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On the theory of linear elasticity in statistically homogeneous media

P. MAZILU (BUCHAREST)

A STATISTICALLY homogeneous elastic medium subjected to non-uniform strain field is considered; the expression of the effective elastic moduli and the equations for the mean values of stress and strain are derived. A particular attention is paid to the model of perfectly disordered composite materials introduced by E. Kröner in order to describe the polycrystalline aggregates having a very fine microstructure.

Rozważany jest statycznie niejednorodny ośrodek sprężysty poddany równomiernemu polu odkształcenia. Wyprowadzono efektywne moduły sprężyste oraz równania na wartości średnie naprężenia i odkształcenia. Szczególną uwagę poświęcono modelowi idealnie nieuporządkowanych materiałów kompozytowych wprowadzonych przez E. Krönera do opisu agregatów polikrystalicznych, cechujących się doskonałą mikrostrukturą.

Рассматривается статистически неоднородная упругая среда подвергнутая равномерному полю деформаций. Выведены эффективные мосули упругости, а также уравнения для средних значений напряжения и деформации. Особенное внимание посвящено модели идеально неупорядоченных композитных материалов, введенной Э. Кренером для описания поликристаллических агрегатов, обладающих идеальной микроструктурой.

1. Introduction

The classical theory of elasticity is devoted to homogeneous media or to media whose mechanical properties are well known at every point. There are however materials whose mechanical properties vary to such extent that only a statistical description is possible. This is the case of heterogeneous materials, like concrete of fiber-reinforced materials and polycristalline aggregates. References on the mathematical and experimental aspects concerning the mechanical behaviour of such kind of materials can be found in the Beran's book [1] and in the works of Kröner [2, 3].

The present paper deals with some problems concerning the mathematical foundation of the theory of statistically homogeneous media. Section 3 presents a mathematical justification of a successive approximation method used in most of the works devoted to the theory of statistically homogeneous media. Section 4, by using Green matrix technique, presents the analytical representation of the solution corresponding to a deterministic nonhomogeneous material. Using this representation, in the next section the mean values of the displacements strain, stress and internal energy are derived. Section 6 is devoted to the perfectly disordered composite material defined in the sense used by Kröner. For this class of materials one can evidence (via balance of energy), in addition the mean value of stress $\langle\sigma\rangle$, a new mean value denoted $\bar{\sigma}$ which can be interpreted as a surface-average stress. This surface-average satisfies Hill condition $\langle W\rangle = \frac{1}{2}\bar{\sigma}\langle\varepsilon\rangle$.

From this form of Hill relation it follows that the variational principles used sometimes in order to derive the effective elastic moduli, leads in fact to a relation between surface-average $\bar{\sigma}$ and ensemble (volume) average $\langle \varepsilon \rangle$.

2. Statistically homogeneous elastic materials

Let us consider a statistically homogeneous elastic medium, filling a bounded domain $D \subset R^3$, whose Hooke tensor L is characterized by the correlation tensors $\langle L^p \rangle$, $p = 1, 2, \ldots$, calculated in the ensemble sense i.e. the tensor of components

$$\langle L::(x_1) \ldots L::(x_p) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \overset{(n)}{L}::(x_1) \ldots \overset{(n)}{L}::(x_p), \; x_1, \ldots x_p \in D, \tag{2.1}$$

where $(\overset{(n)}{L})_{n \in N}$ is a realization of the random Hooke tensor L; (a sequence of deterministic tensor functions defined on D). By $\overset{(n)}{L}::$ one denotes the components of $\overset{(n)}{L}$.

Assume this medium to be subjected to an external body force $F = F(x)$, $x \in D$ (of components F_i, $i = 1, 2, 3$) and to a prescribed displacement h on its boundary S

$$u = h(x); \quad x \in S \tag{2.2}$$

(both F and h are deterministic functions).

The problem is to derive in this case the mean values of the displacements, strains, stress and internal energy. These mean values are calculated in the ensemble sense, and therefore, if one admits the validity of the ergodic hypothesis these are at the same time volume averages. The physical meaning of the stress requires also the determination of the surface-average stress.

In order to solve this problem, the following general strategy will be followed:

i) a particular realization of the random Hooke tensor L i.e. a sequence of tensorial functions $\overset{(\nu)}{L} = \overset{(\nu)}{L}(x)$; $x \in D$ such that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{\nu=1}^{N} \overset{(\nu)}{L}::(x_1) \ldots \overset{(\nu)}{L}::(x_n) = \langle L::(x_1) \ldots L::(x_n) \rangle \tag{2.3}$$

is considered.

ii) The deterministic solutions of the boundary value problems

$$\begin{gathered} (\overset{(\nu)}{L}{}^{hk}_{ij} \overset{(\nu)}{u}_{hk})_{,j} + F_i = 0, \\ u_i(x) = h_i(x), \quad \text{for } x \in D \end{gathered} \tag{2.4}$$

are derived.

iii) the mean values of the displacements, strains, stress and internal energy are calculated.

For developing this *program*, we shall firstly present some mathematical preliminaries.

3. Solving of equilibrium equations of nonhomogeneous elastic media by an iterative processus

We shall solve the boundary value problem (2.4) by iterative processes.

