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Non-linear mechanics of constrained material continua. II. Ideal constraints for deformations and stresses

Cz. WOŹNIAK (WARSZAWA)

In the first part of this paper [1] foundations of mechanics for material continuum with internal constraints imposed on deformations were discussed. The present note constitute a generalization of an approach given in [1]; the constraints are imposed not only on deformations but also on stresses. The character of constraints in both cases is different; roughly speaking, the constraints for deformations make the body more "rigid" and the constraints for stresses make the body more "slender". The constraints for stresses are applied to formulate theories of slender bodies (such as strings and membranes), in theories of shells and rods (where we neglecte certain stress components), in discretized approaches to continuum mechanics (where certain classes of discretized states of stress are postulated *a priori*), etc. All mentioned problems are special cases of the general approach given in the note.

W pierwszej części tej pracy [1] omówiono podstawy mechaniki ośrodków ciągłych z więzami dla deformacji. Obecna nota jest uogólnieniem podejścia przedstawionego w [1]; rozpatrywane są zarówno więzy dla deformacji jak i dla naprężeń. Charakter więzów w obu przypadkach jest różny; z grubsza biorąc, więzy dla deformacji „usztywniają" ciało, a więzy dla naprężeń czynią ciało bardziej „wiotkim". Więzy dla naprężeń występują przy formułowaniu teorii ciał wiotkich (takich jak cięgna i membrany), w teoriach powłok i prętów (gdzie pomijamy pewne składowe naprężenia), w dyskretyzowanych podejściach do mechaniki kontinuum (gdzie są postulowane a priori pewne klasy dyskretyzowanych stanów naprężenia), etc. Wymienione tu problemy są przypadkami szczególnymi podejścia przedstawionego w pracy.

В первой части данной работы [1] обсуждены основы механики сплошных сред со связями для деформаций. Настоящая заметка является обобщением подхода представленного в [1]; рассматриваются так связи для деформаций, как и для напряжений. Характер связей в обоих случаях различен; грубо принимая связи для деформаций „придают жесткости "телу, а всязи для напряжений делают тело более „гибким". Связи для напряжений выступают при формулировке теории гибких тел (таких как связи и мембраны), в теориях оболочек и стержней (где пренебрегаем некоторыми составляющими напряжений), в дискретизированных подходах к механике континуум (где постулируются а приори некоторые классы дискретизированных напряженных состояний), и т. п. Перечисленные здесъ проблемы являются частными случаями подхода представленного в работе.

Notations

Indices α, β and k, l run over the sequence $\{1, 2, 3\}$, and indices K, L, M, N run over the sequence $\{1, 2\}$; the summation convention holds. The inertial coordinates in the space-time are denoted by x^k, t, where the system $\{x^k_\blacksquare\}$ is assumed to be Cartesian and orthogonal. The material coordinates are denoted by $\mathbf{X} = (X^\alpha)$ and coincide with the coordinates $\mathbf{x} = (x^k)$ when a body is in the reference configuration. Subscripts preceded by a comma or by a vertical line denote partial or covariant differentiation, respectively, symbol $[a]$ denotes a jump of the field $\mathbf{a}$ across the surface.

1. Laws of dynamics

LET B_R be a region occupied by a body in the reference configuration. We assume that there is given a partition $\overline{B}_R = \cup \overline{B}_a$, $a = 1, \ldots, m$, $B_a \cap B_b = \phi$ for each $a \neq b$, and we denote $B_0 \equiv \cup B_a$, $S_R \cup \equiv \partial B_a$. Let the surface S_R be oriented by a unit normal vector $\mathbf{n}_R = (n_{R\alpha})$ and divided into two separate parts: S_t, S_χ, where $S_t \subset B_R$ (cf. Fig. 1).

To formulate the basic laws of dynamics we introduce the following primitive concepts:

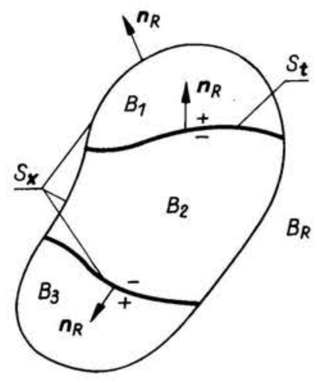

FIG. 1

1. The deformation function $x^k = \chi^k(\mathbf{X}, t)$, $\mathbf{X} \in \overline{B}_R$, $t \in R$; we assume that $\boldsymbol{\chi}$ may suffer discontinuities on S_t only (i.e., S_t is not a material surface, but there exists a pair of moving surfaces $\boldsymbol{\chi}^{(+)}(\mathbf{X}, t)$ and $\boldsymbol{\chi}^{(-)}(\mathbf{X}, t)$, $\mathbf{X} \in S_t$, where $\boldsymbol{\chi}^{(+)}$, $\boldsymbol{\chi}^{(-)}$ are boundary values of $\boldsymbol{\chi}$ on S_t).

2. The mass distribution $\varrho_R(\mathbf{X}) > 0$, $\mathbf{X} \in B_0$, in the reference configuration.

3. The body forces $b^k(\mathbf{X}, t)$, $\mathbf{X} \in B_0$, $t \in R$.

4. The surface tractions $p_R^k(\mathbf{X}, t)$, $\mathbf{X} \in S_\chi$, $t \in R$, related to the reference configuration and defined nearly everywhere on S_χ.

