.

For the fields uncorrelated with the phases, in particular for the deterministic $\boldsymbol{\epsilon}^0(\mathbf{x})$, we simply average term by term in the operator expansions (2.12), (2.13) and obtain for the phase mean fields

$$\bar{\boldsymbol{\epsilon}}^{(0)}_{K_0} = \bar{B}_{K_0}\boldsymbol{\epsilon}^0, \quad \bar{\boldsymbol{\sigma}}^{(0)}_{K_0} = \bar{B}_{K_0}\boldsymbol{\epsilon}^0 \tag{2.14}$$

where

$$\bar{B}_{K_0} = J + \boldsymbol{\Lambda}_{01}\langle c_1\rangle_{K_0} + \boldsymbol{\Lambda}_{012}\langle c_1 c_2\rangle_{K_0} + \ldots,$$

$$\bar{B}_{K_0} = \bar{C}_{K_0} + \boldsymbol{\Lambda}_{01}\langle C_0 c_1\rangle_{K_0} + \boldsymbol{\Lambda}_{012}\langle C_0 c_1 c_2\rangle_{K_0} + \ldots.$$

Here, for instance, $\langle c_1 c_2\rangle_{K_0}$ is the two-point conditional correlation moment i.e. the mean value of the product $c(\mathbf{x}_1)\otimes c(\mathbf{x}_2)$ for the given points $\mathbf{x}_1, \mathbf{x}_2$, provided that the point $\mathbf{x}_1$ covers the phase K_0 (only those "specimens" in the ensemble are taken into account where it does).

For the fields correlated with phases (that is, with stochastic operators) we cannot separate averaging, therefore we obtain

$$\bar{\boldsymbol{\epsilon}}^{(d)}_{K_0} = \langle A\boldsymbol{\sigma}^d\rangle_{K_0}, \quad \bar{\boldsymbol{\sigma}}^{(d)}_{K_0} = \langle A'\boldsymbol{\sigma}^d\rangle_{K_0}, \tag{2.15}$$

more explicitly,

$$\bar{\boldsymbol{\epsilon}}^{(d)}_{K_0} = \boldsymbol{\Lambda}_{01}\langle\boldsymbol{\sigma}^d_1\rangle_{K_0} + \boldsymbol{\Lambda}_{012}\langle c_1\boldsymbol{\sigma}^d_2\rangle_{K_0} + \boldsymbol{\Lambda}_{0123}\langle c_1 c_2\boldsymbol{\sigma}^d_3\rangle_{K_0} + \ldots,$$

$$\bar{\boldsymbol{\sigma}}^{(d)}_{K_0} = \boldsymbol{\Lambda}_{01}\langle C_0\boldsymbol{\sigma}^d_1\rangle_{K_0} + \boldsymbol{\Lambda}_{012}\langle C_0 c_1\boldsymbol{\sigma}^d_2\rangle_{K_0} + \Lambda_{0123}\langle C_0 c_1 c_2\boldsymbol{\sigma}^d_3\rangle_{K_0} + \ldots.$$

In the sequel we shall base our argument on the following assumptions:

(i) the reference field $\boldsymbol{\epsilon}^0$ is homogeneous, i.e., constant throughout the body and we restrict ourselves to the infinite medium consequently the operators do not depend on position;

(ii) the fields $\boldsymbol{\sigma}^d$ are region-wise homogeneous, in particular, they are constant in a phase.

The assumption (i) is indispensable if we want to deal with material constants, i.e., quantities depending on the substance only and neither on the geometry of the body nor on the properties of the loading field. The assumption (ii) will be needed for the approximation procedure to be outlined in Sect. 3. It is exact for some kinds of distortion fields e.g. for thermal dilatations.

