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On continuum modeling of the dynamic behavior of layered composites

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A HIERARCHY of new approximate continuum theories to model the dynamic behavior of layered composites, termed matched effective stiffness theories, is advanced. In each approximation the approximate frequency spectra are matched as closely as possible with the exact spectra, and the range of validity of each approximation is ascertained. The procedure followed is conceptually not unlike that employed in deriving approximate plate theories. It requires an intimate knowledge of exact frequency spectra, together with the associated mode shapes. With the example of SH-waves propagating normal to the layering, it is shown that rather simple approximate field equations are capable of describing accurately the important filtering property of a composite, which none of the other approximate theories proposed by previous investigators were able to do. Similarities to certain phenomena in solid state physics are pointed out and, finally, an analogy to a system with one degree of freedom is drawn.

Dokonano znacznego postępu w doskonaleniu nowych przybliżonych teorii kontynualnych zwanych teoriami wyrównywanych efektywnych sztywności. W każdej aproksymacji przybliżone widma częstości są dobierane możliwie jak najbliżej widm ścisłych, a zakres ważności każdej aproksymacji jest ściśle określany. Dalsza procedura jest w koncepcji zbliżona do podejścia, które się wykorzystuje przy wyprowadzaniu przybliżonych teorii płyt. Wymaga ona pewnej znajomości ścisłych widm częstości i towarzyszących im postaci drgań. Na przykładzie płaskich fal ścinania (SH waves) rozprzestrzeniających się w kierunku normalnym do uwarstwienia wykazano, że już dosyć proste przybliżone równania pola są w stanie opisać poprawnie ważną własność kompozytu jaką jest filtrowanie, czego żadna z pozostałych teorii przybliżonych proponowanych przez poprzednich badaczy nie jest w stanie dokonać. Wskazano na podobieństwa do pewnych zjawisk występujących w fizyce ciała stałego, a w końcu pokazano analogię do układu z jednym stopniem swobody.

Проделан значительный прогресс в усовершенствовании новых, приближенных континуальных теорий, называемых теориями выравниваемых эффективных жесткостей. В каждой аппроксимации приближенные спектры частот подбираются возможно самым близким образом к точным спектрам, интервал же справедливости каждой аппроксимации точно определяется. Дальнейшая процедура в принципе сближена к подходу, который используется при выводе приближенных теорий плит. Требует она некоторого знания точных спектров частот и сопутствующих им типов колебаний. На примере плоских волн сдвига (SH-волны), распространяющихся в нормальном направлении к слоям, показано, что уже довольно простые приближенные уравнения поля в состоянии описать правильно важное свойство композита, каком является фильтрация, а чего ни одна из остальных приближенных теорий, предложенных предыдущими исследователями, не в состоянии сделать. Указаны аналогии с некоторыми явлениями выступающими в физике твердого тела, а наконец показана аналогия с системой с одной степенью свободы.

1. Introduction

It is by now well recognized that composite materials, such as fiber-reinforced or laminated solids, possess several features which make them more attractive for certain structural applications than conventional single-phase solids such as steel or aluminum. If linearly elastic behavior is assumed, it is conceptually quite straightforward to form ulate the prob-

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lem of dynamic response of a structural element, such as a beam or a plate, fabricated from a composite and subjected to prescribed time-dependent loads. Such a problem could be posed as one governed by classical field equations of elasticity, but with variable coefficients, to account for different material properties (elastic moduli, mass density) in the two materials which make up the composite. Conversely, classical field equations of elasticity with constant coefficients within the domain of each material could be used as a basis for the analysis, but then at each interface suitable continuity (or jump) conditions of traction and displacement would have to be imposed. In spite of the conceptual ease with which such problems could be formulated, it becomes very soon quite evident that only a very limited class of idealized problems would become analytically tractable on the basis of either of the two formulations sketched above.

Thus, a need arises to construct suitable, effective continuum models of approximate nature which would, on one hand, retain some of the essential physical features of the original system, while, on the other hand, be of sufficient simplicity as to permit analytical treatment of initial value problems of bounded bodies. As is outlined in Sect. 2, numerous attempts have been made by a variety of authors during the last decade to construct such approximate theories to describe the dynamic behavior of composites. Yet, in the view of the present authors, no definitive stage of development concerning approximate theories has been reached and there is still considerable room for proposing efficient, and in some sense optimized, approximate theories. It appears, in fact, that much work in this area is devoted to the question as to how to establish an approximate theory, rather than raising the question as to what is to be approximated, and to what degree. It is indicated briefly in Sect. 3 what the requirements of an approximate theory should or could be, using the example of the well-explored dynamic behavior of plates. It is the central thesis of this approach that, before attempting any construction of approximate theories, certain features of the system to be described approximately must be known in detail on the basis of an exact treatment, as discussed in Sect. 4 for layered two-phase composites. Similarities of this system to the behavior of certain systems of key importance in solid-state physics (e.g. motion of a crystal lattice or motion of an electron through a lattice) are touched upon in Sect. 5. With this background, certain simple approximate theories are presented in Sect. 6 and compared with other theories. It is found that among all the approximate theories proposed, the present one is the only one which is capable of reproducing, within a certain range, important phenomena predicted by the exact theory with astonishing accuracy, yet retaining relative simplicity as regards the structure of the governing field equations and the number of dependent functions involved. Specifically, the characteristic filtering property of a periodic composite has not been modeled at all by any of the previous approximate continuum theories.

