Archives of Mechanics • Archiwum Mechaniki Stosowanej • 28, 3, pp. 571–581, Warszawa 1976

On the relation between discrete and continuum mechanics of certain material systems

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THE main purpose of the paper is to present some ways of deriving the discrete and the continuous models of material systems with complex (in general, discrete-continuous) structure. To this aim we assume that the motion of the small parts of the system which are known can be approximated by the family of homogeneous deformations. Next, the conditions of continuity of deformations and stresses among those parts are taken into account. It is shown that the latter conditions lead to a special form of ideal constraints imposed on deformations and internal forces. The material systems under consideration are assumed to be hyperelastic and only binary interactions are investigated.

Głównym celem pracy jest przedstawienie pewnych sposobów konstruowania dyskretnych i ciągłych modeli układów materialnych o złożonej (dyskretno-ciągłej) strukturze. Ruch dostatecznie małych lecz danych z góry części ciała aproksymuje się jednorodnymi deformacjami, biorąc następnie pod uwagę warunki ciągłości deformacji i oddziaływań pomiędzy częściami. Warunki te są równoznaczne narzuceniu pewnych idealnych więzów dla deformacji i sił wewnętrznych. Ograniczono się do układów hipersprężystych i tylko binarnych oddziaływań.

Главной целью работы является представление некоторых способов построения дискретных и сплошных моделей материальных систем со сложной (дискретно-сплошной) структурой. Движение достаточно малых, но заданных априори, частей тела аппроксимируется однородными деформациями, имея затем в виду условия непрерывности деформаций и взаимодействие между частиями тела. Эти условия равносильны накладыванию некоторых идеальных связей для деформаций и внутренних сил. Ограничиваются гиперупругими системами и только бинарными взаимодействиями.

Introduction

As the object of our investigations we shall take a certain complex (in general, discrete-continuous) material hyperelastic system which will be denoted by M. We are to develop the mechanics of M exclusively by means of the approximate, discrete or continuous, models of M. Such an approach is necessary when the material system M is too complex to be successfully analysed in the direct way. To give an example of such a system we can take a homogeneous material continuum with a great number of inclusions or concentrated masses, a body with fibrous structure, etc. This paper shows how to construct discrete, continuous and continuous-discrete models of M according to the concept of ideal constraints for deformations and interactions.

1. General field equations

All models of M we are to deal with will be obtained under the assumption that the partition $M = \cup P(Z)$, $Z \in \Pi$, Π being the finite, countable or uncountable set, so that the motion of each part $P(Z)$ can be approximated (for a class of problems we are interested in) by the family of homogeneous deformations. Let x_k, t be inertial coordinates in the space-time. Thus the motion of any part $P(Z)$ can be described, in a sufficiently accurate way, by the deformation function of the form(¹)

$$x_k = \chi_k(Z, t)+F_{k\alpha}(Z, t)Y^\alpha, \quad \mathbf{Y}\in \bar{P}_R(Z), t \in R, \tag{1.1}$$

where $\bar{P}_R(Z)$ is a part of the physical space E^3 occupied by $P(Z)$ in the fixed reference configuration, and $\mathbf{Y}$ is a position vector in E^3 of the material points belonging to $\bar{P}_R(Z)$ (the domains of $\mathbf{Y}$ may be different for the different values of Z). We assume that each $\bar{P}_R(Z)$ contains at least four material points not situated on the same plane and that $\det\mathbf{F} > 0$ for any time instant t. Functions $\boldsymbol{\chi}$ and $\mathbf{F}$, defined on $\Pi\times R$, are unknown fields which determine the motion of the model of M. Mind, that if $P(Z_1)\cap P(Z_2) \neq \phi$ for $Z_1 \neq Z_2$, then the motion of the common part of $P(Z_1)$ and $P(Z_2)$ may not be uniquely described since the two deformation functions of the form (1.1) (for $Z = Z_1$ and $Z = Z_2$) represent two different approximations of the motion of $P(Z_1)\cap P(Z_2)$.

The equations of motion of any part $P(Z)$ of the material system M will be postulated in the form

$$\ddot{x}^k d\hat{\varrho}_R = d\hat{t}^k_R+d\hat{b}^k_R-d(\hat{\sigma}_{R,x_k}), \tag{1.2}$$

where the measures $\hat{\varrho}_R = \hat{\varrho}_R(Z; \mathbf{Y})$, $\hat{\sigma}_R = \hat{\sigma}_R(Z; \mathbf{Y}, |\mathbf{x}-\boldsymbol{\chi}|)$ describe the inertial and hyperelastic properties of $P(Z)$, respectively; the measure $\hat{\mathbf{b}}_R = \hat{\mathbf{b}}_R(Z; \mathbf{Y}, t)$ characterizes the known external loads acting at $P(Z)$; and $\hat{\mathbf{t}}_R = \hat{\mathbf{t}}_R(Z; \mathbf{Y}, t)$ is the measure of internal forces due to the interaction between $P(Z)$ and $M-P(Z)$. We have tacitly assumed that only binary interactions exist between material points in M.