The correlation moment, say $\langle c_1 c_2 \dots \sigma_n^d \rangle_{K_0}$, is expressed by the following formula

$$\langle c_1 c_2 \dots \sigma_n^d \rangle_{K_0} = \int \dots \int c_1 c_2 \dots \sigma_n^d f(c_1 \dots \sigma_n^d / x_0 \subset D_{K_0}) dc_1 \dots d\sigma_n^d, \tag{2.16}$$

where f is the corresponding conditional probability density function, given under the assumption (ii) by

$$f = \sum P_{K_1 \dots K_n / K_0} \delta(c_1 - c_{K_1}) \delta(c_2 - c_{K_2}) \dots \delta(\sigma_n^d - \sigma_{K_1}^d), \tag{2.17}$$

$P_{K_1 \dots K_n / K_0}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ being the conditional probability of "hitting" the phase K_1 by the point $\mathbf{x}_1$, the phase K_2 by $\mathbf{x}_2$ etc., provided the point $\mathbf{x}_0$ hits the phase K_0 (for a definite system of points); the delta functions point out passing to a discrete probability distribution, the variates assuming only fixed values c_{K_l}, $\sigma_{K_l}^d$ for the respective phases. For $P_{K_1 \dots K_n / K_0}$ we obtain, in particular,

$$P_{K_1 \dots K_n / K_0} = \frac{P_{K_0 \dots K_n}}{P_{K_0}} = \frac{P_{K_0 \dots K_n}}{v_{K_0}}, \tag{2.18}$$

$P_{K_0} = v_{K_0}$ denoting the concentration (i.e. volume ratio) of the phase K_0. Substituting (2.17) in (2.16) we obtain

$$\langle c_1 c_2 \dots \sigma_n^d \rangle = \sum P_{K_1 \dots K_n / K_0}(\mathbf{x}_0, \dots, \mathbf{x}_n) c_{K_1} c_{K_2} \dots \sigma_{K_n}^d. \tag{2.19}$$

Taking this into account we obtain for the n-th term of the expansion (appearing, for instance, in $\bar{B}_{K_0}$)

$$\Lambda_{012 \dots n} \langle c_1 c_2 \dots \sigma_n^d \rangle_{K_0} = \frac{1}{v_{K_0}} \sum (\Lambda_{01 \dots n} P_{K_0 K_1 \dots K_n}) \times c_{K_1} c_{K_2} \dots \sigma_{K_n}^d = \frac{1}{v_{K_0}} \sum \lambda_{K_0 K_1 \dots K_n}^{(n+1)} c_{K_1} c_{K_2} \dots \sigma_{K_n}^d \tag{2.20}$$

where

$$\lambda_{K_0 K_1 \dots K_n}^{(n+1)} = \Lambda_{01 \dots n} P_{K_0 K_1 \dots K_n} \tag{2.21}$$

is a constant tensor coefficient [of rank 4 $(n+1)$] with regard to $n+1$ phases $K_0, \dots, K_n$ depending, in general, on the order of indices, describing the internal geometry of the medium at a $(n+1)$-th level of correlation.

Using (2.21) we obtain for the mean of, say, $\bar{\epsilon}_{K_0}^{(d)}$ (over the phase K_0) the expansion (cf. (2.15))

$$\bar{\epsilon}_{K_0}^{(d)} = \frac{1}{v_{K_0}} \left[\sum \lambda_{K_0 K_1}^{(2)} \sigma_{K_1}^d + \sum \lambda_{K_0 K_1 K_2}^{(3)} c_{K_1} \sigma_{K_2}^d + \sum \lambda_{K_0 K_1 K_2 K_3}^{(4)} c_{K_1} c_{K_2} \sigma_{K_3}^d + \dots \right].$$

More systematically, if in each term we replace the first phase index by J and the last one by L (these indices not summing out) and introduce the following notations

$$\begin{aligned}
\sum_{K_1 \dots K_n} \lambda_{JK_1 \dots K_n L}^{(n+2)} c_{K_1} \dots c_{K_n} &= W_{JL}^{(n)}(c_{K_1}, \dots, c_{K_n}), \\
\sum_{K_1 \dots K_n} \lambda_{JK_1 \dots K_n}^{(n+1)} c_{K_1} \dots c_{K_n} &= W_J^{(n)}(c_{K_1}, \dots, c_{K_n}), \\
\sum_{K_0 \dots K_1} \lambda_{JK_1 \dots K_n L}^{(n+2)} C_J c_{K_1} \dots c_{K_n} &= W_{JL}'^{(n)}(c_{K_1}, \dots, c_{K_n}), \\
\sum_{K_1 \dots K_n} \lambda_{JK_1 \dots K_n}^{(n+1)} C_J c_{K_1} \dots c_{K_n} &= W_J'^{(n)}(c_{K_1}, \dots, c_{K_n}),
\end{aligned} \tag{2.22}$$