For the sake of simplicity, attention is confined in the present paper to anti-plane strain motions only.

2. Some earlier work

A variety of diverse approaches has been taken by numerous investigators to construct approximate theories describing the dynamic behavior of composite materials, such as laminated two-phase elastic media. A customary early approach consisted in replacing the composite by a homogeneous, but usually anisotropic medium, whose material constants are determined in terms of the geometry and in terms of the material properties of the constituents of the composite. For a laminated medium the effective elastic constants have been computed by POSTMA [1](1), WHITE and ANGONA [2] and RYTOV [3], among others. Theories of this type are termed *effective modulus theories*. Without attempting to provide here a complete survey, some of the other later efforts are briefly mentioned in this Section.

A conceptually different approach was first proposed by HERRMANN and ACHENBACH [4] and elaborated upon with regard to its different aspects in several subsequent publications [5, 6, 7, 8, 9, 10]. Instead of introducing a representative homogeneous medium by means of *effective moduli*, elastic and geometric properties of individual layers are combined into *effective stiffnesses*. By means of a smoothing operation, representative kinetic and strain energy densities can be obtained and application of Hamilton's principle yields the equations of motion.

Another approach is based on a generalization of the elementary theory of gas mixtures in which the constituents coexist, each exerting its own partial pressure. Examples of this approach may be found in the work of BEDFORD and STERN [11, 12]. Extensions have been advanced by HEGEMIER *et al* under the name of *theories of interacting continua* and have been laid down in numerous publications [13, 14, 15, 16].

Several investigators have proposed continuum theories based on asymptotic expansions. As examples of this class of theories, the recent work by BEN-AMOZ [17, 18] and BALANIS [19] might be mentioned.

Other workers have attempted to model the dispersive behavior of harmonic waves in composites by means of analogies to other phenomena. The viscoelastic analogy has been employed by BARKER [20] and the dielectric analogy by CHRISTENSEN [21].

Finally, it might be worth mentioning some investigations where the mathematical modeling of a composite as a continuum has been abandoned in favor of at least partially discrete theory, as in the papers by CHAO and LEE [22] and NELSON and NAVI [23].

Even the partial listing of recent work on approximate mathematical modeling of composite materials is indicative of a great variety of attempts to supply an efficient and simple theory which is supposed to be useful in solving initial and boundary value problems for bonded composite bodies. Yet, it appears that there still is ample room for improvement. For example, it is well known that a periodically-layered composite will exhibit an important filtering property, as experimentally verified by ROBINSON and LEPPELMEIER [24], but so far this phenomenon has not been discussed within the framework of an approximate theory. Further, the range of validity and degree of accuracy of the approximate and exact frequency spectrum are often times discussed but insufficiently, preventing thus a more precise assessment of the capabilities and limitations of a given approximate theory.

(1) Numbers in brackets designate References at end of paper.

3. Requirements to be imposed on approximate theories

To assess the requirements which have to be imposed on approximate theories to be proposed for two-phase laminated composites, it may be advisable to seek guidance from a related but more fully explored problem area such as, for example, the elastic plate theory. The subject of waves and vibrations in isotropic and anisotropic plates is concisely but rather completely presented in the article by PAO and KAUL [25]. The objective of an approximate plate theory is to describe the dynamic behavior in terms of quantities which would be functions only of coordinates in the middle plane of the plate, the dependence on the coordinate normal to that plane having been eliminated. To achieve this objective it is necessary to establish the frequency spectrum on the basis of the exact treatment, as given by solutions of the Rayleigh-Lamb frequency equation. Next, a decision has to be reached as to the largest frequency or the largest wave number up to which the approximate theory should be valid. In either case, this decision determines a rectangular region in the frequency-wave number plane, the lower left-hand corner being the origin. One has now to study in detail the spectral lines within the rectangle to be approximated, together with associated mode shapes. These mode shapes have to be included approximately in the theory to be constructed. Knowing now what is to be approximated, there exists a variety of procedures by which to construct approximate theories. Timoshenko-type beam theories and Mindlin's plate theory may suffice as examples, which can be derived themselves in a variety of ways.

4. Exact treatment

For the sake of conciseness of presentation, attention will be confined, as mentioned, to anti-plane strain motion. With reference to Fig. 1, let u be the component of the displacement vector parallel to the x-axis and assume it to be of the form

$$u(y, z) = U(y)e^{i(K_z z+\omega t)},$$

while the other two displacement components vanish. It is noted that u is independent of x and thus represents horizontally polarized SH-waves. Floquet's quasi-periodicity

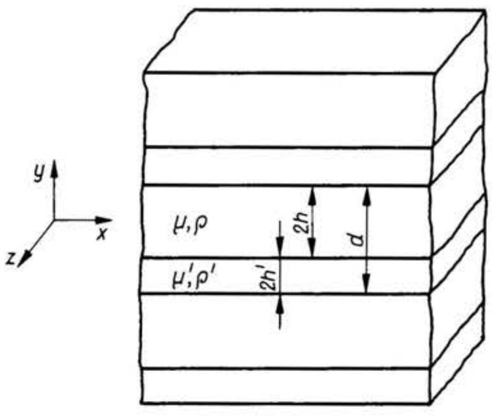

FIG. 1. Laminated medium.