To obtain equations for $\boldsymbol{\chi}$ and $\mathbf{F}$ we shall apply the known orthogonalization approach. By virtue of Eqs. (1.1) we are to analyse not the motion of M governed by Eqs. (1.2) but the motion of the model of M, putting

$$\begin{gathered}\int\limits_{\bar{P}_R(Z)} [\ddot{x}^k d\hat{\varrho}_R-d\hat{t}^k_R-d\hat{b}^k_R+d(\hat{\sigma}_{R,x_k})] = 0,\\ \int\limits_{\bar{P}_R(Z)} Y^\alpha[\ddot{x}^k d\hat{\varrho}_R-d\hat{t}^k_R-d\hat{b}^k_R+d(\hat{\sigma}_{R,x_k})] = 0,\end{gathered} \tag{1.3}$$

where the motion x_k is given by Eq. (1.1) and all integrals over $P_R(Z)$ have to be interpreted in the Stielties sense.

Let us select $\bar{P}_R(Z)$ in such a way that the relation

$$\int\limits_{\bar{P}_R(Z)} Y^\alpha d\hat{\varrho}_R = 0$$

holds for each $Z\in\Pi$ (the point $\mathbf{Y} = 0$ is a mass centre for each $\bar{P}_R(Z)$). Let us also observe that

$$\frac{\partial\check{\sigma}_R}{\partial F_{k\alpha}} = \frac{\partial\hat{\sigma}_R}{\partial x_k} Y^\alpha,$$

(¹) Indices $k, l, \ldots$ and $\alpha, \beta, \ldots$ run over the sequence 1, 2, 3. The summation convention holds.

where $\check{\sigma}_R = \check{\sigma}_R(Z; \mathbf{Y}, \mathbf{F}) \equiv \hat{\sigma}_R(Z; \mathbf{Y}, |F_{k\alpha} Y^\alpha|)$, and

$$\int\limits_{\bar{P}_R(Z)} d\hat{\sigma}_{R,\chi_k} = \frac{\partial}{\partial \chi_k} \int\limits_{\bar{P}_R(Z)} d\hat{\sigma}_R = 0.$$

We shall also introduce the characteristic length dimension $l = l(Z)$ and a suitably chosen measure $v = v(Z)$ for an arbitrary part $\bar{P}_R(Z)$ of the physical space; if $P_R(Z)$ is a region in E^3, then $v(Z)$ can be assumed to be its volume. Taking into account the relations given above, we shall obtain from (1.3) and (1.1) the following system of equations

$$\begin{gathered} \varrho_R(Z) \ddot{x}^k(Z,t) = d_R^k(Z,t) + b_R^k(Z,t), \\ l^2 J_R^{\alpha\beta}(Z) \ddot{F}^k{}_\beta(Z,t) = T_R^{k\alpha}(Z,t) - \varrho_R(Z) \frac{\partial \sigma(Z,F)}{\partial F_{k\alpha}} + B_R^{k\alpha}(Z,t); \quad Z \in \Pi,\ t \in R, \end{gathered} \tag{1.4}$$

where we have denoted

$$\begin{gathered} \varrho_R \equiv \frac{1}{v} \int\limits_{\bar{P}_R(Z)} d\hat{\varrho}_R, \quad J_R^{\alpha\beta} \equiv \frac{1}{v} \int\limits_{\bar{P}_R(Z)} \frac{Y^\alpha Y^\beta}{l^2} d\hat{\varrho}_R, \quad \varrho_R \sigma \equiv \frac{1}{v} \int\limits_{\bar{P}_R(Z)} d\check{\sigma}_R; \\ d_R^k \equiv \frac{1}{v} \int\limits_{\bar{P}_R(Z)} d\hat{t}_R^k, \quad b_R^k \equiv \frac{1}{v} \int\limits_{\bar{P}_R(Z)} d\hat{b}_R^k; \\ T_R^{k\alpha} \equiv \frac{1}{v} \int\limits_{\bar{P}_R(Z)} Y^\alpha d\hat{t}_R^k, \quad B_R^k \equiv \frac{1}{v} \int\limits_{\bar{P}_R(Z)} Y^\alpha d\hat{b}_R^k. \end{gathered} \tag{1.5}$$

If $v = v(Z)$ are properly chosen, then the fields defined by Eqs. (1.5) are densities related to the reference configuration of $P(Z)$. If $\bar{P}_R(Z)$ is a region in E^3 ($P(Z)$ being the material continuum), then, using the same procedure as in [1], we can prove that the sum $\mathbf{T}_R + \mathbf{B}_R$ can be interpreted as the mean value of the first Piola-Kirchhoff stress tensor in $P(Z)$.

Equations (1.4), in which χ_k, $F_{k\alpha}$, d_R^k and $T_R^{k\alpha}$ are unknown fields defined on $\Pi \times R$ and the functions ϱ_R, $l^2 J_R^{\alpha\beta}$, b_R^k and $B_R^{k\alpha}$ are assumed to be known(²), will be called the general field equations. For any given partition $M = \cup P(Z)$, $Z \in \Pi$, they characterize the model of the material system M. If Π is the discrete set D, then, the corresponding model will be called discrete. If Π is a differentiable manifold B (of one, two or three dimensions) and the fields in Eqs. (1.4) are sufficiently smooth, then, the model of M is said to be continuous. If $\Pi = D \times B$ we shall be dealing with the discrete-continuous model of M.