5. The stress vector $\mathbf{t}^k_{(\mathbf{n}_R)}(\mathbf{X}, t)$, $\mathbf{X} \in B_0$, $t \in R$, related to the reference configuration; we assume that $[\mathbf{t}_{(\mathbf{n}_R)}] = \mathbf{0}$ on S_t (it means that there exist two systems of contact forces defined on $S_t \times R$: the system $t_{(\mathbf{n}_R)}$ which acts upon the body across the surface $\boldsymbol{\chi}^{(-)}$ and the system $-\mathbf{t}_{(\mathbf{n}_R)}$ which acts upon the body across the surface $\boldsymbol{\chi}^{(+)}$).

All functions mentioned above must satisfy the suitable conditions of regularity in the domains of their definitions: the functions ϱ_R, $\mathbf{b}$ are assumed to be continuous in each B_a, $a = 1, \ldots, m$, the material derivatives of $\boldsymbol{\chi}$ may suffer discontinuities only on $S_R \cap B_R$, stresses $\mathbf{t}_{(\mathbf{n}_R)}$ have continuous material derivatives in each B_a and the principle of impenetrability of the material is postulated (i.e., the function $x^k = \chi^k(\mathbf{X}, t)$ is assumed to have an inverse $X^\alpha = X^\alpha(\mathbf{x}, t)$ for each time instant $t \in R$ and for each $\mathbf{X} \in B_R - S_t$).

As basic axioms of dynamics we shall take:

1. The laws of conservation of momentum and moment of momentum

$$
\begin{aligned}
&\frac{d}{dt}\int_P \varrho_R \dot{\boldsymbol{\chi}}\, dv_R = \oint_{\partial P} \mathbf{t}_{(\mathbf{n}_R)}\, d\sigma_R + \int_P \varrho_R \mathbf{b}\, dv_R,\\
(1.1)\qquad&\\
&\frac{d}{dt}\int_P \varrho_R \dot{\boldsymbol{\chi}} \times \boldsymbol{\chi}\, dv_R = \oint_{\partial P} \mathbf{t}_{(\mathbf{n}_R)} \times \boldsymbol{\chi}\, d\sigma_R + \int_P \varrho_R \mathbf{b} \times \boldsymbol{\chi}\, dv_R,
\end{aligned}
$$

which have to hold for any regular region $P \subset B_0$.

2. Newton's third law of mechanics

$$\mathbf{p}_R = \begin{cases} \mathbf{t}_{(\mathbf{n}_R)} & \text{on} \quad S_\chi \cap \partial B_R, \\ [\mathbf{t}_{(\mathbf{n}_R)}] & \text{on} \quad S_\chi \cap B, \end{cases} \tag{1.2}$$

on the surface S_χ (we denote $[\mathbf{a}] \equiv \mathbf{a}^{(-)} - \mathbf{a}^{(+)}$ on $S_R - \partial B_R$, cf. Fig. 1).

By virtue of (1.1) we obtain $\mathbf{t}_{(\mathbf{n}_R)} = \mathbf{T}_R \, \mathbf{n}_R$, $\mathbf{T}_R = (T_R{}^{k\alpha})$ being the first Piola-Kirchhoff stress tensor, and the known field equations

$$T_{RP}{}^{k\alpha}{}_{,\alpha} + \varrho_R l^k = \varrho_R \ddot{\chi}^k, \quad T_R{}^{k[\alpha} X^{\beta]}{}_{,k} = 0 \quad \text{in } B_0. \tag{1.3}$$

From (1.2) we derive then bou dary and jump conditions

$$T_R{}^{k\alpha} n_{R\alpha} = p_R{}^k \quad \text{on} \quad S_\chi \cap \partial B_R; \; [T_R{}^{k\alpha}] n_{R\alpha} = p_R{}^k \quad \text{on} \quad S_\chi \cap B_R \tag{1.4}$$

After introducing the convective stress tensor defined by $\overline{T}^{\alpha\beta} \equiv J^{-1} X^\alpha{}_{,k} T_R{}^{k\beta}$, $J \equiv \det \nabla \boldsymbol{\chi}$, we shall obtain the following alternative form of Eqs. (1.3), (1.4), taking into account that $[T_R{}^{k\alpha}] n_{R\alpha} = 0$ on S_t:

$$\begin{gathered} \overline{T}^{\alpha\beta}|_\beta + \varrho b^\alpha = \varrho \ddot{\chi}^k X^\alpha{}_{,k} \quad \overline{T}^{[\alpha\beta]} = 0 \quad \text{in } B_0, \\ \overline{T}^{\alpha\beta} n_\beta = p^\alpha, \quad [\overline{T}^{\alpha\beta}] n_\beta = p^\alpha \text{ on } S_\chi, [\overline{T}^{\alpha\beta}] n_\beta = 0 \quad \text{on} \quad S_t, \end{gathered} \tag{1.5}$$

where $\varrho \equiv J^{-1} \varrho_R$, $b^\alpha \equiv b^k X^\alpha{}_{,k}$ and $p^\alpha d\sigma = p_R{}^\alpha d\sigma_R$, $d\sigma$ being the element of the surface $\boldsymbol{\chi}(S_\chi, t)$. Equations (1.5) can be written down in the Cartesian coordinate system $\{x^k\}$ in the physical space:

$$\begin{gathered} T^{kl}{}_{,l} + \varrho b^k = \varrho \ddot{\chi}^k, \quad T^{[kl]} = 0 \quad \text{in } B_0, \\ T^{kl} n_l = p^k, \quad [T^{kl}] n_l = p^k \text{ on } S_\chi, \quad [T^{kl}] n_l = 0 \quad \text{on } S_t, \end{gathered} \tag{1.6}$$

$\mathbf{T} = (T^{kl})$ being the Cauchy stress tensor, $p^k = \chi^k{}_{,\alpha} \, p^\alpha$.