To this end, we shall assume the existence of two positive constants C_0 and $C_1 (C_0 > > C_1 > 0)$ such that for all $x \in D$ and $\nu \in \mathcal{N}$

$$C_0 \xi_{ij} \xi_{ij} \geqslant \overset{(\nu)}{L}{}_{ij}^{hk}(x) \xi_{ij} \xi_{hk} \geqslant C_1 \xi_{ij} \xi_{ij} \tag{3.1}$$

for any symmetric matrix ξ of components ξ_{ij} $(i, j = 1, 2, 3)$, so that the equality holds only for $\xi = 0$.

Let us consider a general tensor $L(x)$ satisfying (3.1). Denote by $L(x)$ the difference

$$L^*(x) = \bar{L} - L(x),$$

where $\bar{L}$ is the constant tensor

$$\bar{L}_{ij}^{hk} = \frac{1}{2} C_0 (\delta_{ih} \delta_{jk} + \delta_{ik} \delta_{jh}). \tag{3.2}$$

Let us consider the following sequence of boundary value problems:

P_0. Determine $u^{(0)} \in C^2(D) \cap C^1(D+S)$ such that

$$(\bar{L}_{ij}^{hk} u_{hk}^{(0)})_{,j} + F_i = 0, \tag{3.3}$$
$$u_i^{(0)}(x) = h_i(x) \quad \text{for } x \in S.$$

. . . .

P_n. Determine $u^{(n)} \in C^2(D) \cap C^1(D+S)$ such that

$$(\bar{L}_{ij}^{hk} u_{hk}^{(n)})_{,j} = (\overset{*}{L}{}_{ij}^{hk} u_{h,k}^{(n-1)})_{,j}, \tag{3.4}$$
$$u_1^{(n)}(x) = 0 \quad \text{for } x \in S.$$

Let us assume that each of these boundary value problems has a uniquely determined solution(1). Consider the series

$$u = u^{(0)} + u^{(1)} + u^{(2)} + \ldots$$

We have the following theorem:

THEOREM. *The series* $\sum_{n=1}^{\infty} u^{(n)}$ *is convergent with respect to the norm* $||u||^2 = \int_D u_{i,j} u_{i,j} d\tau$ *of the space*

$$W_{12}^0 = \left\{ u \colon D \to R^3 \Big/ \int_D u_{i,j} u_{i,j} d\tau < +\infty, u(x) = 0 \text{ on } S \right\}.$$

Proof. Let us multiply both terms of (2.5) by $u^{(n)}$ and let us integrate over D. According to Green-Ostrogradski formula we find

$$\int_D \bar{L}_{ij}^{hk} u_{h,k}^{(n)} u_{i,j}^{(n)} d\tau = \int_D \overset{*}{L}{}_{ij}^{hk} u_{h,k}^{(n-1)} u_{i,j}^{(n)} d\tau,$$

(1) P_n $(n = 0, 1, 2, \ldots)$ is in fact a Dirichlet boundary value problem for the equations of elastic equilibrium for a material having Lamé constants $\lambda = 0$, $\mu = C_0/2$.

or taking into account the expression of the tensor $\bar{L}$

$$(3.3) \qquad ||u^{(n)}||^2 = \frac{1}{C_0} \int_D \mathring{L}_{ij}^{hk} u_{h,k}^{(n-1)} u_{i,j}^{(n)} d\tau .$$

Let us consider the bilinear form defined on $W_{12}^0 \times W_{12}^0$

$$A(u, v) = \int_D \mathring{L}_{ij}^{hk} u_{h,k} u_{i,j} d\tau .$$

From (3.2) and (3.1) and from the symmetry of Hooke tensor L it follows that $A(u, v)$ defines an inner product in W_{12}^0. Therefore we have

$$|A(u^{n-1}, u^{(n)})| \leqslant \Big(\int_D \mathring{L}_{ij}^{hk} u_{h,k}^{(n-1)} u_{i,j}^{(n-1)} d\tau\Big)^{1/2} \Big(\int_D \mathring{L}_{ij}^{hk} u_{h,k}^{(n)} u_{i,j}^{(n)} d\tau\Big)^{1/2} .$$

But from (3.1) it follows

$$\int_D \mathring{L}_{ij}^{hk} u_{h,k}^{(n-1)} u_{i,j}^{(n-1)} d\tau \leqslant (C_0 - C_1) ||u^{(n-1)}||^2$$

and

$$\int_D \mathring{L}_{ij}^{hk} u_{h,k}^{(n)} u_{i,j}^{(n)} d\tau \leqslant (C_0 - C_1) ||u^{(n)}||^2 .$$

Consequently,

$$|A(u^{(n-1)}, u^{(n)})| \leqslant (C_0 - C_1) ||u^{(n-1)}|| \, ||u^{(n)}|| .$$

From (3.3) it follows

$$||u^{(n)}|| \leqslant \frac{C_0 - C_1}{C_0} ||u^{(n-1)}|| = x||u^{(n-1)}||, \qquad x < 1$$

and therefore the terms of the series $\sum u^{(n)}$ are majorated by the terms of the geometrical progression

$$||u^{(0)}|| \sum_{n=1}^{\infty} x^n .$$

This proves the theorem.