We may observe that the number of unknown functions is equal to 24 while the number of the general field equations (1.4) is equal to 12. The missing equations will be obtained by taking into account the interactions among the different parts $P(Z)$ of the material system M. This will be done in the following sections of the paper. The main idea of the approach we shall to present is to introduce the special kind of constraints imposed on the kinematic fields $\boldsymbol{\chi}$, $\mathbf{F}$ as well as on the kinetic fields $\mathbf{d}_R$, $\mathbf{T}_R$.

(²) The fields $\mathbf{b}_R$, $\mathbf{B}_R$ can also be related to the fields $\boldsymbol{\chi}$, $\mathbf{F}$, provided that $\hat{\mathbf{b}}_R$ depends on the motion.

2. Discrete models based on the geometric constraints

In this section we shall assume that Π is a rectangular lattice L (or a part of this lattice) of points in the physical space E^3 with a fixed vector basis given by three orthogonal vectors $\Delta_\alpha \mathbf{Z} = (l_\alpha \delta_\alpha^\beta)$ (no summation with respect α). Let each $\overline{P}_R(\mathbf{Z})$, $\mathbf{Z} \in L$, be a set of material points situated inside the cell of the lattice with a centre $\mathbf{Z} = (Z^\alpha)$. Assuming in Eq. (1.1) that $-\frac{1}{2} l_\alpha \leqslant Y^\alpha \leqslant \frac{1}{2} l_\alpha$, we shall realize the continuity of the deformation function in points $\mathbf{Z}+\frac{1}{2}\Delta_\alpha \mathbf{Z}$, putting

$$\chi_k(\mathbf{Z}, t)+\frac{1}{2} F_{k\alpha}(\mathbf{Z}, t) l_\alpha = \chi_k(\mathbf{Z}+\Delta_\alpha \mathbf{Z}, t)-\frac{1}{2} F_{k\alpha}(\mathbf{Z}+\Delta_\alpha \mathbf{Z}, t) l_\alpha,$$

for any fixed value of the index α, provided that the cells of the lattice with the centres $\mathbf{Z}$, $\mathbf{Z}+\Delta_\alpha \mathbf{Z}$ are occupied by the material points of M in the reference configuration. The foregoing relation can be written in the form

$$\Delta_\alpha \chi_k(\mathbf{Z}, t)-\mu_\alpha^\beta F_{k\beta}(\mathbf{Z}, t) = 0 \tag{2.1}$$

where we use the known difference and mean value operators on the lattice:

$$\Delta_\alpha \boldsymbol{\chi}(\mathbf{Z}) \equiv \frac{1}{l_\alpha}[\boldsymbol{\chi}(\mathbf{Z}+\Delta_\alpha \mathbf{Z})-\boldsymbol{\chi}(\mathbf{Z})], \quad \overline{\Delta}_\alpha \boldsymbol{\chi}(\mathbf{Z}) \equiv \frac{1}{l_\alpha}[\boldsymbol{\chi}(\mathbf{Z}-\Delta_\alpha \mathbf{Z})-\boldsymbol{\chi}(\mathbf{Z})],$$

$$\mu_\alpha^\beta \mathbf{F}_\beta(\mathbf{Z}) \equiv \frac{1}{2}\delta_\alpha^\beta[\mathbf{F}_\beta(\mathbf{Z}+\Delta_\alpha \mathbf{Z})+\mathbf{F}_\beta(\mathbf{Z})], \quad \overline{\mu}_\alpha^\beta \mathbf{F}_\beta(\mathbf{Z}) \equiv \frac{1}{2}\delta_\alpha^\beta[\mathbf{F}_\beta(\mathbf{Z}-\Delta_\alpha \mathbf{Z})+\mathbf{F}_\beta(\mathbf{Z})].$$

To make our considerations more general we shall also take into account the extra constraints of the form

$$h_\nu(\mathbf{Z}, t, \boldsymbol{\chi}, \Delta_\alpha \boldsymbol{\chi}, \overline{\Delta}_\alpha \boldsymbol{\chi}) = 0, \quad \nu = 1, \dots, r, \tag{2.2}$$

where h_ν are known differentiable functions. We shall postulate that the constraints (2.1), (2.2) imposed on the fields $\boldsymbol{\chi}, \mathbf{F}$, are ideal; this means that the following relation

$$\sum_L (d_R^k \delta\chi_k + T_R^{k\alpha}\delta F_{k\alpha}) = 0 \tag{2.3}$$

hold for any $\delta\chi_k$, $\delta F_{k\alpha}$ consistent with Eqs. (2.1), (2.2). From (1.3) and (2.1)–(2.3) follows that