Denoting $\mathbf{f} \equiv \mathbf{b} - \ddot{\boldsymbol{\chi}}$, we can prove the following variational theorems:

$$\begin{gathered} \int\limits_{S_\chi} p_R{}^k \delta\chi_k d\sigma_R + \int\limits_{B_B} \varrho_R f^k \delta\chi_R dv_R = \int\limits_{B_B} T_R{}^{k\alpha} \delta\chi_{k,\alpha} dv_R - \int\limits_{S_t} T_R{}^{k\alpha} [\delta\chi_k] n_{R\alpha} d\sigma_R, \\ \int\limits_{S_\chi} \chi_k \delta p_R{}^k d\sigma_R + \int\limits_{B_B} \varrho_R \chi_k \delta f^k dv_R = \int\limits_{B_B} \chi_{k,\alpha} \delta T_R{}^{k\alpha} dv_R - \int\limits_{S_t} [\chi_k] \delta T_R{}^{k\alpha} n_{R\alpha} d\sigma_R, \end{gathered} \tag{1.7}$$

which have to be satisfied for any smooth vector field $\delta\boldsymbol{\chi}$ in $\overline{B}_R - S_t$ and for any system of fields $\delta\mathbf{p}_R$, $\delta\mathbf{f}$, $\delta\mathbf{T}_R$, satisfying the relations

$$\begin{aligned} \delta T_R{}^{k\alpha}{}_{,\alpha} + \varrho_R \delta f^k &= 0 \quad \text{in } B_0, \\ \delta T_R{}^{k\alpha} n_{R\alpha} - \delta p_R{}^k &= 0 \quad \text{on } S_\chi \cap \partial B_R, \\ [\delta T_R{}^{k\alpha}] n_{R\alpha} - \delta p_R{}^k &= 0 \quad \text{on } S_\chi \cap B. \end{aligned} \tag{1.8}$$

The Eqs. (1.7), (1.8) can also be expressed in terms that appear in Eqs. (1.5), (1.6).

Relations (1.1)–(1.8) are valid for any material continuum (simple and non-polar), being independent of material properties of the body, its interaction with external fields and constraints imposed on the deformation function or on the stress tensor.

2. Relations defining the body

To formulate the axioms which define the continuous body under consideration as well as its interactions with external fields we must introduce the following primitive concepts, apart from the primitive concepts which were introduced in Sect. 1:

1. The body loads $\bar{\mathbf{b}} = (\bar{b}^k(\mathbf{X}, t))$, $\mathbf{X} \in B_0$, $t \in R$.

2. The external surface loads $\bar{\mathbf{p}}_R = (\bar{p}_R{}^k(\mathbf{X}, t))$, $\mathbf{X} \in S_\chi \cap \partial B_R$, $t \in R$, acting at the body across its boundary.

3. The field of local deformations $\mathbf{F} = (F^k{}_\alpha(\mathbf{X}, t))$, $\mathbf{X} \in B_0$, $t \in R$, where $\det \mathbf{F} > 0$.

The fields $\bar{\mathbf{b}}, \bar{\mathbf{p}}_R, \mathbf{F}$ have to satisfy conditions of regularity like the fields $\mathbf{b}$, $\mathbf{p}_R$, $\nabla \boldsymbol{\chi}$, respectively.

We postulate that for each body under consideration the external loads are related to the deformation function by means of the formulas:

$$\bar{\mathbf{b}}(\mathbf{X}, t) = \boldsymbol{\beta}(\mathbf{X}, t, \boldsymbol{\chi}), \quad \bar{\mathbf{p}}_R(\mathbf{X}, t) = \boldsymbol{\pi}(\mathbf{X}, t, \boldsymbol{\chi}), \tag{2.1}$$

where $\boldsymbol{\beta}$, $\boldsymbol{\pi}$ are known differential or integral operators. In the special case, the right-hand sides of (2.1) are known functions of $\mathbf{X}$ ("dead" loads).

We shall assume that the material of the body is simple and write the stress relation in the form

$$\mathbf{T}_R(\mathbf{X}, t) = \mathop{\mathfrak{h}}_{s=0}^{\infty} \left(\mathbf{X}, \mathbf{F}^{(t)}(s)\right), \quad \mathbf{X} \in B_0, \tag{2.2}$$

where $\mathfrak{h}$ is the known response functional for the first Piola-Kirchhoff stress tensor, $\mathbf{F}^{(t)}(s) \equiv \mathbf{F}(\mathbf{X}, t-s)$ being the history of the local deformation field. The Eq. (2.2) has to satisfy the principle of material frame indifference.