we obtain the final formulae

$$\bar{\boldsymbol{\epsilon}}_J^{(0)} = b_J \boldsymbol{\epsilon}^0, \quad \bar{\boldsymbol{\sigma}}_J^{(0)} = b'_J \boldsymbol{\epsilon}^0,$$

$$\bar{\boldsymbol{\epsilon}}_J^{(d)} = \sum_L a_{JL} \boldsymbol{\sigma}_L^d, \quad \bar{\boldsymbol{\sigma}}_J^{(d)} = \sum_L a'_{JL} \boldsymbol{\sigma}_L^d, \tag{2.23}$$

where

$$b_J = \frac{1}{v_J} \sum_{n=0}^{\infty} W_J^{(n)}, \quad b'_J = \frac{1}{v_J} \sum_{n=0}^{\infty} W_J'^{(n)},$$

$$a_{JL} = \frac{1}{v_J} \sum_{n=0}^{\infty} W_{JL}^{(n)}, \quad a'_{JL} = \frac{1}{v_J} \sum_{n=0}^{\infty} W_{JL}'^{(n)}. \tag{2.24}$$

Notice that the summing in (2.22) spreads over all combinatorial variations with repetitions of n elements K_i out of N (total number of phases). The fourth-rank tensor coefficients (2.24) yield average phase quantities (2.23) for homogeneous $\boldsymbol{\epsilon}^0$ and phase-homogeneous $\boldsymbol{\sigma}^d$ fields.

The terms W in (2.24) are seen to have a polynomial structure with regard to the variables c_{K_i}, and taking into account more terms in the sums (2.24) is equipollent to adding corrective polynomials of higher degrees. There are two possible cases: (i) we are given, on the basis of some theoretical assumptions, a general expression for the n-th correlation moment (i.e., we establish a stochastic model of the medium enabling us to generate all the probabilistic characteristics needed); supposing we know how to sum up the series in (2.24) we are able to calculate exactly the coefficients (2.24); (ii) we take correlation moments experimentally which, of course, is possible up to a moment of a certain order only; then we truncate the series and, consequently, make some mistake being, however, exactly as large as our lack of information.

Having found mean phase quantities (2.23) we easily calculate total averages and obtain the following macroquantities

$$\bar{\boldsymbol{\epsilon}}^{(d)} = \sum_J v_J \bar{\boldsymbol{\epsilon}}_J^{(d)} = \sum_{J,L} v_J a_{JL} \boldsymbol{\sigma}_L^d,$$

$$\bar{\boldsymbol{\epsilon}}^{(0)} = \sum_J v_J \bar{\boldsymbol{\epsilon}}_J^{(0)} = \sum_J v_J b_J \boldsymbol{\epsilon}^0; \tag{2.25}$$

$$\bar{\boldsymbol{\sigma}}^{(0)} = \sum_J v_J \bar{\boldsymbol{\sigma}}_J^{(0)} = \sum_J v_J b'_J \boldsymbol{\epsilon}^0. \tag{2.26}$$

Remark that $\bar{\boldsymbol{\sigma}}^{(d)} = 0$, the eigenstresses being self-equilibrated (we may use this relation to check the correctness of calculations), while (2.26) yields the solution to the elastic problem making it possible to determine the macro elasticity tensor. The formulae (2.23)–(2.26) complete the solution for phase-homogeneous distortions independent of the loading path (e.g. the thermal ones).

2.4. Strain energy

The macrostress cannot be used as a measure of the intensity of eigenstresses because it was seen to disappear nor can it be the macrostrain, since it depends on phase elastic

properties (and might eventually also disappear in spite of non-zero microstresses). The most convenient measure is the mean elastic strain energy of eigenstresses, describing the internal effort of a multiphase medium. It will be needed, besides, for the analysis of the plastic behaviour, to be based on energetic principles. The problem may be solved in an approximate or an exact manner. The approximate solution consists in assuming the field $\bar{\boldsymbol{\epsilon}}_J^{(d)}$ to be phase-wise homogeneous; this will prove to be sufficient for the calculation procedure to be outlined in the sequel, so we discuss it first.