$$\begin{gathered} \overline{\Delta}_\alpha S_R^{k\alpha}+b_R^k+r_R^k = \varrho_R \ddot{\chi}^k, \\ l^2 J_R^{\alpha\beta}\ddot{F}_\beta^k = \overline{\mu}_\beta^\alpha S_R^{k\beta}-\varrho_R \frac{\partial\sigma}{\partial F_{k\alpha}}+B_R^{k\alpha}, \end{gathered} \tag{2.4}$$

where $T_R^{k\alpha} = \overline{\mu}_\beta^\alpha S_R^{k\beta}$ and $S_R^{k\beta}$ are unknown functions defined on $L\times R$ (they are Lagrange's multipliers for the constraints (2.1)). At the same time we obtain the relation

$$\sum_L r_R^k \delta\chi_k = 0 \tag{2.5}$$

which is assumed to hold for any variation $\delta\chi_k$, consistent exclusively with Eqs. (2.2). If there are no constraints (2.2), then, $\mathbf{r}_R = 0$ and Eqs. (2.1), (2.4) represent the basic

system of equations for the unknown fields $\boldsymbol{\chi}$, $\mathbf{F}$ and $\mathbf{S}_R$. In the more general case, Eqs. (2.1),
system of equations for the unknown fields $\boldsymbol{\chi}$, $\mathbf{F}$ and $\mathbf{S}_R$. In the more general case,
Eqs. (2.1), (2.2), (2.4) and (2.5) represent the discrete model of the material system M.

3. Continuous models based on geometric constraints

Now, let us assume that Π is a regular region B in the physical space E^3 with its boundary ∂B, i.e., $\Pi = \overline{B}$. To each $\mathbf{Z} \in B$ a set of material points $\overline{P}_R(\mathbf{Z})$ is assigned in E^3 (cf. Sect. 1); at the same time each $\mathbf{Z} \in \partial B$ will be assigned one material point of M in the reference configuration. Neglecting the inertial properties of this point we can assume that for $\mathbf{Z} \in \partial B$ Eqs. $(1.4)_2$ are identities and Eqs. $(1.4)_1$ reduce to

$$d_R^k(\mathbf{Z}, t) + b_R^k(\mathbf{Z}, t) = 0, \quad \mathbf{Z} \in \partial B, \, t \in R. \tag{3.1}$$

The mapping of the points of $\overline{B}$ into a set of subsets of the material system M in the reference configuration we are to deal with, is not arbitrary. We shall confine ourselves only to those material systems M and those mappings where all known fields defined by Eqs. (1.5) can be approximated, with a sufficient accuracy, by the smooth functions of $\mathbf{Z} \in B$; moreover, we shall construct these fields as the densities related to a region B in E^3. The smooth fields introduced will be denoted in what follows by the same symbols as the fields defined by Eqs. (1.5). Thus the general field equations (1.4) are assumed to hold for each $\mathbf{Z} \in B$ and $t \in R$. We shall also assume that the unknown fields $\boldsymbol{\chi}$, $\mathbf{F}$, $\mathbf{d}_R$ and $\mathbf{T}_R$ in Eqs. (1.4) are also smooth. To obtain the continuous model of the material system M we shall postulate the following form of the internal geometric constraints

$$\chi_{k,\alpha}(\mathbf{Z}, t) - F_{k\alpha}(\mathbf{Z}, t) = 0, \quad \mathbf{Z} \in B, \quad t \in R, \tag{3.2}$$

which can be treated as the continuous "approximation" of the relation (2.1). To make our analysis more general we shall also introduce extra constraints of the form

$$\begin{aligned} &h_\nu(\mathbf{Z}, t, \boldsymbol{\chi}, \nabla\boldsymbol{\chi}, \ldots, \nabla^m\boldsymbol{\chi}) = 0, \quad \mathbf{Z} \in B, \quad t \in R, \quad \nu = 1, \ldots, r, \\ &R_\varrho(\mathbf{Z}, t, \boldsymbol{\chi}, \overline{\nabla}\boldsymbol{\chi}) = 0, \quad \mathbf{Z} \in \partial B, \quad t \in R, \quad \varrho = 1, \ldots, s, \end{aligned} \tag{3.3}$$

where h_ν, R_ϱ are known differentiable functions and $\overline{\nabla}$ is a material deformation gradient on ∂B. It shall be postulated that the constraints (3.2), (3.3) are ideal, i.e., that the relation

$$\int_B (d_R^k \delta\chi_k + T_R^{k\alpha} \delta F_{k\alpha}) dv + \oint_{\partial B} d_R^k \delta\chi_k d\sigma = 0 \tag{3.4}$$

holds for any variations $\delta\chi_k$, $\delta F_{k\alpha}$, admissible by Eqs. (3.2), (3.3). From equations (1.4) (where the domain of the definition of all functions is restricted to $B \times R$) and from (3.1)–(3.4) we obtain the following field equations in $B \times R$

$$\begin{aligned} &T_R{}^{k\alpha}{}_{,\alpha} + b_R^k + r_R^k = \varrho_R \ddot{\chi}^k, \\ &l^2 J_R^{\alpha\beta} \ddot{F}^k{}_\beta = T_R^{k\alpha} - \varrho_R \frac{\partial \sigma}{\partial F_{k\alpha}} + B_R^{k\alpha}, \end{aligned} \tag{3.5}$$