Apart from the load and stress relations we postulate that constraints for deformation are imposed on the motion of the body.

$$\begin{aligned} &h_\nu(\mathbf{X}, t, \boldsymbol{\chi}, \nabla\boldsymbol{\chi}, \boldsymbol{\psi}, \nabla\boldsymbol{\psi}) = 0; \quad \mathbf{X} \in B_0, t \in R; \quad h_\nu = 1, \ldots, r, \\ &R_\varrho(\mathbf{X}, t, \boldsymbol{\chi}, \bar{\nabla}\boldsymbol{\chi}, \boldsymbol{\psi}, \bar{\nabla}\boldsymbol{\psi}) = 0; \quad \mathbf{X} \in S_\chi, t \in R; \quad \varrho = 1, \ldots, s, \end{aligned} \tag{2.3}$$

where h_ν, R_ϱ are known functions, $\boldsymbol{\psi} = (\psi^a(\mathbf{X}, t))$, $a = 1, \ldots, n$, $\mathbf{X} \in \bar{B}_R$, $t \in R$, is an unknown vector and $\bar{\nabla}$ is a material gradient on the surface S_χ, i.e. $\bar{\nabla}\boldsymbol{\chi} = (\chi^k{}_{,K})$, $K = 1, 2$, where $\bar{X}^K$ are local parameters on S_χ. We assume that $\boldsymbol{\psi}$ is not a new primitive concept, the definition of $\boldsymbol{\psi}$ being included in (2.3). In some cases we can eliminate $\boldsymbol{\psi}$ from (2.3) and obtain equations of constraints in the form of differential equations of a higher order, cf. [1] the Eqs. $(2.3)_2$ can be either independent of $(2.3)_1$ or generated by $(2.3)_1$ (1).

We shall also postulate that there exist constraints imposed on the stress tensor. Firstly, we assume that these constraints are imposed on the first Piola-Kirchhoff stress tensor

$$\begin{aligned} &h^\mu(\mathbf{X}, t, \mathbf{T}_R, \nabla\mathbf{T}_R, \bar{\boldsymbol{\psi}}, \nabla\bar{\boldsymbol{\psi}}) = 0; \quad \mathbf{X} \in B_0, \ t \in R; \quad \mu = 1, \ldots, r, \\ &R^\pi(\mathbf{X}, t, \mathbf{T}_R\mathbf{n}_R, \quad \bar{\nabla}(\mathbf{T}_R\mathbf{n}_R), \bar{\boldsymbol{\psi}}, \bar{\nabla}\bar{\boldsymbol{\psi}}) = 0; \quad \mathbf{X} \in S_t^*, \ t \in R; \quad \pi = 1, \ldots, s, \end{aligned} \tag{2.4}$$

$S_t^* \subset S_t$, where h^μ_t, R^π are known functions (they can also depend on $\mathbf{F}$) and $\boldsymbol{\psi} = (\bar{\psi}^b(\mathbf{X}, t))$, $\mathbf{X} \in B_R$, $t \in R$, $b = 1, \ldots, \bar{n}$, is an unknown vector which has the same meaning as the

(1) For example, if the constraints $\boldsymbol{\chi}_{,11} = 0$ are given in B_0, and a part S_χ^* of S_χ coincide with a congruence of parametric lines X^1, then the constraints $\boldsymbol{\chi}_{,11} = 0$ have also to be introduced on the surface $S_\chi^* \subset S_\chi$.

vector $\boldsymbol{\psi}$ in (2.3). The Eqs. $(2.4)_1$ have to be consistent with the stress relations (2.2); it means that after substituting right-hand sides of (2.2) into $(2.4)_1$ we arrive at the equations which have to be satisfied by at last one history $\mathbf{F}^{(t)}$ at each point $\mathbf{X} \in B_0$. Some from the relations $(2.4)_2$ can be generated by $(2.4)_1$.

3. Principles of reaction for constraints. Ideal constraints

Let us observe that the Eqs. (2.4) can be interpreted as the constraints for deformations if the stress relation is given in the form $\mathbf{T}_R = \mathfrak{h}_R(\nabla \boldsymbol{\chi}^{(t)})$. The other interpretation of these constraints can be obtained by treating (2.4) as certain restriction on the class of materials (on the form of the response functional). Neither of these two interpretations will be assumed in what follows; the constraints for stresses will be treated independently of the constraints for deformations and material properties of the body.

We will postulate the following statement which will be called the principle of reaction for constraints imposed on deformation:

If on the deformation function $\boldsymbol{\chi}(\mathbf{X}, t)$ *are imposed constraints* (2.3), *then there exist the reaction forces* $\mathbf{r}(\mathbf{X}, t)$ *in* B_0 *and the reaction forces* $\mathbf{s}_R(\mathbf{X}, t)$ *on* S_χ, *which maintain these constraints such that the relations*

$$\mathbf{b} = \bar{\mathbf{b}} + \mathbf{r} \quad \text{in } B_0, \qquad \mathbf{p}_R = \bar{\mathbf{p}} + \mathbf{s}_R \quad \text{on } S_\chi, \tag{3.1}$$

hold.

From (3.1) it follows that the system of forces $\mathbf{b}, \mathbf{p}_R$, introduced in Sect. 1, is given by the sum of external loads and reaction forces. Because of $\bar{\mathbf{p}}_R = \mathbf{0}$ on $S_\chi \cap B_R$, we also have $\bar{\mathbf{p}}_R = \mathbf{s}_R$ on $S_\chi \cap B_R$.