Let us consider the displacement field

$$u = u^{(0)} + \sum u^{(n)} .$$

We shall prove that u is the weak solution of the boundary value problem (2.4), i.e.

$$(3.4) \qquad \int_D L_{ij}^{hk} u_{h,k} v_{i,j} d\tau = \int_D F \cdot v d\tau ,$$

for any $v \in W_{12}^0$.

Indeed, from (3.3) and (3.4) it follows

$$\int_D \bar{L}_{ij}^{hk} u_{h,k}^{(0)} v_{i,j} d\tau = \int_D F_i v_i d\tau,$$

$$\int_D \bar{L}_{ij}^{hk} u_{h,k}^{(n)} v_{i,j} d\tau = \int_D \dot{L}_{ij}^{hk} u_{h,k}^{((n-1)} v_{i,j} d\tau,$$

whence, calculating the sum with respect to n, one finds (3.4).

Now, let us assume that Hooke tensor L has the form

$$L(x, \alpha) = \tilde{L} - \alpha \mathring{L}(x), \tag{3.5}$$

where α is a positive parameter and $\tilde{L}$ is an arbitrary constant tensor. Assume that $L(x, \alpha)$ satisfies (3.1) for all $\alpha \in [0, 1]$.

Let us write $L(x, \alpha)$ in the form

$$L(x, \alpha) = \bar{L} - (\bar{L} - \tilde{L} + \alpha L(x)) \tag{3.6}$$

with $\bar{L}$ defined by (3.2). Now, denoting $\dot{L} = \bar{L} - \tilde{L} + \alpha L(x)$ we can solve the boundary value problem

$$\begin{gathered} (L_{ij}^{hk}(x, \alpha) u_{h,k})_{,j} + F_i = 0, \\ u_i(x) = h_i(x) \quad \text{for } x \in S \end{gathered} \tag{3.7}$$

by the iterative procedure described above. Therefore the series

$$u = u^{(0)} + u^{(1)} + \dots, \tag{3.8}$$

where u is the solution of the boundary value problem

$$\begin{gathered} (\bar{L}_{ij}^{hk} u_{h,k}^{(0)})_{,j} + F_i = 0, \\ u_i^{(0)}(x) = h_i(x) \quad \text{for } x \in S \end{gathered}$$

and $u^{(n)}$, $(n = 1, 2, \dots)$ are the solutions of the boundary value problems

$$\begin{gathered} (\bar{L}_{ij}^{hk} u_{h,k}^{(n)})_{,j} = (\bar{L}_{ij}^{hk} - \tilde{L}_{ij}^{hk} + \alpha \bar{L}_{ij}^{hk}(x)) u_{h,k}^{(n-1)})_{,j}, \\ u_i^{(n)}(x) = 0 \quad \text{for } x \in S \end{gathered}$$

is uniformly convergent with respect to $\alpha \in [0, 1]$ in W_{12}.

We shall prove that the sum $u(x, \alpha)$ is analytical for $\alpha \in [0, 1]$. Indeed, let us look for a solution of the form

$$u = v^{(0)} + \alpha v^{(1)} + \alpha^2 v^{(2)} + \dots \tag{3.9}$$

for the boundary value problem (3.7). Taking into account (3.5) it follows for $u^{(n)}$ $(n = 0, 1, 2 \dots)$ that the boundary value problems have forms

$$\begin{gathered} (\tilde{L}_{ij}^{hk} v_{h,k}^{(0)})_{,j} + F_i = 0, \\ v_i^{(0)}(x) = h_i(x) \quad \text{for } x \in S \end{gathered} \tag{3.10}$$

and for $n = 1, 2, \dots$

$$\begin{gathered} (\tilde{L}_{ij}^{hk} v_{h,k}^{(n)})_{,j} = (\dot{L}_{ij}^{hk} v_{h,k}^{(n-1)})_{,j}, \\ v_i(x) = 0 \quad \text{for } x \in S. \end{gathered} \tag{3.11}$$

According to the above results each of these problems have a uniquely determined solution.

In the same way as in the precedent theorem we obtain the following results:

For sufficiently small $0 \leqslant \alpha \leqslant \alpha_1$ the series (3.9) is convergent in W_{12}^0 and represents the weak solution of (3.7). Since the weak solution of (3.7) is uniquely determined, it follows that (3.8) is analytical on $[0, \alpha_1]$.

Now writing the tensor $L(x, \alpha)$ under the form

$$L(x, \alpha) = \tilde{L} - \alpha_1 \mathring{L}(x) - (\alpha - \alpha_1)\mathring{L}(x) \tag{3.10}$$

and looking for a solution of the form

$$u = w^{(0)} + (\alpha - \alpha_1)w^{(1)} + (\alpha - \alpha_1)^2 w^{(2)} + \dots$$

it follows that (3.8) is analytical for $\alpha \in [\alpha_1, \alpha_2]$.

Remark. The proof of this latter fact is quite similar to the proof of the precedent theorem if we take $\tilde{L} - \alpha_1 L(x)$ instead $\bar{L}$ and $\mathring{L}$ instead of $\dot{L}$. (The fact that L is a constant tensor has no importance in the proof).