and the kinetic boundary conditions on $\partial B \times R$

$$T_R^{k\alpha} n_{R\alpha} = b_R^k + s_R^k, \tag{3.6}$$

where $\mathbf{n}$ is the unit vector normal to ∂B. At the same time we also obtain the relation

$$\int_B r_R^k \delta\chi_k dv + \oint_{\partial B} s_R^k \delta\chi_k d\sigma = 0, \tag{3.7}$$

which is assumed to hold for any $\delta\chi_k$ exclusively consistent with Eqs. (3.3). The system $\{\mathbf{r}_R, \mathbf{s}_R\}$ represents the reaction forces due to the extra-constraints (3.3). Equations (3.3), (3.5)–(3.7), after substituting $\mathbf{F} = \nabla\boldsymbol{\chi}$, characterize the continuous model of the material system M. Let us observe that in Eqs. (3.5) we are dealing with three partial differential equations $(3.5)_1$ which, in the absence of constraints (3.3) (then $\mathbf{r}_R = \mathbf{s}_R = 0$), have the well known form of the Cauchy equations of motion. If the characteristic length dimension l is sufficiently small, we can put $l \to 0$; moreover, if the external load leads to the condition $B_R^{k\alpha} = 0$, then, we obtain from $(3.5)_2$ the known stress relation of the hyperelastic bodies. On the other hand, if $l \sim 0$ and $B_R^{k\alpha} = 0$, we can derive from (3.3), (3.5)–(3.7) the equations of elastic continuum with geometric constraints [2].

4. Discrete models based on the kinetic constraints

Let Π be a lattice L in E^3 (or a part of this lattice) and the structure of M be described, in the reference configuration analogously as in Sect. 2. Let us also assume that the interactions among the different parts $P(Z)$ of M can be expressed as the densities related to the boundaries S_α, $\bar{S}_\alpha$ of the cell which is situated on the parametric planes $Y^\alpha = \frac{1}{2} l_\alpha$, $Y^\alpha = -\frac{1}{2} l_\alpha$, respectively. Putting $v = l_1 l_2 l_3$, we obtain, from $(1.5)_4$,

$$d_R^k = \frac{1}{l_1 l_2 l_3} \sum_\alpha \Big(\int_{S_\alpha} d\hat{t}_R^k + \int_{\bar{S}_\alpha} d\hat{t}_R^k \Big) = \sum_\alpha \frac{S_R^{k\alpha} - \bar{S}_R^{k\alpha}}{l_\alpha}, \tag{4.1}$$

where we have denoted

$$S_R^{k\alpha} \equiv \frac{1}{l_\beta l_\gamma} \int_{S_\alpha} d\hat{t}_R^k, \quad \bar{S}_R^{k\alpha} \equiv -\frac{1}{l_\beta l_\gamma} \int_{\bar{S}_\alpha} d\hat{t}_R^k, \quad \alpha \neq \beta \neq \gamma \neq \alpha.$$

Let us also assume that for each $\alpha \neq \beta$ we have

$$\int_{S_\beta} Y^\alpha d\hat{t}_R^k = 0, \quad \int_{\bar{S}_\beta} Y^\alpha d\hat{t}_R^k = 0; \quad \alpha \neq \beta.$$

Then, from (1.5) we conclude that

$$T_R^{k\alpha} = \frac{1}{2} (S_\alpha^k + \bar{S}_R^{k\alpha}). \tag{4.2}$$

Equations (4.1), (4.2) hold for any $\mathbf{Z} \in L$ and $t \in R$.

We shall assume that the part $P(\mathbf{Z})$ of M can interact with parts $P(\mathbf{Z} \pm \Delta_\alpha \mathbf{Z})$; the continuity of interactions leads to the conditions

$$S_R^{k\alpha}(\mathbf{Z}, t) = \bar{S}_R^{k\alpha}(\mathbf{Z} + \Delta_\alpha \mathbf{Z}, t), \quad \bar{S}_R^{k\alpha}(\mathbf{Z}, t) = S_R^{k\alpha}(\mathbf{Z} - \Delta_\alpha \mathbf{Z}, t). \tag{4.3}$$

From (4.1)–(4.3) we obtain

$$(4.4)\qquad \begin{aligned} d_R^k(\mathbf{Z}, t) &= \Delta_\alpha \bar{S}_R^{k\alpha}(\mathbf{Z}, t) = \bar{\Delta}_\alpha S_R^{k\alpha}(\mathbf{Z}, t),\\ T_R^{k\alpha}(\mathbf{Z}, t) &= \mu_\beta^\alpha \bar{S}_R^{k\beta}(\mathbf{Z}, t) = \bar{\mu}_\beta^\alpha S_R^{k\beta}(\mathbf{Z}, t). \end{aligned}$$