In what follows we are to deal with what are called ideal constraints [1];

The constraints imposed on the deformation function are said to be ideal if the sum of the works of the reaction forces due to the constraints on any virtual displacement $\delta\boldsymbol{\chi}$ *is equal to zero, i.e., if the relation*

$$\int_{S_\chi} s_R{}^k \delta\chi_k d\sigma_R + \int_{B_0} \varrho_R r^k \delta\chi_k dv_R = 0, \tag{3.2}$$

holds for any field $\delta\boldsymbol{\chi} \in D^1(B_R - S_t)$ *such that* $\{\delta\boldsymbol{\chi}, \delta\boldsymbol{\psi}\}$ *is the solution of the system*

$$\begin{aligned} &\frac{\partial h_\nu}{\partial \chi^k}\delta\chi^k + \frac{\partial h_\nu}{\partial \chi^k{}_{,\alpha}}\delta\chi^k{}_{,\alpha} + \frac{\partial h_\nu}{\partial \psi^a}\delta\psi^a + \frac{\partial h_\nu}{\partial \psi^a{}_{,\alpha}}\delta\psi^a{}_{,\alpha} = 0, \quad X \in B_0,\ t \in R,\\ &\frac{\partial R_\varrho}{\partial \chi^k}\delta\chi^k + \frac{\partial R_\varrho}{\partial \chi^k{}_{,K}}\partial\chi^k{}_{,K} + \frac{\partial R_\varrho}{\partial \varphi^a}\delta\varphi^a + \frac{\partial R_\varrho}{\partial \psi^a{}_{,K}}\delta\psi^a{}_{,K} = 0, \quad X \in S_\chi,\ t \in R, \end{aligned} \tag{3.3}$$

defined for $\{\boldsymbol{\chi}, \boldsymbol{\psi}\}$ *which satisfy* (2.3)(²).

The ideal constraints imposed on deformations were analysed in [1]. In this note we will deal mainly with the constraints imposed on stresses. The following statement will be called the principle of reaction for constraints (2.4):

(²) Remember that the symbol $\delta\boldsymbol{\chi}$ has different meanings in the Eqs. (3.3) and $(1.7)_1$. The local coordinates on S_χ are denoted by X^K, $K = 1, 2$.

If on the first Piola-Kirchhoff stress tensor $\mathbf{T}_R$ *are imposed constraints* (2.4), *then there exist the strain incompatibilities* $\mathbf{I} = \left(I^k{}_\alpha(\mathbf{X}, t)\right)$ *in* B_0 *and the jump* $\mathbf{d} = \left(d^k(\mathbf{X}, t)\right)$ *of the deformation function across* $S_t^* \subset S_t$, *due to these constraints such that the relations*

$$\nabla\boldsymbol{\chi} = \mathbf{F}+\mathbf{I} \text{ in } B_0, \quad [\boldsymbol{\chi}] = \mathbf{d} \quad \text{on } S_t^*, \tag{3.4}$$

hold and the stress relation in the form (2.2) *is valid*[3].

Let us observe that in general the field $\mathbf{F}$ of local deformation cannot be integrable. Analogously as in the case of constraints imposed on deformations, if nothing is known about the constraints (2.4), then nothing is known about the stress incompatibilities $\mathbf{I}$ and jump $\mathbf{d}$ of the deformation function. In this note we shall confine ourselves to the concept of ideal constraints for stresses. To this end we shall formulate the following definition.

The constraints imposed on the first Piola-Kirchhoff stress tensor $\mathbf{T}_R$ *are termed ideal if the relation*

$$\int\limits_{R_B} I_{k\alpha}\,\delta T_R{}^{k\alpha}\,dv_R + \int\limits_{S_t^*} d_k\,\delta T_R{}^{k\alpha} n_{R\alpha}\,\delta\sigma_R = 0, \tag{3.5}$$

holds for any field $\delta\mathbf{T}_R \in D^1(\overline{B}_R - S_\chi)$ *such that* $\{\delta T_R, \delta\overline{\boldsymbol{\psi}}\}$ *is the solution of the system*

$$\begin{aligned}
&\frac{\partial h^\mu}{\partial T_R{}^{k\alpha}}\delta T_R{}^{k\alpha} + \frac{\partial h^\mu}{\partial T_R{}^{k\alpha}{}_{,\beta}}\delta T_R{}^{k\alpha}{}_{,\beta} + \frac{\partial h^\mu}{\partial\overline{\psi}^b}\delta\overline{\psi}^b + \frac{\partial h^\mu}{\partial\overline{\psi}^b{}_{,\alpha}}\delta\overline{\psi}^b{}_{,\alpha} = 0, \quad X \in B_0, \quad t \in R,\\
&\frac{\partial R^\pi}{\partial t_R{}^k}\delta t_R{}^k + \frac{\partial R^\pi}{\partial t_R{}^k{}_{,K}}\delta t_R{}^k{}_{,K} + \frac{\partial R^\pi}{\partial\overline{\psi}^b}\delta\overline{\psi}^b + \frac{\partial R^\pi}{\partial\overline{\psi}^b{}_{,K}}\delta\overline{\psi}^b{}_{,K} = 0; \; t_R{}^k \equiv T_R{}^{k\alpha} n_{R\alpha}; \; X \in S_t^*, \; t \in R,
\end{aligned} \tag{3.6}$$

defined for $\{\mathbf{T}_R, \overline{\boldsymbol{\psi}}\}$ *satisfying* (2.4).