It is possible to show that by this procedure we cover the whole interval $[0, 1]$. Indeed, let us assume the contrary, that is there exists $\alpha_0 \in [0, 1]$ such that $u(x, \alpha)$ is not analytical in the neighbourhood of α_0.

Writing $L(x, \alpha)$ under the form

$$L(x, \alpha) = (\tilde{L} - \alpha_0 \mathring{L}(x)) - (\alpha - \alpha_0)\mathring{L}$$

it follows, by the procedure already used, that there exists a vicinity of α_0 where $u(x, \alpha)$ is analytical. This contradiction proves our assertion.

We can conclude therefore that the solution $u(x, \alpha)$ corresponding to the boundary value problem (3.1) with $L(x, \alpha)$ having the form (3.5) is analytical in $\alpha \in [0, 1]$. This implies that the series

$$u = v^{(0)} + \alpha v^{(1)} + \dots$$

with $v^{(n)}$ satisfying (3.10) or (3.11) is convergent for sufficiently small α and represents the weak solution of (3.1). The above results show that for the Hooke tensor splitted under the form $L(x) = \bar{L} - \dot{L}(x)$, where L is defined by (3.2), the above iterative processes are convergent. Generally, if a different splitting $L(x) = \tilde{L} - \mathring{L}(x)$ is used, the above iterative procedure does not converge. However, in this second case, for sufficiently small α, we can prove the convergence of the series corresponding to Hooke tensor $\tilde{L} - \alpha\mathring{L}$ and the possibility of prolonging analytically this series over $[0, 1]$.

This fact allows us to develop any proof for small α and to extend "aposteriori" the final results to $\alpha = 1$.

4. Representation of the solution by Green matrix

Let us consider a Hooke tensor of the form (3.5) with $\tilde{L}$ isotropic

$$\tilde{L}_{ij}^{hk} = \lambda\delta_{ij}\delta_{hk} + \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}), \tag{4.1}$$

where λ and μ are positive constants. Corresponding to the domain D and to the isotropic Hooke tensor consider Green matrix G of components:

$$G_{ij}(x_1, x_2) = g_{ij}(x_1, x_2) + G^0_{ij}(x_1, x_2), \tag{4.2}$$

where g_{ij} are the components of Somigliana tensor

$$g_{ij}(x_1, x_2) = a\frac{\delta_{ij}}{(x_1 - x_2)} + b\frac{(x_{1i} - x_{2i})(x_{1j} - x_{2j})}{|x_1 - x_2|^3}$$

with

$$a = -\frac{3\lambda + 7\mu}{16\pi\mu(\lambda\mu)}, \quad b = -\frac{\lambda + \mu}{16\pi\mu(\lambda + 2\mu)}$$

and $G^0_{ij}(x_1, x_2)$ solutions of the equations

$$\tilde{L}^{hk}_{pq}\frac{\partial G^0_{ij}}{\partial x_{2k}\partial x_{2q}} = 0, \quad G^0_{ij}(x_1, x_2) = -g_{ij}(x_1, x_2) \quad \text{for } x \in S.$$

Using Green matrix the solution of the problem P_n can be written

$$u^{(n)}_i(x_1) = \int_D G_{ip}(x_1, x_2)\frac{\partial \mathring{L}^{hk}_{pq} u^{(n-1)}_{hk}}{dx_{2q}}\partial x_2, \tag{4.3}$$

or

$$u^{(n)}_i(x_1) = \int_D G_{ip}(x_1, x_2)\frac{\partial \mathring{L}^{hk}_{pq} \varepsilon^{(n-1)}_{h,k}}{dx_{2q}}\partial x_2, \tag{4.4}$$

where

$$\varepsilon^{(n-1)}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

According to the flux divergence formula this can be written

$$u^{(n)}_i(x_1) = -\int_D \frac{\partial G_{ip}}{\partial x_{2q}}\mathring{L}^{hk}_{pq}\varepsilon^{(n-1)}_{hk} d\tau_2, \tag{4.5}$$

or in a more compact form

$$u^{(n)} = -G'_{12} * \mathring{L}_2 \varepsilon^{(n-1)}_2.$$

Since Green matrix has a weak singularity for $x_1 = x_2$ having the order of magnitude $0\left(\frac{1}{|x_1 - x_2|}\right)$ $\left(\text{(this means that } |G_{ij}| < \frac{k}{|x_1 - x_2|} \text{ for all } x_1, x_2 \in D\right)$, the flux divergence formula must be applied in $D - \Omega_\eta$, where Ω_η is a sphere centred in x_1 having η as radius. In the flux divergence formula the contribution of the boundary Σ_η of Ω_η is

$$I_\eta = \int_{\Sigma\eta} G_{ip}\mathring{L}^{hk}_{pq}\varepsilon^{(n-1)}_{hk} n_q ds.$$

It follows that

$$|I_\eta| \leqslant Mk\int_{\Sigma\eta}\frac{1}{|x_1 - x_2|}ds = 4\pi Mk\eta,$$

where $M = \max|L^{hk}_{ij}\varepsilon^{(n-1)}_{hk}|$.