across a periodic cell, continuity of stress and displacement at the bonded interfaces between any two layers, together with the displacement equation of motion, lead to the dispersion relation(²)

$$4\gamma\alpha\beta\cos\pi(1+t)\eta+(\gamma\alpha-\beta)^2\cos\pi(\alpha-t\beta)-(\gamma\alpha+\beta)^2\cos\pi(\alpha+t\beta) = 0.$$

In this relation the following dimensionless quantities have been introduced:

$$\gamma = \mu/\mu', \quad \sigma^2 = \mu\varrho'/(\mu'\varrho), \quad t = h'/h,$$
$$\Omega = 2h/\pi\times\omega/\sqrt{\mu/\varrho}, \quad \eta = 2hK_y/\pi, \quad \zeta = 2hK_z/\pi$$

and

$$\alpha = \sqrt{\Omega^2-\zeta^2}, \quad \beta = \sqrt{(\sigma\Omega)^2-\zeta^2}.$$

The dispersion relation is evaluated most conveniently in terms of the dimensionless frequency Ω as a function of the dimensionless wave numbers η and ζ for given values of the ratios $\gamma = \mu/\mu'$, $t = h'/h$ and $\sigma^2 = \mu\varrho'/(\mu'\varrho)$. Thus $f(\Omega, \eta, \zeta) = 0$ will represent implicitely a surface which might be thought as having evolved from the conical surface for

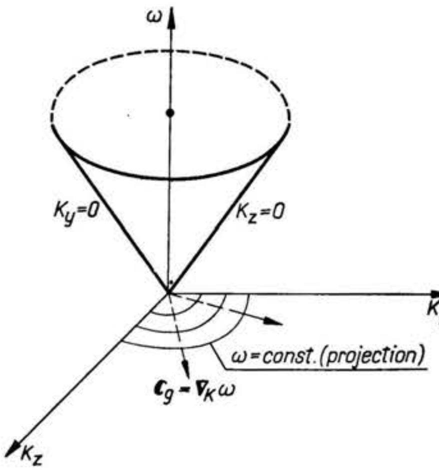

FIG. 2. Spectral surface for homogeneous body.

a homogeneous body sketched in Fig. 2, where the parameters γ and σ would have to be set equal to unity.

For a layered composite the surface $f(\Omega, \eta, \zeta) = 0$ is represented qualitatively in Fig. 3. It is seen that the surface features a structure given by the Brillouin zones, whose width is $2h/d$ or $1/(1+t)$ if $d = 2(h+h')$ and $t = h'/h$. The surface is single-valued, except at the ends of the Brillouin zones $\eta = 1/(1+t)$, $\eta = 2/(1+t)$, $\eta = 3/(1+t)$ labeled B_1, B_2, B_3, respectively.

For these values of η the surface is in general double-valued, except for certain isolated values of ζ. These might be called conical points, or points of coalescence of frequencies and their properties are discussed in detail in Ref. [26].

A quantitative representation of the surface $f(\Omega, \eta, \zeta) = 0$ for $\sigma^2 = 0.06$ and $t = 0.2$ in terms of projections on the $\Omega-\eta$ plane for various values of ζ, on the $\Omega-\zeta$ plane for

(²) Details of this derivation will be given in a separate forthcoming paper.

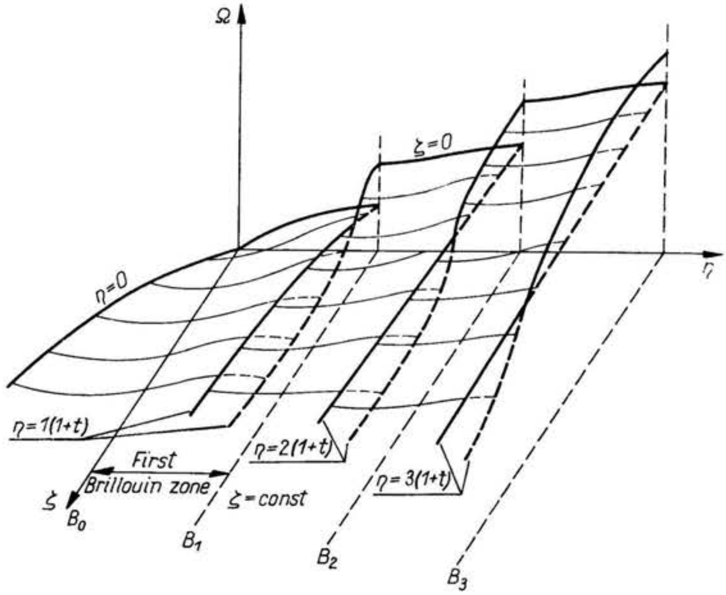

FIG. 3. Qualitative representation of spectral surface for laminated medium.

various values of η and on the $\eta-\zeta$ plane for various values of Ω is given in Figs. 4, 5, and 6, respectively. Figure 4 clearly exhibits the banded structure for $\zeta = 0, 0.5, 1.0$ and 1.5. The complex branches are not shown. For $\zeta = 0$, the low frequency limit, which is given by the *effective modulus theory*, as is well-known, as well as the high frequency limit, which is discussed in Ref. [27], are given as straight lines. Thus, it is to be noted that even