In what follows we shall take as the unknown generalized forces the fields $S_R^{k\alpha}(\mathbf{Z}, t)$. For the purpose of generalization we postulate extra conditions on those forces, as follows:

$$(4.5)\qquad h^\mu(\mathbf{Z}, t, \mathbf{S}_R, \Delta_\alpha \mathbf{S}_R, \bar{\Delta}_\alpha \mathbf{S}_R) = 0, \quad \mu = 1, \dots, l,$$

where h^μ are known differentiable functions. We assume that the constraints (4.4), (4.5) are ideal; this means that the relation

$$(4.6)\qquad \sum_L (\chi_k \delta d_R^k + F_{k\alpha} \delta T_R^{k\alpha}) = 0$$

holds for any δd_R^k, $\delta T_R^{k\alpha}$ consistent with (4.4), (4.5). The foregoing relation can also be transformed into the form

$$(4.7)\qquad \sum_L J_{k\alpha} \delta S_R^{k\alpha} = 0, \quad J_{k\alpha} \equiv \Delta_\alpha \chi_k - \mu_\alpha^\beta F_{k\beta},$$

which has to hold for any $\delta S^{k\alpha}$ consistent with Eqs. (4.5). From general field equations (1.4) and equations of kinetic constraints (4.4) we obtain the equations of motion

$$(4.8)\qquad \begin{aligned} \bar{\Delta}_\alpha S_R^{k\alpha} + b_R^k &= \varrho_R \ddot{\chi}^k,\\ l^2 J_R^{\alpha\beta} \ddot{F}_\beta^k &= \bar{\mu}_\beta^\alpha S_R^{k\beta} - \varrho_R \frac{\partial \sigma}{\partial F_{k\alpha}} + B_R^{k\alpha}. \end{aligned}$$

Equations (4.5), (4.7) and (4.8) represent the discrete model of the material system M, which is based on the kinetic constraints (4.4), (4.5).

5. Continuous models based on kinetic constraints

Let us assume that $\Pi = \bar{B}$, where B is a regular region in the physical space E^3 and that all the assumptions given at the beginning of the Sect. 3, including the relation (3.1), are satisfied. To obtain the continuous model of M we shall postulate the kinetic constraints

$$(5.1)\qquad d_R^k(\mathbf{Z}, t) - T_R{}^{k\alpha}{}_{,\alpha}(\mathbf{Z}, t) = 0, \quad \mathbf{Z} \in B, t \in R.$$

Equations (5.1) can be interpreted as the "continuous" approximation of the discrete kinetic constraints $(4.4)_1$. We shall also take into consideration extra kinetic constraints in the form

$$(5.2)\qquad h^\mu(\mathbf{Z}, t, \mathbf{T}_R, \nabla \mathbf{T}_R) = 0, \quad \mathbf{Z} \in B, t \in R, \quad \mu = 1, \dots, l,$$

where h^μ are known differentiable functions. Assuming that the constraints (5.1), (5.2) are ideal we postulate that the relation

$$(5.3)\qquad \int_B (\chi_k \delta d_R^k + F_{k\alpha} \delta T_R^{k\alpha}) dv + \oint_{\partial B} \chi_k \delta d_R^k d\sigma = 0$$

holds for any δd_R^k, $\delta T_R^{k\alpha}$, consistent with Eqs. (5.1), (5.2). The latter relation can be taken in the form of

$$\int_B J_{k\alpha}\delta T_R^{k\alpha}dv = 0, \quad J_{k\alpha} \equiv \chi_{k,\alpha} - F_{k\alpha}, \tag{5.4}$$

provided that we also postulate the boundary kinetic constraints

$$T_R^{k\alpha}n_{R\alpha} + d_R^k = 0, \quad \mathbf{Z} \in \partial B, \quad t \in R. \tag{5.5}$$

Relation (5.4) has to hold for any $\delta T_R^{k\alpha}$ consistent with the extra constraints (5.2). From the general field equations (1.4) (defined on $B \times R$) and from Eqs. (3.1), (5.1), (5.5), we obtain the equations of motion in $B \times R$

$$\begin{aligned} &T_R{}^{k\alpha}{}_{,\alpha} + b_R^k = \varrho_R \ddot{\chi}^k,\\ &l^2 J_R^{\alpha\beta}\ddot{F}^k{}_\beta = T_R^{k\alpha} - \varrho_R \frac{\partial\sigma}{\partial F_{k\alpha}} + B_R^{k\alpha}, \end{aligned} \tag{5.6}$$

and the kinetic boundary conditions on $\partial B \times R$

$$T_R^{k\alpha} n_{R\alpha} = b_R^k. \tag{5.7}$$

Equations (5.6), (5.7) and (5.2), (5.4) characterize the continuous model of the material system M, based on the kinetic constraints. If $l \approx 0$ and $B_R^{k\alpha} \approx 0$, we obtain from the equations mentioned above, the equations of the material continuum with the constraints imposed on the first Piola-Kirchhoff stress tensor, [2].