Passing to the limit for $\eta \to 0$ one obtains

$$I_\eta \to 0.$$

Now, in order to derive the expression of $\varepsilon^{(n)}$ one differentiates (4.5). According to the formula of differentiation of integrals with a weak singularity (S. G. MIKHLINE [4]), we find

$$\varepsilon^{(n)} = g_{i(pq)j} \mathring{L}^{hk}_{pq} \varepsilon^{(n-1)}_{hk} - \int\limits_D G_{i(p,q)j} \mathring{L}^{hk}_{pq} \varepsilon^{(n)}_{hk} d\tau_2,$$

where the parantheses around subscript indicate symmetrization and g_{ipqj} denote the following integrals over the unit sphere Ω

$$g_{ipqj} = \int\limits_\Omega r^2 g_{ip,qj} \cos(r, x_2) dw.$$

Let us denote by γ the tensor of components

$$g_{i(pq)j} \equiv \frac{1}{2}(g_{ipqj} + g_{iqpj}),$$

by φ the tensorial field of components

$$-\frac{1}{2} r^3 (g_{ip,qj} + g_{iq,pj}),$$

and by Φ the matrix of components

$$G^0_{i(pq)j} = -\frac{1}{2}(G^0_{ip,qj} + G^0_{iq,pj}).$$

For the sake of compactness, we shall introduce the operator defined for all tensor functions $e\colon D \to R^{3\times 3}$

$$\Gamma_{12} * e_2 = \gamma(x_1) e(x_1) + \int\limits_D \frac{\varphi(x_1, x_2)}{r^3_{12}} e(x_2) d\tau_2 + \int\limits_D \Phi(x_1, x_2) e(x_2) d\tau_2.$$

Using this notation $\varepsilon^{(n)}$ can be expressed as

$$\varepsilon^{(n)} = \Gamma_{12} * \varepsilon^{(n-1)}.$$

Remark. If $D \equiv R^3$ (whole Euclidean space), then $G^0 = 0$ and Γ reduces to $\mathring{\Gamma}$ defined by

$$\mathring{\Gamma}_{12} * e_2 = \gamma(x_1) e(x_1) + \int\limits_{R^3} \frac{\varphi(x_1, x_2)}{r^3_{12}} e(x_2) d\tau_2.$$

The solution of the boundary value problem (3.7) is given by the series

$$\begin{aligned} u &= u^{(0)}_1 - G^1_{12} * \mathring{L}_2 \varepsilon^{(0)}_2 - G^1_{12} * \mathring{L}_2 \varepsilon^{(1)}_2 + \dots, \\ \varepsilon &= \varepsilon^{(0)}_1 + \Gamma_{12} * \mathring{L}_2 \varepsilon^{(0)}_2 + \Gamma_{12} * \mathring{L}_2 \varepsilon^{(1)}_2 + \dots, \\ \sigma &= L_1 \varepsilon^0_1 + \Gamma_{12} * L_1 \mathring{L}_2 \varepsilon^{(0)}_2 + \Gamma_{12} * L_1 \mathring{L}_2 \varepsilon^{(1)}_2 + \dots, \end{aligned} \tag{4.6}$$

or expressing all in terms of $u^{(0)}$, $\varepsilon^{(0)}$

$$(4.7)\qquad \begin{aligned} u &= u_1^0 - G_{12}^1 * \mathring{L}_2 \varepsilon_2^{(0)} - G_{12}^1 * \Gamma_{23} * \mathring{L}_2 \varepsilon_{(2)}^{(1)} + \ldots, \\ \varepsilon &= \varepsilon_1^{(0)} + \Gamma_{12} * \mathring{L}_2 \varepsilon_2^{(0)} + \Gamma_{12} * \Gamma_{23} * \mathring{L}_2 \mathring{L}_3 \varepsilon_2^{(1)} + \ldots, \\ \sigma &= L_1 \varepsilon_1^{(0)} + \Gamma_{12} * L_1 \mathring{L}_2 \varepsilon_2^{(0)} + \Gamma_{12} * \Gamma_{23} * L_1 \mathring{L}_2 \mathring{L}_3 \varepsilon_3^{(1)} + \ldots. \end{aligned}$$

5. Mean values for statistically homogeneous media

Let us consider a statistically homogeneous elastic medium whose random Hooke tensor is characterized by its correlation tensors $\langle L \rangle$, $\langle L^2 \rangle$, ... Assume that the mean value $\langle L \rangle$ is an isotropic tensor.

$$\langle L_{ij} \rangle = \langle \lambda \rangle \delta_{ij} \delta_{hk} + \langle \mu \rangle (\delta_{ih} \delta_{jk} + \delta_{ih} \delta_{jk}).$$

Let $\overset{(\nu)}{L}(x)_{\nu \in \mathcal{N}}$ be a realization of the random Hooke tensor. Consider for each tensor of this realization the boundary value problem

$$(5.1)\qquad (\overset{(\nu)hk}{L_{ij}}(x) u_{hk})_{,j} + F_i = 0, \quad u_i(x) = h_i(x) \quad \text{for } x \in S.$$