The local elastic strain energy (per unit volume) is expressed by

$$E = \frac{1}{2}(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}^d)^T C(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}^d), \tag{2.27}$$

$$= \frac{1}{2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}^d)^T S(\boldsymbol{\sigma}-\boldsymbol{\sigma}^d),$$

where matrix-vector notation is being used. Now, if we assume $\boldsymbol{\epsilon}_J^{(d)}(\mathbf{x}) \approx \bar{\boldsymbol{\epsilon}}_J^{(d)}$, we obtain for the phase energy

$$E_J = \frac{1}{2}(\bar{\boldsymbol{\epsilon}}_J^{(d)}-\boldsymbol{\epsilon}_J^d)^T C(\bar{\boldsymbol{\epsilon}}_J^{(d)}-\boldsymbol{\epsilon}_J^d)$$

$$= \frac{1}{2}\Big(\sum_L a_{JL}\boldsymbol{\sigma}_L^d-\boldsymbol{\epsilon}_J^d\Big)^T C_J\Big(\sum_L a_{JL}\boldsymbol{\sigma}_L^d-\boldsymbol{\epsilon}_J^d\Big)$$

$$= \frac{1}{2}\Big(\sum_L \alpha_{JL}\boldsymbol{\sigma}_L^d\Big)^T C_J\Big(\sum_M \alpha_{JM}\boldsymbol{\sigma}_M^d\Big),$$

where $\alpha_{JL} = a_{JL}+S_J\delta_{JL}$ (Kronecker delta!). Consequently,

$$E_J = \frac{1}{2}\sum_{L,M}(\boldsymbol{\sigma}_L^d)^T D_{LM}^{(J)}\boldsymbol{\sigma}_M^d, \tag{2.28}$$

where

$$D_{LM}^{(J)} = \alpha_{JL}^T C_J \alpha_{JM}$$

is the energy coefficients matrix exhibiting the interaction energy contributions of particular phases.

The total unit strain energy of eigenstresses is expressed by

$$E = \sum_J v_J E_J = \frac{1}{2}\sum_{L,M}(\boldsymbol{\sigma}_L^d)^T E_{LM}\boldsymbol{\sigma}_M^d, \tag{2.29}$$

where

$$E_{LM} = \sum_J v_J D_{LM}^{(J)}.$$

The exact expression would always lead to a greater value, since at different points of a phase we would obtain for $\boldsymbol{\epsilon}^{(d)}$ some excess or defect contributions to $\bar{\boldsymbol{\epsilon}}^{(d)}$ and the matrix C in the above argument is positive definite; this, in turn, will be seen to yield a better final approximation. In order to obtain exact formulae, express the probability p in (2.21) in the following way

$$p_{K_0K_1\ldots K_n} = \langle \Phi_{K_0}\Phi_{K_1}\ldots\Phi_{K_n}\rangle, \tag{2.30}$$

where Φ_M are set characteristic functions,

$$\Phi_M(\mathbf{x}) = \begin{cases} 1 \text{ for } \mathbf{x} \in D_M, \\ 0 \text{ for } \mathbf{x} \notin D_M, \end{cases} \tag{2.31}$$

and D_M is the region occupied by the phase M. Now, before we perform averaging, we may introduce coefficients, analogous to λ [cf. (2.21)],

$$\tilde{\lambda}^{(n+1)}_{K_0 \dots K_n} = \Lambda_{0 \dots n} \Phi_{K_0} \dots \Phi_{K_n} \tag{2.32}$$

and manage them exactly in the same way we did for λ except for that the $\tilde{\lambda}$'s are random quantities. Finally, we would come at the formula analogous to (2.28) where (only at this stage we average)

$$D^{(J)}_{LM} = \langle \tilde{\alpha}^T_{0L} C_0 \tilde{\alpha}_{0M} \rangle_J, \tag{2.33}$$

i.e., we average but for specimens where $\mathbf{x}_0 \in D_J$, index "0" replacing "J". It is obvious that cross-correlation terms (including the Φ's in the left-hand and right-hand $\tilde{\alpha}$'s) would appear which would lead, generally, to correlation moments of twice as high order as in the approximate formulae.