From (3.5) it follows that if there are no constraints for $\mathbf{T}_R$, then $\mathbf{I} \equiv 0$ and $\mathbf{d} \equiv 0$, i.e., the local deformation field coincides with the deformation gradient $\nabla\boldsymbol{\chi}$ and the field $\boldsymbol{\chi}$ is continuous across S_t^* (denoting $S_t^0 \equiv S_t - S_t^*$, we can also put $S_t^* = \phi$, $S_t = S_t^0$).

The dynamical axioms (1.1), (1.2), load relations (2.1), stress relation (2.2), equations of constraints for deformation (2.3), (3.1), (3.2), equations of constrains for stresses (2.4), (3.4), (3.5) characterize continuum mechanics with constraints imposed on deformations and on the first Piola-Kirchhoff stress tensor.

4. Alternative forms of ideal constraints for stresses

The constraints for stresses can also be given as the restrictions imposed on the convective stress tensor $\overline{\mathbf{T}} = (\overline{T}^{\alpha\beta})$ or on the Cauchy stress tensor $\mathbf{T} = (T^{kl})$. In the former case we have

$$\begin{aligned}
&\overline{h}^\mu(\mathbf{X}, t, \overline{\mathbf{T}}, \nabla\overline{\mathbf{T}}, \overline{\boldsymbol{\psi}}, \nabla\overline{\boldsymbol{\psi}}) = 0; \quad \mathbf{X} \in R; \quad \mu = 1, \ldots, \overline{r},\\
&\overline{R}^\pi(\mathbf{X}, t, \overline{\mathbf{t}}, \overline{\nabla}\overline{\mathbf{t}}, \overline{\boldsymbol{\psi}}, \overline{\nabla}\overline{\boldsymbol{\psi}}) = 0; \quad \mathbf{X} \in S_t^*, \; t \in R; \quad \pi = 1, \ldots, \overline{s},
\end{aligned} \tag{4.1}$$

(3) The concept of non-material surface across which the deformation function is discontinuous but the stress vector is assumed to be continuous has a physical sense only for special kinds of discontinuities $[\boldsymbol{\chi}]$ and constraints $(2.4)_2$. Such a surface secures the existence of deformation function $\boldsymbol{\chi}$ when (2.4) holds (cf. Sect. 9).

and in the latter we shall write down

$$(4.2)\qquad \begin{aligned} &h^{\mu}(\mathbf{X}, t, \mathbf{T}, \operatorname{grad} \mathbf{T}, \bar{\boldsymbol{\psi}}, \operatorname{grad} \bar{\boldsymbol{\psi}}) = 0; \quad \mathbf{X} \in B_0,\ t \in R; \quad \mu = 1, \ldots, \bar{r}, \\ &R^{\pi}(\mathbf{X}, t, \mathbf{t}, \overline{\operatorname{grad}}\, \mathbf{t}, \bar{\boldsymbol{\psi}}, \overline{\operatorname{grad}}\ \bar{\boldsymbol{\psi}}) = 0; \quad \mathbf{X} \in S_t^*, t \in R; \quad \mu = 1, \ldots, \bar{s}, \end{aligned}$$

where $\bar{\mathbf{t}} \equiv (\bar{T}^{\alpha\beta} n_\beta)$, $\mathbf{t} \equiv (T^{kl} n_l)$, $\nabla\bar{\mathbf{T}}$ can be treated as covariant derivative, $\nabla\bar{\mathbf{T}} \equiv (\bar{T}^{\alpha\beta}|_\gamma)$, grad $\mathbf{T}$ is a spatial derivative, grad $\mathbf{T} \equiv (T^{kl}{}_{,m})$, and $\overline{\operatorname{grad}}\, T$ is a spatial derivative on the surfaces $\boldsymbol{\chi}^{(+)}(S_t, t)$, $\boldsymbol{\chi}^{(-)}(S_t, t)$, where $\boldsymbol{\chi}^{(+)}$, $\boldsymbol{\chi}^{(-)}$ are boundary values of the deformation function.

The stress relations for the stress tensors $\bar{\mathbf{T}}$ and $\mathbf{T}$ are given by

$$(4.3)\qquad \bar{T}^{\alpha\beta}(\mathbf{X}, t) = \mathop{\bar{\mathfrak{h}}^{\alpha\beta}}_{s=0}^{\infty} (\mathbf{X}, \mathbf{C}^{(t)}(s)), \qquad T^{kl}(\mathbf{X}, t) = \mathop{\mathfrak{h}^{kl}}_{s=0}^{\infty} (\mathbf{X}, \mathbf{F}^{(t)}(s)),$$