In order to solve these boundary value problems, let us split each $\overset{(\nu)}{L}$ into mean and fluctuating parts according to the formula

$$L(x) = \langle L \rangle - \mathring{L}(x).$$

Now, writing the expression of u, ε and σ given by (4.7) for each term $\overset{(\nu)}{L}$ of the realization $(\overset{(\nu)}{L})_{\nu \in \mathcal{N}}$ and taking the average, it follows (since $\langle \mathring{L} \rangle = 0$)

$$\begin{aligned} \langle u \rangle &= u_1^{(0)} - G_{12}^1 * \Gamma_{23} * \langle \mathring{L}_2 L_3 \rangle \varepsilon_3^{(0)} + \ldots, \\ \langle \varepsilon \rangle &= \varepsilon_1^{(0)} + \Gamma_{12} * \Gamma_{23} * \langle \mathring{L}_2 \mathring{L}_3 \rangle \varepsilon_3^{(0)} + \ldots, \\ \langle \sigma \rangle &= \langle L \rangle \varepsilon_1^0 + \Gamma_{12} * \langle L_1 \mathring{L}_2 \rangle \varepsilon_2^{(0)} + \Gamma_{12} * \Gamma_{23} * \langle L_1 \mathring{L}_2 \mathring{L}_3 \rangle \varepsilon_3^{(0)} + \ldots. \end{aligned}$$

Remark. Here one assumes that the average of our series can be calculated term by term.

We shall derive the expression of the internal energy. In order to better evidence the variables, let us write the strain and stress tensor for an arbitrary term of the realization $(\overset{(\nu)}{L})_{\nu \in \mathcal{N}}$ under the form

$$\begin{aligned} \varepsilon_1 &= \varepsilon_1^{(0)} + \Gamma_{12} * L_2 \varepsilon_2^{(0)} + \Gamma_{12} * \Gamma_{23} * \mathring{L}_2 \mathring{L}_3 \varepsilon_3^{(0)} + \ldots, \\ \sigma_1 &= L_1 \varepsilon_1^{(0)} + \Gamma_{12'} * L_1 L_{2'}^0 \varepsilon_2^{(0)} + \Gamma_{12'} * \Gamma_{2'3'} * L_1 L_{2'} L_{3'} \varepsilon_3^{(0)} + \ldots. \end{aligned}$$

If we consider the product $\sigma\varepsilon = \sigma_{ij} \varepsilon_{ij}$ one finds

$$\begin{aligned} \sigma_1 \varepsilon_1 = \varepsilon_1^{(0)} (L_1 \varepsilon_1^{(0)} + \Gamma_{12} * L_1 \mathring{L}_2 \varepsilon_2^{(0)} + \Gamma_{12} * \Gamma_{23} * L_2 \mathring{L}_3 \mathring{L}_{3'} \varepsilon_3^{(0)} + L_1 \varepsilon_1^{(0)} (\Gamma_{12} \mathring{L}_2 \varepsilon_2^{(0)} \\ + \Gamma_{12} * \Gamma_{23} * \mathring{L}_2 \mathring{L}_3 \varepsilon_3^{(0)} + \ldots) + \Gamma_{12} * \Gamma_{12'} * L_1 L_2 L_{2'} \varepsilon_2^{(0)} \varepsilon_{2'}^{(0)} + \ldots. \end{aligned}$$

Calculating the average one obtains

$$(5.2)\quad \langle\sigma\varepsilon\rangle_1 = \varepsilon_1^{(0)}\langle\sigma_1\rangle+\langle L_1\rangle\varepsilon_1^{(0)}(\Gamma_{12}\langle L_2^{(0)}\rangle\varepsilon_2^{(0)}+\Gamma_{12}*\Gamma_{13}*\langle\mathring{L}_2\mathring{L}_3\rangle\varepsilon_3^{(0)}+\ldots)$$
$$+\varepsilon_1^{(0)}(\Gamma_{12}\langle L_1\mathring{L}_2\rangle\varepsilon_2^{(0)}+\Gamma_{12}*\Gamma_{23}*\langle L_1\mathring{L}_2L_3\rangle\varepsilon_3^{(0)}+\ldots)$$
$$+\Gamma_{12}*\Gamma_{12'}\langle L_1\mathring{L}_2\mathring{L}_{2'}\rangle\varepsilon_2^{(0)}\varepsilon_{2'}^{(0)}+\ldots.$$

6. Perfectly disordered composite materials

Let us consider a statistically homogeneous elastic material filling the whole Euclidean space R^3.

Assume that the distribution of its material components over two very distant regions is statistically independent. From the mathematical point of view this implies

$$\lim_{\substack{|x_i-x_j|\to\infty\\ i\neq j}}\langle L^{k_1}(x_1)\ldots L^{k_p}(x_p)\rangle = \langle L^{k_1}\rangle\ldots\langle L^{k_p}\rangle.$$

Now, let us consider a statistically homogeneous medium defined by the following correlation tensors

$$\langle\lambda^{k_1}(x_1)\ldots\lambda^{k_p}(x_p)\rangle = \lim_{n\to\infty}\langle L^{k_1}(nx_1)\ldots L^{k_p}(nx_p)\rangle.$$

We note that, if $x_i\neq x_j$, $(i\neq j)$, then

$$(6.1)\qquad \langle\lambda^{k_1}(x_1)\ldots\lambda^{k_p}(x_p)\rangle = \langle\lambda^{k_1}\rangle\ldots\langle\lambda^{k_p}\rangle,$$

A statistically homogeneous material whose correlation tensors satisfy (6.1) is called a perfectly disordered composite material.