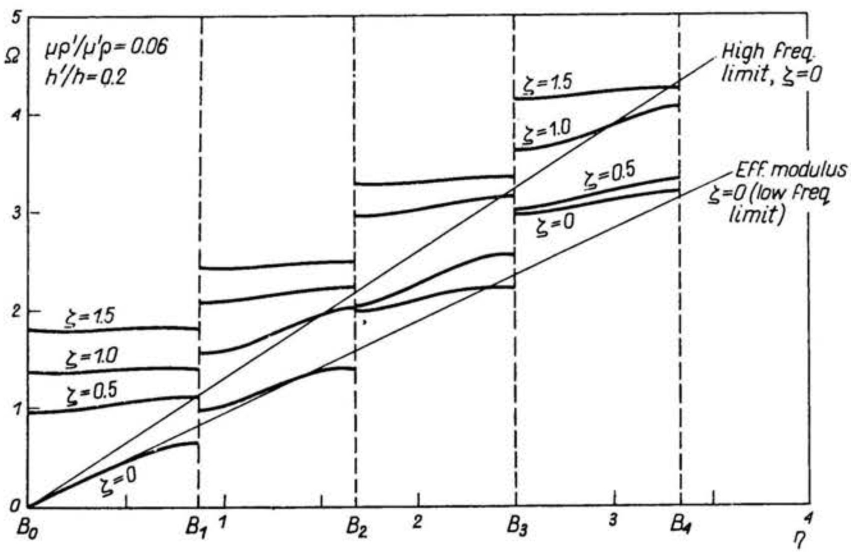

FIG. 4. Spectral lines for constant ζ of laminated medium.

for $\zeta = 0$, the banded structure of the dispersion curves does not *oscillate* around a single straight line, as is erroneously indicated in Fig. 1 of Ref. [24].

Figure 5 shows the dependence of Ω on ζ, the wave number parallel to the layers for values of η at the ends of the Brillouin zones, i.e, $\eta = 0$, $\eta = 1/(l+t)$, $\eta = 2/(1+t)$ and

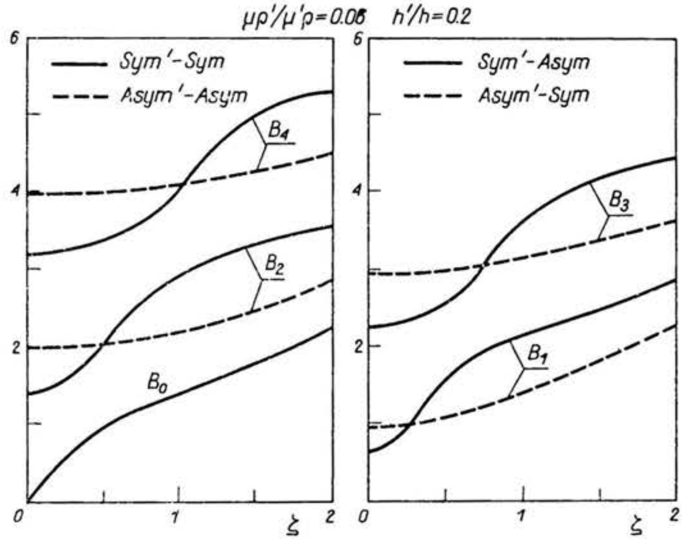

FIG. 5. Spectral lines and associated mode shapes along ends of Brillouin zones for laminated medium.

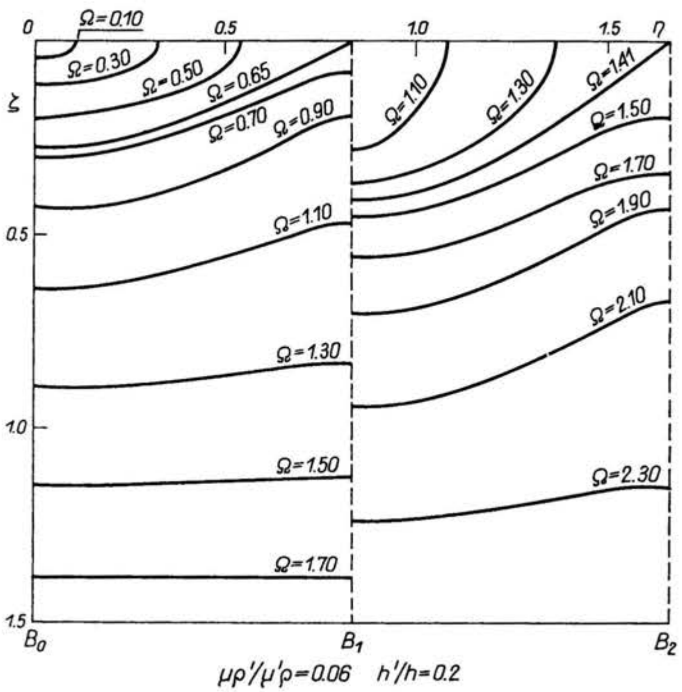

FIG. 6. Curves of constant frequency (Fermi lines).