3. The plastic strain problem

The solution in Sect. 2 is sufficient for all kinds of distortions provided the distortion fields are known i.e. given in advance. This is not the case of plastic distortions which depend on the loading path, in other words, on the present and the past fields of stress. Increasing external loads by a small amount (at a sufficiently intense state of stress) produces new elementary distortion sources disseminated, in general, all over the body in a random way; however, to find their positions (even in a statistic description) and partial fields generated by those would be a complicated problem. Therefore, the basic concept of solution consists in approximating the actual stress field by a region-wise homogeneous one where the regions may be selected arbitrarily (to some extent only, as it will be seen in the sequel) and using extremum principles of plasticity. This procedure will be discussed in Sect. 3.2 and now we start off with explaining the techniques of mathematical (non-linear) programming used in the argument for the first approximation, i.e., the approximation with phase-wise homogeneous stress fields.

3.1. The non-linear programming method

Let us restrict ourselves, for the time being, to a simple loading path, and assume approximately

$$\boldsymbol{\sigma}^{(0)}(\mathbf{x}) = \bar{\boldsymbol{\sigma}}^{(0)}_J, \quad \boldsymbol{\sigma}^{(d)}(\mathbf{x}) = \bar{\boldsymbol{\sigma}}^{(d)}_J \tag{3.1}$$

for $\mathbf{x} \in D_J$. According to the Haar-Kármán principle in Prager-Symonds formulation [2], the actual state of equilibrium minimizes the strain energy of plastic distortions,

provided the stresses do not exceed the plastic yield limit (at any point). The latter reads, under our assumption,

$$F_J(\overline{\boldsymbol{\sigma}}_J) = F_J(\overline{\boldsymbol{\sigma}}_J^{(0)} + \overline{\boldsymbol{\sigma}}_J^{(d)}) \geqslant 0, \quad J = 1, 2, \ldots, N, \tag{3.2}$$

where $F_J(\boldsymbol{\sigma}) = 0$ is the equation of the yield hypersurface in the stress space and (3.2) is the yield criterion (e.g. Huber-Mises, Tresca criterion, etc.) for the J-th phase. In a more explicit way, substituting (2.23) we obtain

$$F_J\left(b'_J\,\boldsymbol{\epsilon}^0 + \sum a'_{JL}\boldsymbol{\sigma}_L^d\right) \geqslant 0. \tag{3.3}$$

The function to be minimized is (2.29), consequently the whole of the problem is expressed as follows

$$\min \sum_{L,M} (\boldsymbol{\sigma}_L^d)^T E_{LM} \boldsymbol{\sigma}_M^d$$

$$F_J\left(b'_J\,\boldsymbol{\epsilon}^0 + \sum a'_{JL}\boldsymbol{\sigma}_L^d\right) \geqslant 0, \quad J = 1, 2, \ldots, N. \tag{3.4}$$

This is a constrained problem of non-linear programming where (2.29) is the objective function, (3.3) are the constraints and $\boldsymbol{\sigma}_J^d$ are the variables searched for (to be optimized). Having solved this problem and determined $\boldsymbol{\sigma}_J^d$ (as functions of $\boldsymbol{\epsilon}^0$, i.e., of external loads), we calculate easily macro strains and macrostresses by means of (2.23). We obtain

$$\begin{aligned} \overline{\boldsymbol{\sigma}} &= \overline{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}^0) = \sum_J v_J(\overline{\boldsymbol{\sigma}}_J^{(0)} + \overline{\boldsymbol{\sigma}}_J^{(d)}), \\ \overline{\boldsymbol{\epsilon}} &= \overline{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}^0) = \sum_J v_J(\overline{\boldsymbol{\epsilon}}_J^{(0)} + \overline{\boldsymbol{\epsilon}}_J^{(d)}), \end{aligned} \tag{3.5}$$

that is, a parametric equation of the $\overline{\boldsymbol{\sigma}}$ versus $\overline{\boldsymbol{\epsilon}}$ (generalized) curve. Of course, we shall obtain, in general, a strain-hardening law although the phase behaviour was assumed to be perfectly elastic-plastic.