6. Discrete-continuous models

Let us assume now that $\Pi = B \times L$, where B is a regular region in E^N, N is equal to 1 or 2, and L is a lattice of points in E^{3-N}. We can interpret E^N and E^{3-N} as the subspaces of the physical space E^3. Let us denote $\mathbf{Z} = ((X^K), (X^A))$, where $(X^K) \in B$, $(X^A) \in L$, and the indices K, A run over the sequences $1, \ldots, N$ and $N+1, \ldots, 3$, respectively, and where X^α are Carthesian orthogonal coordinates in E^3. Combining the results given in Secst. 2 and 3, we shall postulate the geometric constraints

$$\begin{aligned} &\Delta_\alpha \chi_k(\mathbf{Z}, t) - \mu_\alpha^\beta F_{k\beta}(\mathbf{Z}, t) = 0,\\ &\chi_{k,K}(\mathbf{Z}, t) - \delta_K^\beta F_{k\beta}(\mathbf{Z}, t) = 0. \end{aligned} \tag{6.1}$$

These constraints are said to be ideal if the relation

$$\sum_L \Big[\int_B (d_R^k \delta\chi_k + T_R^{k\alpha}\delta F_{k\alpha})dv + \oint_{\partial B} d_R^k \delta\chi_k d\sigma\Big] = 0 \tag{6.2}$$

holds for any $\delta\chi_k$, $\delta F_{k\alpha}$ admissible by Eqs. (6.1) (if $B = (0, h) \subset R$, then the integral over ∂B has to be replaced by the expression of the form $[d_R^k \delta\chi_k]_{X^1=0}^{X^1=h}$). From Eq. (6.2) and from the general field equations (1.4) (where the functions are now defined on $B \times L \times$ $\times R$ or $B \times L$) we obtain in $B \times L \times R$ the following equations of motion

$$\begin{aligned} &\bar{\Delta}_A S_R^{kA} + T_R{}^{kK}{}_{,K} + b_R^k = \varrho_R \ddot{\chi}^k,\\ &l^2 J_R^{\alpha\beta}\ddot{F}_\beta^k = \bar{\mu}_A^\alpha S_R^{kA} + \delta_K^\alpha T_R^{kK} - \varrho_R \frac{\partial\sigma}{\partial F_{k\alpha}} + B_R^{k\alpha}, \end{aligned} \tag{6.3}$$

and on $\partial B \times L \times R$ the kinetic boundary conditions of the form

$$T_R^{kK} n_{RK} = b_R^k, \tag{6.4}$$

where Eq. (3.1) was also taken into account.

Equations (6.1)–(6.4) describe the discrete-continuous model of the material system M, which is based on the concept of ideal geometric constraints (6.1). We shall obtain an analogous result after including the kinetic constraints

$$d_R^k = \bar{\Delta}_A S_R^{kA} + T_R{}^{kK}{}_{,K}, \quad T_R^{k\alpha} = \bar{\mu}_A^\alpha S_R{}^{kA} + \delta_K^\alpha T_R{}^{kK}, \tag{6.5}$$

and assuming that they are ideal, i.e., that the relation

$$\sum_L \Big[\int_B (\chi_k \delta d_R^k + F_{k\alpha} \delta T_R^{k\alpha}) dv + \oint_{\partial B} \chi_k \delta d_R^k d\sigma \Big] = 0 \tag{6.6}$$

holds for any δd_R^k, $\delta T_R^{k\alpha}$ admissible by Eqs. (6.5). However, models of M based on geometric constraints are different from those based on kinetic constraints if the conditions of the form (2.2), (3.3), (4.5) or (5.2) are included. Discrete-continuous models can be used, among others, to describe the multi-layered structures (then $K = 1, 2$ and $A = 3$) or bodies reinforced with the family of non-intersecting regularly distributed fibres (then $K = 1$ and $A = 2, 3$).

7. Conclusions

The main problem in the paper, that is, how to construct the approximate mechanical models of certain material systems, was solved in Secs. 2–6 by applying the concept of ideal constraints to the general field equations (1.4). We can now pass to the second problem which concerns the relation between different models of the given material system. This problem is much more complicated and here we shall confine ourselves to a few conclusions which follow from the results obtained in the foregoing sections of the paper.

Comparing Eqs. (2.4) with Eqs. (3.5) and Eqs. (4.8) with Eqs. (5.6), we can see that, from the analytical point of view, the basic equations of the discrete models constitute the finite difference approximation of the partial differential equations of the corresponding continuum models. It follows that the "distance" between the related models can be reckoned, for example, by the methods given in [3]. However, it does not necessarily follow that the continuous models are more or less exact than the discrete models. Using the known methods of function spaces [3], only the "distance" between both kinds of models can be analysed because the discrete and continuous models studied were obtained independently each other. On the other hand, the degree of approximation of any model of M given in Secs. 2–6 with respect to the exact equations (1.2) of mechanics of the material system M, can be traced only for a given kind of a partition $M = \cup\ P(Z)$, $Z \in \Pi$ (cf. Sect. 1), for a known class of motions etc. This problem is connected with the applications of the models dealt with to the special cases of material systems, such as laminated bodies, fibrous media and others.