$\eta = 3/(1+t)$, labeled B_0, B_1, B_2 and B_3, respectively. Calculations show that the full lines for even-subscripted B's correspond to mode shapes which are symmetric with respect to the midplanes in all layers, while the dashed lines in the same case correspond to mode shapes which are anti-symmetric in all layers. By contrast, for odd-subscripted B's, the full lines are associated with mode shapes which are symmetric in layers identified by properties with a prime (μ', ϱ') and anti-symmetric in the others (μ, ϱ), while mode shapes associated with dashed lines are just the other way round. The points of coalescence of frequencies already mentioned are also clearly seen in this figure.

Curves of constant frequency (Fermi lines) in the $\eta-\zeta$ plane are plotted in Fig. 6 for the first two Brillouin zones. These curves may be interpreted as being evolved from concentric quarter circles for a homogeneous body, as sketched in Fig. 2.

Since the group velocity $\mathbf{C}_g$ is the gradient of the frequency ω in the space of the wave number vector, i.e.,

$$\mathbf{C}_g = \nabla_K \omega = \frac{\partial \omega}{\partial K_y} \mathbf{e}_y + \frac{\partial \omega}{\partial K_z} \mathbf{e}_z,$$

where $\mathbf{e}_y$ and $\mathbf{e}_z$ are unit vectors in the y- and z-direction, respectively, it is possible to construct the group velocity field, as indicated merely qualitatively in Fig. 7. The principal

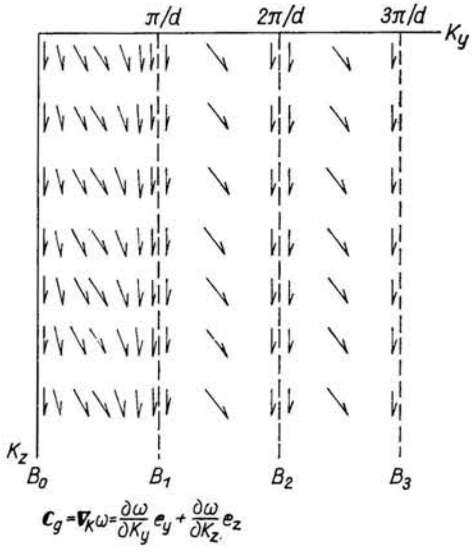

FIG. 7. Qualitative representation of group velocity field.

feature of this field is that along the ends of the Brillouin zones the group velocity vector is parallel to the layers, i.e. it does not have a component in the y-direction. A noteworthy exception are points of coalescence of frequencies mentioned earlier.

5. Similarities to phenomena in solid state physics

As has already become clear, the analysis of wave propagation through a periodically layered composite is closely related to the analysis of certain basic phenomena in solid state physics. Since the latter area has commanded the attention of researchers for a considerably longer period of time, it is understandably more fully explored and, thus, mechanicians who study composites may benefit from the existing similarities. Indeed, the motion of a free electron may be likened to a harmonic wave in a homogeneous elastic body, while an electron moving through a lattice corresponds to a wave propagating through a laminated composite normal to the layering. In both cases banded spectral lines contain many of the same features. Mode shapes at ends of the zones have been determined in solid state physics many years ago.

Surfaces of constant frequency in wave number space have a special significance in studying properties of metals and are referred to as Fermi surfaces. For details reference is made to the texts by ZIMAN [28], SMITH [29], KITTEL [30] and the article by MACKINTOSH [31], as well as to the more classical monograph by BRILLOUIN [32]. Some aspects of the similarity, however, do not carry through. For one thing, each element (layer) in the composite is itself a continuum, by contrast to discrete particles of a lattice. For another, in a laminated medium the width of the Brillouin zones depends in a different manner on the angle of incidence of a wave than in solid state physics, and there is even a singular direction (parallel to the layers) at which this width becomes infinite.

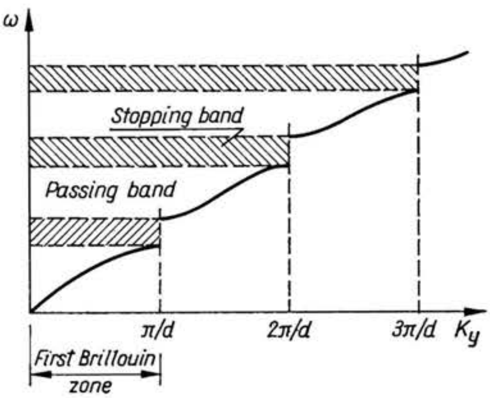

FIG. 8. Spectral lines of a good conductor.

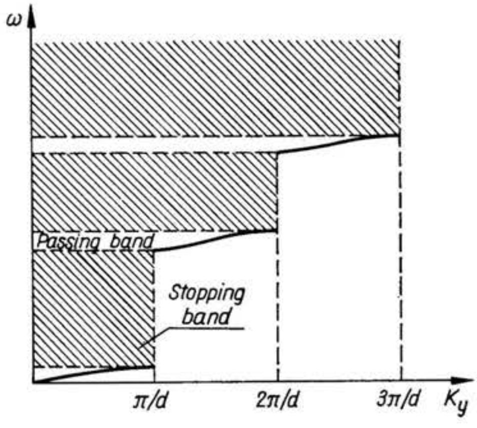

FIG. 9. Spectral lines of a good insulator.