Remark. From physical point of view, the model of perfectly disordered material corresponds to a statistically homogeneous material having a very fine microstructure, so that the distribution of its internal components is statistically independent in any two closed regions.

This concept and its physical interpretation was firstly used by KRÖNER in order to describe the mechanical behaviour of the polycristalline aggregates.

Let us denote by $\langle u\rangle_n$, $\langle\varepsilon\rangle_n$, $\langle\sigma\rangle_n$, $\langle\sigma\varepsilon\rangle_n$ the mean values of the mechanical parameters corresponding to the statistically homogeneous medium whose random Hooke tensor has the correlations

$$\langle L_n^k(x_1)\ldots L_n^k(x_p)\rangle = \langle L^k(nx)\ldots L^k(nx)\rangle.$$

By definition the limits

$$\langle u\rangle = \lim_{n\to\infty}\langle u\rangle_n,\qquad \varepsilon = \lim_{x\to\infty}\langle\varepsilon\rangle_n,$$

$$\langle\sigma\rangle = \lim_{n\to\infty}\langle\sigma\rangle_n,\qquad w = \frac{1}{2}\langle\sigma\varepsilon\rangle = \frac{1}{2}\lim_{n\to\infty}\langle\sigma\varepsilon\rangle_n,$$

will be the mean values for the perfectly disordered material.

It was proved [5] that for $x \in D$

$$\lim_{n\to\infty} \Gamma^{k-1} * \langle \mathring{L}_n^k \rangle \varepsilon^{(0)} = (\mathring{\Gamma}^{k-1} * \langle \mathring{L} \rangle)\varepsilon^{(0)},$$

$$\lim_{n\to\infty} \Gamma^{k} * \langle \mathring{L}_n^k \rangle \varepsilon^{(0)} = (\mathring{\Gamma} * \mathring{L}^k)\varepsilon^{(0)}.$$

Remark. These formulae are no longer valid if $x \in S$.

Let us denote by C and $\tilde{C}$ the constant tensors defined by

$$C = \langle L \rangle - \sum_{k=2}^{\infty} \mathring{\Gamma}^{k-1} * \langle \mathring{L}^k \rangle,$$

$$\tilde{C} = \langle L \rangle - C = \sum_{k=2}^{\infty} \mathring{\Gamma}^{k-1} * \langle \mathring{L}^k \rangle.$$

Using these notations we obtain the following expressions for the mean values of the displacement, strain and stress in a perfectly disordered composite material:

$$\langle u \rangle = u^{(0)} - G' * \tilde{C}\varepsilon^{(0)},$$

$$\langle \varepsilon \rangle = \varepsilon^{(0)} + \Gamma * \tilde{C}\varepsilon^{(0)},$$

$$\langle \sigma \rangle = C\varepsilon^{(0)} + \langle L \rangle \Gamma * \tilde{C}\varepsilon^{(0)}.$$

It is possible to give a mechanical interpretation for the vector $G' * \tilde{C}\varepsilon^{(0)}$ and for the tensor $\Gamma * \tilde{C}\varepsilon^{(0)}$.

Let us consider the following boundary value problem: determine the displacement field $\tilde{u}$: $D \to R^3$, $\tilde{u} \in C(D+S) \cap C^2(D)$ such that

$$(\langle L_{ij}^{hk} \rangle \tilde{u}_{h,k})_{,j} = (\tilde{C}_{ij}^{hk} \varepsilon_{hk}^{(0)})_{,j}, \quad \tilde{u}(x) = 0 \quad \text{for } x \in S.$$

Obviously, the solution of this boundary value problem is

$$\tilde{u} = -G'_{12} * \tilde{C}\varepsilon_2^{(0)}$$

and the corresponding strain field will be

$$\tilde{\varepsilon} = \Gamma_{12} * C\varepsilon_2^{(0)}.$$

Therefore, the above formulae can be written

$$\langle u \rangle = u^{(0)} + \tilde{u},$$

$$\langle \varepsilon \rangle = \varepsilon^{(0)} + \tilde{\varepsilon},$$

$$\langle \sigma \rangle = C\varepsilon^{(0)} + \langle L \rangle \tilde{\varepsilon}.$$

In order to derive the expression of the internal elastic energy for a perfectly disordered composite material, we remark that

$$\lim_{n\to\infty} \Gamma_{12} * \Gamma_{12'} \langle L_{n_1} \mathring{L}_{n_2} L_{n_{2'}} \rangle \varepsilon_2^{(0)} \varepsilon_{2'}^{(0)} = \Gamma_{12} * \lim_{n\to\infty} \Gamma_{12'} \langle L_{n_1} \mathring{L}_{n_2} \mathring{L}_{n_3} \rangle = 0$$

(since $\lim_{n\to\infty} \langle L(n x_1) L(nx_2) L(nx_{2'}) \rangle = 0$ for $x_2 \neq x_{2'}$).