The next problem concerns the relations between models based on the geometric and the kinetic constraints, both in discrete and continuum approximation. If we compare

Eqs. (2.4) with Eqs. (4.8) and Eqs. (3.5) with Eqs. (5.6), we can see that the corresponding models (based on the geometric and kinetic constraints, respectively) coincide in the absence of the extra constraints given either by Eqs. (2.2), (4.5) or by Eqs. (3.3), (5.2). In this paper the extra constraints were introduced formaly (we did not analyse the form of the functions h_ν, R_ρ, h^μ); however, their role has to be investigated for each special case of the construction of the model of M. Roughly speaking, the models based on geometric constraints are more "rigid" while the models based on kinetic constraints make the body more "slender". For a more detailed discussion of this problem the reader is referred to [2].

The discrete and continuous models obtained in the paper are rather simple models of the complex material system M. However, because of presence of the length dimension l in the field equations (2.4), (3.5), (4.8) or (5.6), the weak non-local effects are included. For the asymptotic case $l \approx 0$ we obtain from $(1.4)_2$ the stress relation of the hyperelastic material. If $l \approx 0$, $B_R^{k\alpha} \approx 0$, then the models obtained in Secs. 3, 5 are governed by the equations of the classical continua with internal constraints for deformations and stresses [2]. Using the ideas which lead to the models as described in Secs. 2–6, we can also construct more general models. For this purpose we have to introduce, instead of Eqs. (1.1), the nonhomogeneous approximations of the motion of any part $P(Z)$ of the material system M. In this way we arrive at the form of the basic field equations which is more general than that given by Eqs. (1.4). The constraints imposed on the unknown fields in the basic equations will lead to the discrete and continuous models which are governed by the equations of the non-simple or Cosserat type systems with geometric or kinetic ideal constraints.

It must be stressed, that the approach to the problem of construction of approximate models of material systems which was presented in the paper and based on the concept of ideal constraints, is, obviously, not a unique one. Instead of postulating the ideal constraints of the form (2.1), (3.2), (4.4), (5.1), we can also postulate constitutive equations for the kinetic fields d_R^k, $T_R^{k\alpha}$, the arguments of corresponding response functionals are the kinematic fields $\chi_k, F_{k\alpha}$. For the discrete models with potential interactions between parts $P(\mathbf{Z})$, $P(\mathbf{Z}+\Delta_\alpha \mathbf{Z})$ of the material system M, we obtain, after some simple calculations (cf. [4]),

$$d_R^k = \bar{\Delta}_\alpha H_R^{\alpha k}, \quad T_R^{k\alpha} = \bar{\Delta}_\beta H_R^{\alpha k\beta} + G_R^{k\alpha}$$

where

$$H_R^{\alpha k} \equiv \frac{\partial \varepsilon_R}{\partial \Delta_\alpha \chi_k}, \quad H_R^{\alpha k\beta} \equiv \frac{\partial \varepsilon_R}{\partial \Delta_\beta F_{k\alpha}}, \quad G_R^{k\alpha} \equiv -\frac{\partial \varepsilon_R}{\partial F_{k\alpha}},$$

and where $\varepsilon_R = \varepsilon_R(\mathbf{Z}; \Delta_\alpha \chi_k, F_{k\alpha}, \Delta_\beta F_{k\alpha})$ is the potential of interaction of the part $P(\mathbf{Z})$ with parts $P(\mathbf{Z}+\Delta_\alpha \mathbf{Z})$ of M. The foregoing equations along with the general field equations (1.4) give what can be called the discrete elastic system of the Cosserat type. For the continuous model with potential interactions among different parts $P(Z)$ of M, we shall obtain by means of the variational approach the Eqs. (1.4) in which

$$d_R^k = H_R{}^{\alpha k}{}_{,\alpha}, \quad T_R^{k\alpha} = H_R{}^{\alpha k\beta}{}_{,\beta} + G_R^{k\alpha},$$

where

$$H_R^{\alpha k} \equiv \frac{\partial \varepsilon_R}{\partial \chi_{k,\alpha}}, \quad H_R^{\alpha k \beta} \equiv \frac{\partial \varepsilon_R}{\partial F_{k\alpha,\beta}}, \quad G_R^{k\alpha} \equiv -\frac{\partial \varepsilon_R}{\partial F_{k\alpha}},$$

and where $\varepsilon_R = \varepsilon_R(\mathbf{Z}; \chi_{k,\alpha}; F_{k\alpha}; F_{k\alpha,\beta})$ is the density of the potential of interactions. Lagrange's corresponding function was assumed in the form $\frac{1}{2}\varrho_R\, \dot{\chi}^k \dot{\chi}_k + \frac{1}{2} l^2 J_R^{\alpha\beta} \dot{F}^k{}_\alpha \dot{F}_{k\beta} - \varrho_R \sigma - \varepsilon_R$. These models are more complicated than those based on the concept of constraints. Recent literature on the laminated and fibrous media and composite materials also gives evidence of other approaches to the problem of construction of approximate models of complex material systems are applied (cf. [5, 6]). However, the approach based on the notion of constraints seems to be very simple and very general. The direct applications of the discussed models special complex material systems, have not been analysed thus far and will be the subject of separate investigations.

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