As in solid state physics, one can term layered composites either as *conductors*, if the passing bands are wide, as indicated qualitatively in Fig. 8, or as *insulators*, if the passing bands are narrow, Fig. 9. Review of existing results in band theories of solids suggests that different approximate theories may have to be developed for conductors and insulators. The phase velocity $C_p = \omega/K_y$ for a conductor and an insulator is sketched qualitatively for the first three Brillouin zones in Fig. 10. Similarly, the group velocity $C_g = d\omega/dK_y$ as a function of the wave number K_y is sketched in Fig. 11.

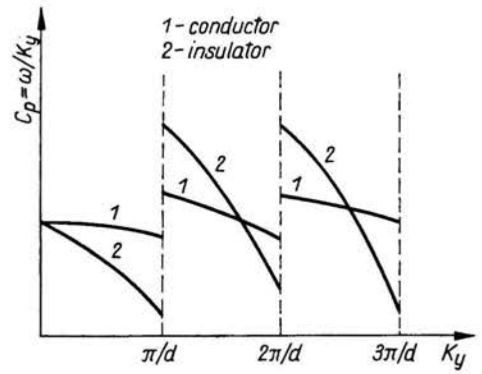

Fig. 10. Qualitative dependence of phase velocity on wave number.

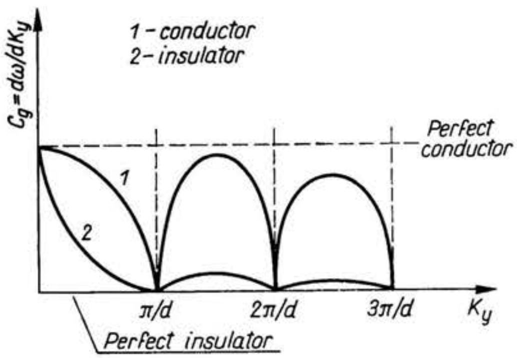

Fig. 11. Qualitative dependence of group velocity on wave number.

6. A hierarchy of matched effective stiffness theories

The knowledge which has now been gained concerning the exact frequency spectrum $f(\Omega, \eta, \zeta) = 0$ together with the associated mode shapes can be the basis for the establishment of a hierarchy of approximate theories. These theories, as regards their derivation, will be somewhat similar to the *old* effective stiffness theories already referenced. An important difference will exist, however, in that the coefficients of the various terms of the governing field equation, or more precisely of the resulting dispersion relation, will be determined by matching the exact and the approximate dispersion curves as closely as possible. It will be recalled that the coefficients of the *old* effective stiffness theories were given completely in terms of the material constants and the geometry of the composite.

For want of a better term, the new theories will be called *Matched Effective Stiffness Theories* or MES Theories for short. To make the present discussion as concise as possible, attention will again be restricted to SH-waves propagating normal to the layers. A hierarchy of three matched effective stiffness theories will be presented, each one identified in terms of the number of parameters available for matching.

6.1. Three-parameter theory

Suppose that in the effective stiffness theory [5] as presented in [33] attention is confined to anti-plane strain motions normal to the layers and, moreover, inertia coupling is neglected. The equations of motion reduce then to, in the notation of [33],

$$a_3 \partial_{22} u_1 - a_6 \partial_2 \psi_{21} = b_2 \ddot{u}_1, \qquad -a_6 \partial_2 u_1 + a_{15} \psi_{21} = -b_4 \ddot{\psi}_{21},$$

where the 2-direction is normal to the layering. u_1 is the gross displacement, ψ_{21} the micro-deformation. The notation $\partial_2 \equiv \partial/\partial x_2$ is used and dots above variables denote partial differentiation with respect to time.

If one assumes solutions proportional to $e^{i(Kx_2 - \omega t)}$, the dispersion relation results

$$\omega^4 - (\alpha K^2 + \beta)\omega^2 + \gamma K^2 = 0.$$

The three parameters α, β and γ are combinations of the coefficients a_3, a_6, a_{15}, b_2 and b_4 which are originally given in terms of geometric and material properties of the composite.

Here, by contrast, these parameters are to be determined in such a way that the approximate theory matches *best* the exact theory. One possibility to determine the three parameters is to require that

a) $d\omega/dK = C_g$ at $\omega = K = 0$,

b) $d\omega/dK = 0$ at $\omega = \omega_B, \quad K = K_B$,

c) $\omega = \omega_B$ at $K = K_B$.

In the above ω_B and K_B indicate the frequency and the wave number, respectively, at the end of the first Brillouin zone.

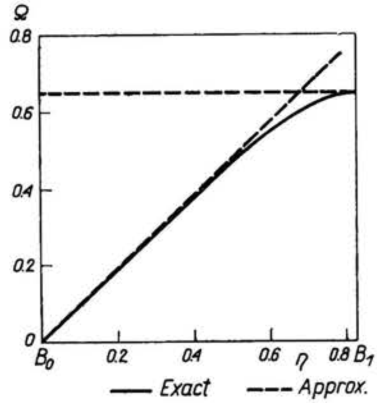

FIG. 12. Spectral lines of three-parameter MES theory.