Suppose, for instance, the objective function and the constraints fulfil the convexity conditions needed for application of the Kuhn-Tucker criterions of non-linear programming. Define consecutively the vectors of variables, of (non-negative) Lagrange multipliers and of constraints

$$\begin{aligned} \boldsymbol{\sigma}^{dT} &= [(\boldsymbol{\sigma}_1^d)^T, \ldots, (\boldsymbol{\sigma}_N^d)^T]^T, \\ \boldsymbol{\mu} &= [\mu, \mu_2, \ldots, \mu_N]^T, \\ \mathbf{F} &= [F_1(\boldsymbol{\sigma}^d), F_2(\boldsymbol{\sigma}^d), \ldots, F_N(\boldsymbol{\sigma}^d)]^T. \end{aligned} \tag{3.6}$$

Denote the Lagrangian

$$L(\boldsymbol{\sigma}^d, \mu) = E(\boldsymbol{\sigma}^d) + \mu^T F(\boldsymbol{\sigma}^d), \tag{3.7}$$

where E is constructed by $(3.4)_1$ and the components of F by $(3.4)_2$. Then, the Kuhn-Tucker conditions read

$$\nabla L(\boldsymbol{\sigma}^d) = 0, \quad \mu_J F_J(\boldsymbol{\sigma}^d) = 0, \quad \boldsymbol{\sigma}^d \text{ feasible}, \tag{3.8}$$

where ∇L is the gradient of the Lagrangean with regard to $\boldsymbol{\sigma}_J^d$. This is the set of algebraic equations for $\boldsymbol{\sigma}_J^d$. Remark that in many cases the constraints are linear (e.g. Tresca criterion) or may be linearized which (together with the objective function being quadratic)

leads to a quadratic programming problem, extensively elaborated in the theory of non-linear programming.

The above argument can easily be extended to the arbitrary loading path. In this case we use the Greenberg principle [3]; according to this the infinitesimal distortions (accompanying the infinitesimal increase or decrease of external loads) should minimize the elastic strain energy provided the total stresses at any point do not exceed the yield limit. We keep track of the loading path by expressing the reference field as a function of a unique parameter, say t, i.e., $\boldsymbol{\epsilon}^0 = \boldsymbol{\epsilon}^0(t)$ followed by a discretization, that is, we take into account the step-by-step changes $\Delta\boldsymbol{\epsilon}^0$. Now we optimize the variables $\Delta\boldsymbol{\sigma}_J^d$ brought about by $\Delta\boldsymbol{\epsilon}^0$ and, to this end, we minimize the function $E(\Delta\boldsymbol{\sigma}^d)$ under the constraints

$$F_J[\overline{\boldsymbol{\sigma}}_J^{(0)}(\boldsymbol{\epsilon}^0+\Delta\boldsymbol{\epsilon}^0)+\overline{\boldsymbol{\sigma}}_J^{(d)}(\boldsymbol{\epsilon}^0)+\Delta\overline{\boldsymbol{\sigma}}_J^{(d)}] \geqslant 0.$$

Consequently, using (2.23), we come at the following non-linear programming problem:

$$\min \sum_{L,M} (\Delta\boldsymbol{\sigma}_L^d) E_{LM} \Delta\boldsymbol{\sigma}_M^d,$$

$$F_J\Big[\overline{\boldsymbol{\sigma}}_J(\boldsymbol{\epsilon}^0)+b_J'\Delta\boldsymbol{\epsilon}^0+\sum_L a_{JL}\Delta\boldsymbol{\sigma}_L^d\Big] \geqslant 0, \quad J = 1, 2, \ldots, N. \tag{3.9}$$

Having found the optimum $\Delta\boldsymbol{\sigma}_J^d$'s we calculate $\overline{\boldsymbol{\epsilon}}_J^{(d)}(\boldsymbol{\epsilon}^0+\Delta\boldsymbol{\epsilon}^0)$, $\overline{\boldsymbol{\sigma}}_J^{(d)}(\boldsymbol{\epsilon}^0+\Delta\boldsymbol{\epsilon}^0)$ and plot the next point of the $\boldsymbol{\sigma}-\boldsymbol{\epsilon}$ curve. Then we repeat calculations starting with the "new" state as the initial one. Thus, in this case, we obtain the solution, in general, in a step-wise procedure.