Similar formulae are true for higher order products.

Taking into account these equalities one finds, for the internal elastic energy, the expression

$$w = \frac{1}{2}(C\varepsilon^{(0)}\varepsilon^{(0)} + 2\langle L\rangle\varepsilon^{(0)}\tilde{\varepsilon} - \tilde{C}\varepsilon^{(0)}\varepsilon^{(0)}).$$

For an increment of the displacement field $\delta\langle u\rangle = \delta u^{(0)} + \delta\tilde{u}$, the variation of this internal energy will be

$$\delta w = (C\varepsilon^{(0)} + \langle L\rangle\tilde{\varepsilon})\delta\varepsilon^{(0)} + \langle L\rangle\varepsilon^{(0)}\delta\tilde{\varepsilon} - \tilde{C}\varepsilon^{(0)}\delta\varepsilon^{(0)}. \tag{6.2}$$

According to the theorem of the virtual work one must find

$$\int_D \delta w d\tau = \int_D F\delta\langle u\rangle d\tau + \int_S \overline{\sigma}^{(n)}\delta\langle u\rangle ds, \tag{6.3}$$

where $\overline{\sigma}^{(n)}$ is the mean stress on the boundary S of D.

If we introduce (6.2) in (6.3), after a simple integral transformation (flux-divergence formula) it follows

$$\int_S \overline{\sigma}^{(n)}\delta\langle u\rangle ds = \int_S (\langle\sigma\rangle - \tilde{C}\varepsilon^{(0)})n\delta\langle u\rangle ds + \int_D \psi\delta u^0 dv, \tag{6.4}$$

where ψ denotes the vector of components

$$\psi_i = (\tilde{C}_{ij}^{hk}\varepsilon_{hk}^{(0)})_{,j}.$$

R e m a r k. By virtue of Riesz-Fréchet theorem and Sobolev-Kondrasev imbedding theorem, the relation (6.4) defines uniquely the vector $\overline{\sigma}^{(n)}$.

However, because of the last term of (6.4) it is not generally possible to express $\overline{\sigma}^{(n)}$ in terms of $\langle\sigma\rangle$ or $\langle\varepsilon\rangle$ (or equivalently in terms of $\varepsilon^{(0)}$ and $\tilde{\varepsilon}$) calculated on the boundary. We shall see that this will be possible in the particular case when the mean strain-field is uniform ($\varepsilon^{(0)} = \text{const}$). In this case ψ is zero and from (5.8) it follows

$$\overline{\sigma}^{(n)} = (\langle\sigma\rangle - \tilde{C}\varepsilon^{(0)})n.$$

Let us consider, generally, the tensor

$$\overline{\sigma} = \langle\sigma\rangle - \tilde{C}\varepsilon^{(0)}. \tag{6.5}$$

We note that

$$\overline{\sigma}_{ij} = \frac{\partial w}{\partial\varepsilon_{ij}^{(0)}}$$

and

$$\overline{\sigma}_i^{(n)} = \overline{\sigma}_{ij}n_j.$$

For this reason we shall interprete the tensor $\overline{\sigma}$ as the mean value of strain calculated over oriented surfaces.

R e m a r k. According to the ergodic hypothesis, $\langle\sigma\rangle$ can be regarded as the mean value of stress calculated over volume-elements.

It is easily seen that the mean value of stress $\langle\sigma\rangle$ verifies the equilibrium equations

$$\text{div}\langle\sigma\rangle + F = 0.$$

From (6.5) it follows:

$$\operatorname{div}\bar{\sigma}+F+\psi = 0.$$

This relation suggests to interprete the vector field ψ as an internal body force field. This internal body force field occurs only for non-uniform strain state.

Now, we can evidence the following expression for the variation of the internal energy

$$\int\limits_D \delta w = \int\limits_D (F\delta\langle u\rangle+\psi\delta u^{(0)})d\tau \int\limits_S \bar{\sigma}n\,\delta\langle u\rangle ds.$$

Therefore the variation of the internal energy is equal to the virtual work performed by the external body force F, internal body force ψ and the surface-average of stress $\bar{\sigma}^{(n)} = \bar{\sigma}n$. From (6.5) it follows, for uniform mean strain field, that $\bar{\sigma} = (C-\tilde{C})\langle\varepsilon\rangle$ and consequently the surface-average $\bar{\sigma}$ and the volume average of the strain $\langle\varepsilon\rangle$ satisfy Hooke law, having $C_{ef} = C-\tilde{C}$ as effective tensor of elastic moduli.

We note that the internal elastic energy also can be expressed under the form

$$w = \frac{1}{2}\bar{\sigma}\langle\varepsilon\rangle.$$

This shows that Hill condition is satisfied if instead of volume average of stress one takes the surface-average of stress.

Therefore if in the deriving of the effective elastic moduli tensor one uses a variational principle, then one obtains $C_{ef} = C-\tilde{C}$ as the effective tensor of elastic moduli, corresponding to a relation between surface-average of stress $\bar{\sigma}$ and the volume-average of strain $\langle\varepsilon\rangle$.

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