With these requirements one obtains

$$\alpha = C_g^2, \quad \beta = \omega_B^2, \quad \gamma = \omega_B^2 C_g^2$$

and the dispersion relation factors into

$$(\omega^2 - \omega_B^2)(\omega^2 - C_g^2 K^2) = 0.$$

Thus both branches of the frequency spectrum consist of straight lines in the $\omega - K$ plane (or the $\Omega - \eta$ plane). The lower (acoustic) branch has slope C_g and passes through the origin, while the upper (optical) branch has vanishing slope. The two branches are uncoupled, cf. Fig. 12.

6.2. Four-parameter theory

In this theory the equations of motion for anti-plane strain normal to the layers are the same as in the *old* effective stiffness theory, i.e., cf. [33];

$$a_3 \partial_{22} u_1 - a_6 \partial_2 \psi_{21} = -b_1 \partial_{22} \ddot{u}_1 + b_2 \ddot{u}_1 + b_3 \partial_2 \ddot{\psi}_{21},$$
$$-a_6 \partial_2 u_1 + a_{15} \psi_{21} = b_3 \partial_2 \ddot{u}_1 - b_4 \ddot{\psi}_{21}.$$

The equations lead to the dispersion relation

$$(\alpha K^2 + \beta)\omega^4 + (\gamma K^2 + \delta)\omega^2 + K^2 = 0.$$

In addition to the three requirements in the Three-parameter Theory, the additional fourth parameter can be used to require the acoustic branch to pass through the point $\omega = \omega_B$, $K = K_B$. The acoustic and the optic branch are still uncoupled. It is seen from Fig. 13 that the acoustic branch of the approximate theory is very close to the exact curve in the whole Brillouin zone.

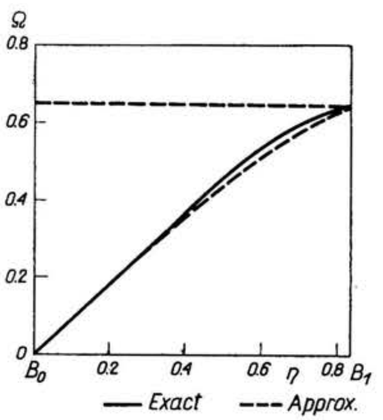

Fig. 13. Spectral lines of four-parameter MES theory.

6.3. Six-parameter theory

Assume next that in addition to the terms of the *old* effective stiffness theory, there is a term in the strain energy with the square of the gradient of the micro-deformation, thus generating a term proportional to $\partial_{22}\psi_{21}$ in the equations of motion. The equations of motion are then

$$a_3\partial_{22}u_1 - a_6\partial_2\psi_{21} = -b_1\partial_{22}\ddot{u}_1 + b_2\ddot{u}_1 + b_3\partial_2\ddot{\psi}_{21}$$

$$-a_6\partial_2 u_1 + a_{15}\psi_{21} + a_{18}\partial_{22}\psi_{21} = b_3\partial_2\ddot{u}_1 - b_4\ddot{\psi}_{21}.$$

This leads to the dispersion relation which contains six parameters

$$(\alpha K^2+\beta)\omega^4+(\gamma K^2+\zeta K^4+\delta)\omega^2+\varepsilon K^2+K^4 = 0.$$

The six independent parameters can be used to satisfy, in addition to the three conditions of the Three-parameter Theory, the conditions that the optical branch, on the periodic zone scheme, have the same frequency for zero wave number and that at the end of the first Brillouin zone the exact and the approximate frequencies coincide, and that there, in addition, $d\omega/dK = 0$.

The satisfaction of these requirements leads to a Matched Effective Stiffness Theory which reproduces not only the acoustic and the optic branches rather accurately now within the first two Brillouin zones, but contains in addition the complex loop corresponding to the stopping band, as plotted in Fig. 14. It is to be noted that the *old* effective

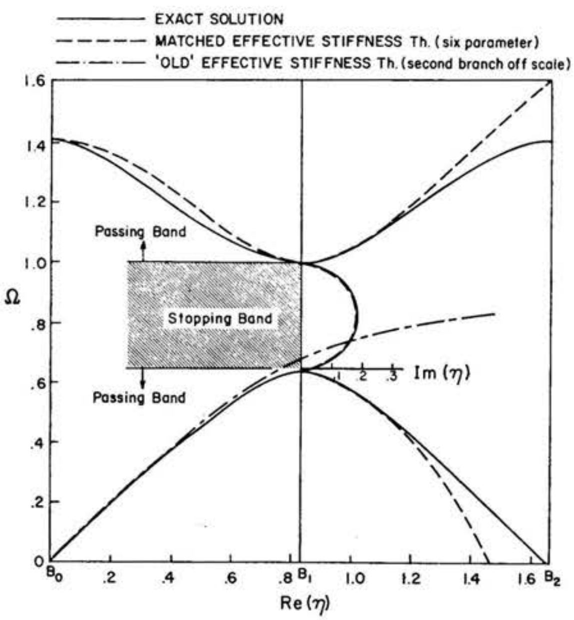

FIG. 14. Spectral lines of six-parameter MES theory.

stiffness theory is very inaccurate as regards the acoustic branch in the second Brillouin zone, while the optic branch of the *old* effective stiffness theory, even in the first Brillouin zone, is so inaccurate that it could not be shown on the scale of Fig. 14.