where $\mathbf{C} \equiv (C_{\alpha\beta}) \equiv \mathbf{F}^T\mathbf{F}$, and where $\mathfrak{h}^{\alpha\beta}$, $\mathfrak{h}^{kl}$ are known response functionals. Because the right-hand sides of Eqs. (2.2), (4.3) are related to $\mathfrak{h}^{kl} = F^k{}_\alpha F^l{}_\beta \bar{\mathfrak{h}}^{\alpha\beta}$, $\mathfrak{h}_R{}^{\alpha\beta} = F^k{}_\varrho \bar{\mathfrak{h}}^{\beta\alpha} \det \mathbf{F}$, and the left-hand sides of (2.2), (4.3) are related by $T^{kl} = \chi^k{}_{,\alpha}\chi^l{}_{,\beta}\, \bar{T}^{\alpha\beta}$, $T_R{}^{k\alpha} = \chi^k{}_{,\beta} \bar{T}^{\beta\alpha} J$, we conclude that the constraints (2.4), (4.1) and (4.2) are not equivalent. Thus we have to assume that a suitable form of the stress relation corresponds to each form of constraints for stresses [this assumption is included to each principle of reaction for constraints imposed on stresses, cf. (3.4) and (4.4)]. Moreover, the Eqs. $(4.3)_{1,2}$ must be consistent with the Eqs. (4.1), (4.2), respectively (cf. Sect. 3).

Now we shall postulate the principles of reaction for constraints (4.1), (4.2) and formulate the suitable definitions of ideal constraints.

If the constraints are imposed on the convective stress tensor $\bar{\mathbf{T}} = (\bar{T}^{\alpha\beta})$ (*on the Cauchy stress tensor* $\mathbf{T} = (T^{kl})$), *then there exists the strain incompatibilities* $\mathbf{D} = (D_{\alpha\beta})$ *in* B_0 *and the jump* $\mathbf{d} = (d_\alpha)$ *of the function* χ_k *across* S_t^* (the strain incompatibilities E_{kl} in B_0 and the jump $\mathbf{d} = (d_k)$ of the function χ_k across S_t^*) *due to these constraints such that the relation*

$$(4.4)\qquad \nabla\boldsymbol{\chi}^T\nabla\boldsymbol{\chi} = \mathbf{C} + \mathbf{D} \ \text{in}\ B_0, \quad [\nabla\boldsymbol{\chi}^T\boldsymbol{\chi}] = \mathbf{d} \quad \text{on}\ S_t^*$$

(the relations; $\nabla\boldsymbol{\chi}\nabla\boldsymbol{\chi}^T = \mathbf{B} + \boldsymbol{\Phi}$ and $[\boldsymbol{\chi}] = \mathbf{d}$, where $\mathbf{B} \equiv \mathbf{F}\mathbf{F}^T$) *hold and the stress relation* $(4.3)_1$ (the stress relation $(4.3)_2$) *is valid*.

The constraints (4.1), (4.2) *are termed ideal if the relations*

$$(4.5)\qquad \int_{B_B} D_{\alpha\beta}\, \delta\bar{T}^{\alpha\beta} dv + \int_{S_t^*} d_\alpha\, \delta\bar{T}^{\alpha\beta} n_\beta d\sigma = 0, \qquad \int_{B_B} E_{kl}\, \delta T^{kl} dv + \int_{S_t^*} d_k\, \delta T^{kl} n_l d\sigma = 0,$$

hold for any field $\delta\bar{\mathbf{T}} \in \mathscr{D}^1(\bar{B}_R - S_\chi)$, $\delta\mathbf{T} \in \mathscr{D}^1(\bar{B}_R - S_\chi)$ *respectively such that* $\{\delta\bar{\mathbf{T}}, \delta\bar{\boldsymbol{\psi}}\}$ *and* $\{\delta\bar{\mathbf{T}}, \delta\bar{\boldsymbol{\psi}}\}$ *are solutions of the systems*

$$(4.6)\qquad \delta\bar{h}^{\mu} = 0, \quad \delta\bar{R}^{\pi} = 0 \quad \text{and}\ \delta h^{\mu} = 0,\ \delta R^{\pi} = 0,$$

respectively (symbol δ in (4.6) denotes the variation of the corresponding function).

Systems $(4.6)_{1,2}$ of equations are determined for $\{\bar{\mathbf{T}}, \bar{\boldsymbol{\psi}}\}$, $\{\bar{\mathbf{T}}, \bar{\boldsymbol{\psi}}\}$ which satisfy (4.1) and (4.2), respectively.

From the definition given above follows that $D_{\alpha\beta} = 0$, $d_\alpha = 0$ ($E_{kl} = 0$, $d_k = 0$) if there are no constraints imposed on the convective stress tensor $\bar{\mathbf{T}} = (\bar{T}^{\alpha\beta})$ [on the Cauchy stress tensor $\mathbf{T} = (T^{kl})$]; in this cases we obtain $C_{\alpha\beta}(B_{kl})$ as the components of the right (left) Cauchy-Green deformation tensor.

The dynamic axioms (1.1), (1.2), load relations (2.1), the Eqs. (2.3), (3.1), (3.2) which define the constraints for deformations and the Eqs. (4.1), (4.4), (4.5) defining the constraints for the convective stress tensor, with the stress relation $(4.3)_1$, characterize the continuum mechanics with constraints imposed on deformations and convective stress tensor. Analogously, we can formulate the basic equations of continuum mechanics when the constraints are imposed on the Cauchy stress tensor, which is determined by the stress relation $(4.3)_2$. The principles of reaction for constraints are not physical laws and can be treated in the same manner as, for example, the principle of material frame indifference. We have also assumed that the jump $[\boldsymbol{\chi}]$ of the deformation function is due to the constraints for stresses only on the given *a priori* part S_t^* of the surface S_χ; on the other part S_t^0 of this surface we can deal with discontinuities of the deformation function $\boldsymbol{\chi}$ which are independent of constraints (S_t^0 may not be given *a priori*).