3.2. Convergence procedure

The approximate solution in the precedent section was based on the assumption (3.1) (phase-wise homogeneous fields). It would be desirable to have an algorithm for any accuracy of approximation, i.e. convergent to the exact solution. Performing finer and finer partitions of inclusions into regions where the above homogeneity assumption holds, we approach the actual state as close as is needed (the fields being, in fact, variable within a phase). Suppose we divide the phase L in partial regions called sub-phases, denoted by $L^{(i)}$, where $\bigcup_i D_{L^{(i)}} = D_L$, $\bigcap_i D_{L^{(i)}} = \emptyset$ and $D_{L^{(i)}}$ is an open set. Keeping in mind that this is a completely fictious operation, sub-phases being no physical reality, we may select the regions from the point of view of mathematical effectiveness only. Assume that in each sub-phase the fields $\boldsymbol{\sigma}^d$, $\boldsymbol{\sigma}^{(d)}$, $\boldsymbol{\sigma}^{(0)}$ are homogeneous, i.e., (3.1) holds for any particular sub-phase. Now, if we replace a "phase" by a "sub-phase", all the results obtained so far remain valid for an adequately increased number of "phases".

Under such a partition of, say, the L-th phase, the probability p (cf. (2.17), (2.21), etc.) will be decomposed in the "partial" probabilities

$$p_{J\ldots L\ldots M} = p_{J\ldots L^{(1)}\ldots M}+p_{J\ldots L^{(2)}\ldots M}+\cdots,$$

the events of "hitting" a particular sub-phase being mutually exclusive and exhausting all possibilities. Note, however, that in particular polynomials (2.23) the partitions of

"intermediate" phases (denoted there by $K_1 \ldots, K_n$) do not influence the result since, in view of invariable c_{K_j}, the sub-terms with $K_j^{(i)}$ sum out. Thus, only the first (J) and the last phase have to be considered and we obtain

$$p_{JK_1 \ldots K_n L} = \sum_{i,l} p_{J^{(i)} K_1 \ldots K_n L^{(l)}},$$

$$\lambda_{JK_1 \ldots K_n L} = \sum_{i,l} \lambda_{J^{(i)} K_1 \ldots K_n L^{(l)}}. \tag{3.10}$$

Consequently, we obtain $W_{J^{(i)} L^{(l)}}$ instead of W_{JL}, $a_{J^{(i)} L^{(l)}}$ instead of a_{JL}, etc. Using adequately (3.4) (and analogically for (3.9)) we determine the optimum $\boldsymbol{\sigma}_{L^{(l)}}^{(d)}$'s and obtain the $\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\epsilon}}$ curve with an accuracy the better the larger was the number of sub-phases.

The important feature of the procedure is that there is no need to determine sub-phases in reality (say, design sub-phase regions on microsection enlargements of a specimen). All we have to do is to select "partial" correlation moment functions $p_{J^{(i)} \ldots L^{(l)}}(\mathbf{x}_0, \ldots \ldots \mathbf{x}_{n+1})$ accomplishing (3.10). Choosing these functions we, at the same time, perform actual partition of all inclusions up to the accuracy governed by correlation moments of the order considered. This selection is, in principle, arbitrary, there is, however, a restriction, namely, we could eventually obtain the same optimum $\boldsymbol{\sigma}^d$ for two (or more) sub-phases of a phase. This would mean that both sub-phases merge and the partition yields no better approximation. On the contrary, if we obtain different $\boldsymbol{\sigma}^d$'s we are sure the approximation is better.

In order to avoid the situation described, the partition should not be perfectly random, i.e., sub-regions must be in some way correlated with the internal geometry (say, isolate layers at grain boundaries, and the like). As a matter of fact, should the partition of the L-th phase be random, we would have $p_{J \ldots L^{(l)}} = v_l p_{J \ldots L}$, v_l being the volume ratio of sub-phase $L^{(l)}$ to phase L, for each $\mathbf{x}_{n+1}$ (independent events). Consequently $\lambda_{J \ldots L^{(l)}} = v_l \lambda_{J \ldots L}$, $W_{JL^{(l)}} = v_l W_{JL}$, $a_{JL^{(l)}} = v_l a_{JL}$, etc., and we would obtain for the constraints $F_L(v_l \boldsymbol{\sigma}_L^d) = v_l^q F_L(\boldsymbol{\sigma}_L^d)$, F being a homogeneous function with regard to $\boldsymbol{\sigma}$. Thus the constraints would be identical, while the objective function is invariant to interchanging sub-phase variables and this would result in identical optimum $\boldsymbol{\sigma}^d$ for sub-phases. We conclude that the probability functions $p(\mathbf{x})$ should not be divided into proportional parts. A preferable algorithm of (single or subsequent) partitions is the one leading to possibly different $\boldsymbol{\sigma}^d$'s (finding such optimum algorithms would call for further mathematical investigation).