It is to be noted that instead of matching the upper branch in the first Brillouin zone (on the periodic or reduced zone scheme), it could be matched in the second Brillouin zone (on the extended zone scheme). Since the approximate dispersion relation is not periodic, a different result would be obtained, but considerably more algebraic work would be involved. These and related aspects will be discussed in a separate paper.

It should be noted further that instead of point-wise matching, a more rigorous matching procedure over one or two Brillouin zones could perhaps be devised, rendering the average mean square derivation between the exact and the approximate curves a minimum. Consideration of these possibilities is also deferred to later studies by the present authors.

Another new approach to construct approximate theories is based on the application of the Lie series. Use of this method will be presented in a separate study and is mentioned here merely for completeness.

7. Analogy to a system with one degree of freedom

It might be of interest to suggest an analogy between Floquet waves propagating through a layered composite in the direction normal to the laminates and a system with one degree of freedom consisting of a mass m and spring constant k. It is assumed that k is made to change periodically with time t from a large value to a smaller value, with period T^*, as indicated in Fig. 15. The resulting frequency equation will have a structure

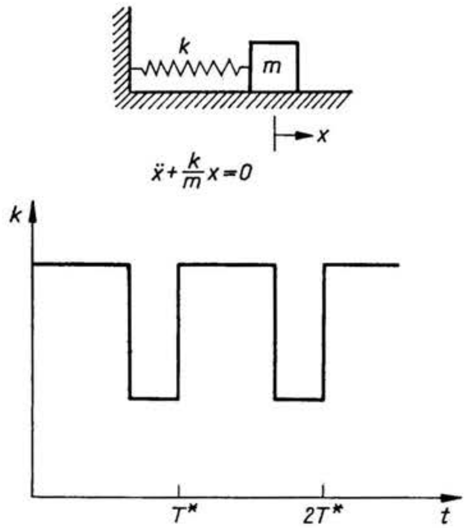

FIG. 15. System with one-degree of freedom as model for layered composite.

similar to the dispersion relation for Floquet waves mentioned above. If the period of free vibration of the system $T = 2\pi/\omega$ is large as compared to the period T^*, the response will essentially be a simple harmonic motion governed by an effective stiffness k_f (Fig. 16). This case corresponds to the long-wave approximation of Floquet waves. By contrast, if T is very much smaller than T^*, then the motion will be simple harmonic during the time intervals when the spring constant is maintained at the higher value, and also simple

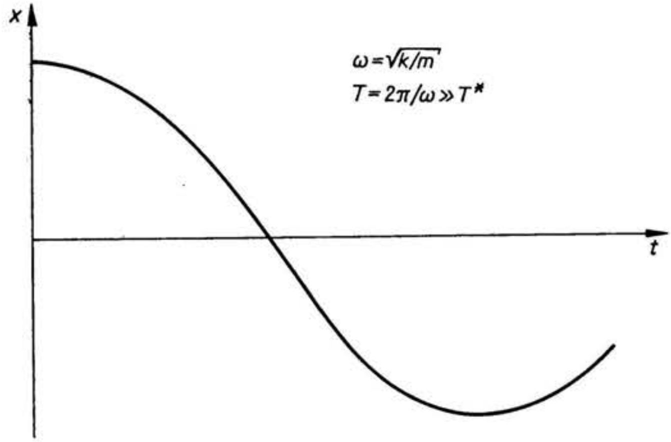

FIG. 16. Effective stiffness governs (long-wave approximation).

harmonic, but with lower frequency, during the remaining time intervals when the spring constant k is maintained at the lower value, Fig. 17. This limiting case corresponds to the short wave approximation. It is of course possible to discuss the complete response

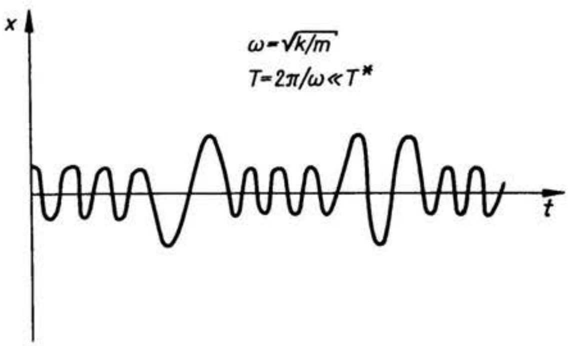

FIG. 17. Separate stiffnesses govern (short-wave approximation).

in all ranges, but this will be deferred to a subsequent study. It might suffice here to mention that the proposed analogy could offer interesting possibilities for experimental investigations.

8. Conclusions

It is quite apparent, particularly from Fig. 14, that the class of matched effective stiffness theories proposed in this paper holds great promise in being more useful, as a result of its simplicity combined with a large range of validity, than other approximate theories advanced in the past. The proposed theories are merely sketched here and a more formal and complete derivation, for motions in anti-plane strain, plane strain, and possibly more general situations, is reserved for future presentations. The questions of suitable boundary and initial conditions, generalized stresses, energy expressions, alternate matching procedures and related considerations are also reserved for these later studies.

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