Let us end up with the following comment. A division into a larger number of sub-phases increases, in general, the energy (the matrices E_{LM} being positive definite), therefore we obtain $\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\epsilon}}$ curves moved in a unique "direction" (within a given order of correlation considered), in other words, we obtain a one-sided limitation. Now, using respective complementary variational principles we could, by an argument analogous to the present one, obtain curves limiting from the other side and comprise the exact solution within both of them.

4. Nonhomogeneity, nonlocality, final conclusions

The results obtained in Sect. 3 were based on the assumption of an infinite medium and a homogeneous reference field $\boldsymbol{\epsilon}^0$. Consequently, the solution (3.5) (and analogically for non-simple loading) was independent of position and the deformation law could be looked upon as a material constant. The parametric representation was a consequence of a not one-to-one stress-strain dependence in plasticity.

For a body of finite size the operator $\boldsymbol{\Lambda}$ (2.6) is no more dependent on the difference of points $\mathbf{x}-\boldsymbol{\xi}$ only (i.e., invariant to translation) and becomes an explicit function of $\mathbf{x}$. Therefore, all results, in particular the coefficients (2.24), become dependent on position and the solution for homogeneous $\boldsymbol{\epsilon}^0$ (analogous to (3.5)) takes the form

$$\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}^0; \mathbf{x}), \quad \bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}^0; \mathbf{x}). \tag{4.1}$$

This discloses the effect of plastic macro-inhomogeneity (although the body is statistically homogeneous with regard to internal geometry), and is an extension of a similar effect known in elasticity of composites. Thus, the $\bar{\boldsymbol{\sigma}}-\bar{\boldsymbol{\epsilon}}$ curves at a given point for an assumed loading path [in (4.1) simple loading] will depend on the position of this point. This is sometimes called the boundary effect because it is especially pronounced near the boundaries of the body.

Take, on the other hand, an inhomogeneous reference field, $\boldsymbol{\epsilon}^0 = \boldsymbol{\epsilon}^0(\mathbf{x})$ (say, for an infinite medium, for the time being). In this case the coefficients b_J, b'_J in (2.23) are seen to become operators which we point out simply by writing $b_J(\boldsymbol{\epsilon}^0(\mathbf{x}))$, $b'_J(\boldsymbol{\epsilon}^0(\mathbf{x}))$. Consequently, the final solution for simple loading would read

$$\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}\big(\boldsymbol{\epsilon}^0(\mathbf{x})\big), \quad \bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}}\big(\boldsymbol{\epsilon}^0(\mathbf{x})\big). \tag{4.2}$$

This means that the $\bar{\boldsymbol{\sigma}}-\bar{\boldsymbol{\epsilon}}$ curves will depend not only on what happens at a point, and this expresses the effect of nonlocality. For arbitrary loading paths, we can no more plot a separate point of the curve because the latter must be constructed step-wise (cf. (3.9)) and (4.2) will generalize to

$$\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}\big(\boldsymbol{\epsilon}^0(\mathbf{x}; t)\big), \quad \bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}}\big(\boldsymbol{\epsilon}^0(\mathbf{x}; t)\big), \tag{4.3}$$

i.e., the shape of the curve depends on the loading path in a large neighbourhood of the point.

Finally, combining the effects (4.1) and (4.3) we come at the extended form

$$\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}\big(\boldsymbol{\epsilon}^0(\mathbf{x}; t), \mathbf{x}\big), \quad \bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}}\big(\boldsymbol{\epsilon}^0(\mathbf{x}; t), \mathbf{x}\big) \tag{4.4}$$

exhibiting the dependence on the loading path as well as on the position, i.e., the effects of macro-nonhomogeneity and nonlocality cumulate.

We want to conclude with an additional commentary on the method of approximation. One could imagine a procedure where the constraints in the non-linear programming problem would be established on a representative (finite) set of points (for a procedure of this kind cf. [4]). However, it is obvious that the number of such points should be extensive as compared to the number of partitions used for correlation functions, since the latter "cover", at a simple step, the whole of the body; besides, no subsequent averaging is needed for partitions concerning already probabilistic quantities.

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