5. General theorems

General theorems which concern the continua with ideal constraints for deformation were given in [1]; here we shall write down only *the principle of virtual work*, which can be derived from $(1.7)_1$ (3.1) and (3.2):

$$(5.1)\qquad \oint_{\partial B_B} \bar{p}^k \delta\chi_k d\sigma_R + \int_{B_B} \varrho_R \bar{f}^k \delta\chi_k dv_R = \int_{B_B} T_R^{k\alpha}(\delta\chi_k)_{,\alpha} dv_R - \int_{S_t} T_R^{k\alpha}[\delta\chi_k] n_{R\alpha} d\sigma_R,$$

where $\bar{\mathbf{f}} \equiv \bar{\mathbf{b}} - \boldsymbol{\chi}$; Eq. (5.1) has to be satisfied by any virtual displacement $\delta\boldsymbol{\chi}$ satisfying (3.3). It is easy to prove that *the constraints for deformations are ideal if the principle of virtual work holds.* By virtue of $(1.7)_2$, (3.4) and (3.5) we arrive at

$$(5.2)\qquad \int_{S_\chi} \chi_k \delta p_R{}^k d\sigma_R + \int_{B_B} \varrho_R \chi_k \delta f^k dv_R = \int_{B_B} F_{k\alpha} \delta T_R{}^{k\alpha} dv_R - \int_{S_t^0} [\chi_k] \delta T_R{}^{k\alpha} n_{R\alpha} d\sigma_R,$$

and the later relation has to be satisfied by any $\delta\mathbf{p}_R$, $\delta\mathbf{f}$, $\delta\mathbf{T}_R$ satisfying (1.8) and (3.6).

This statement is called *the principle of complementary virtual work*. We can also prove that *the constraints* (2.4) *for the first Piola-Kirchhoff stress tensor are ideal if the principle of complementary virtual work holds.* Bearing in mind that the form of principles (5.1), (5.2) is independent of reaction forces $\mathbf{r}$, $\mathbf{s}$ and incompatibilities $\mathbf{I}$, $\mathbf{d}$ (where $\mathbf{d}$ was introduced only on S_t^*, $S_t^* \cap S_t^0 = \phi$).

Using (5.2) one can write the known Castigliano formula, which will be expressed either by the relation

$$(5.3)\qquad \oint_{\partial B_B} \chi_k \delta p_R{}^k d\sigma_R = \int_{B_B} F_{k\alpha} \delta T_R{}^{k\alpha} dv_R - \int_{S_t^1} [\chi_k] \delta T_R{}^{k\alpha} n_{R\alpha} d\sigma_R,$$

which holds for any $\delta\mathbf{p}_R$, $\delta\mathbf{T}_R$ satisfying $(3.6)_1$, $(1.8)_2$ and $\mathrm{Div}(\delta\mathbf{T}_R) = 0$ in B_0, $[\delta\mathbf{T}_R]\mathbf{n}_R = 0$ on $S_\chi \cap B_R$, or by the relation

$$(5.4)\qquad \int_{B_B} F_{k\alpha} \delta T_R{}^{k\alpha} dv_R - \int_{S_t^0} [\chi_k] \delta T_R{}^{k\alpha} n_{R\alpha} d\sigma_R = 0,$$

which holds for any $\delta\mathbf{T}_R$ satisfying $(3.6)_1$ and $\mathrm{Div}(\delta\mathbf{T}_R) = 0$, $\delta\mathbf{T}_R\mathbf{n}_R = \mathbf{0}$, $[\delta\mathbf{T}_R]\mathbf{n}_R = \mathbf{0}$ [cf. (1.8)] in B_0 and on $S_\chi \cap \partial B_R$, $S_\chi \cap B_R$, respectively.

Analogously, we can prove that for the constraints imposed on the convective stress tensor (4.1) the principle of the complementary virtual work is given by

$$(5.5) \qquad \int_{S_\chi} \chi_k \delta p^k d\sigma + \int_{B_B} \varrho\chi_k \delta f^k dv = \int_{B_B} C_{\alpha\beta}\delta\overline{T}^{\alpha\beta} dv - \int_{S_t^0} [\chi_\alpha]\delta\overline{T}^{\alpha\beta} n_\beta d\sigma,$$

where $\chi_\alpha \equiv \chi_k\chi^k{}_{,\alpha}$; the Eq. (5.5) holds for any δp^k, δf^k, $\delta\overline{T}^{\alpha\beta}$ satisfying $(4.6)_1$

$$(\delta\overline{T}^{\alpha\beta})|_\beta + \varrho X^\alpha{}_{,k}\delta f^k = 0 \text{ in } B_0, \quad [\delta\overline{T}^{\alpha\beta}]n_\beta = X^\alpha{}_{,k}\delta p^k \text{ on } S_\chi \cap B_R, \quad \delta\overline{T}^{\alpha\beta}n_\beta = X^\alpha{}_{,k}\delta p^k$$

on $S_\chi \cap \partial B_R$. Formulae (5.2) and (5.5) are not equivalent because the ideal constraints for $\mathbf{T}_R$ are not equivalent to the ideal constraints for $\overline{\mathbf{T}}$. From (5.5) follows the second form of the Castigliano formula

$$(5.6) \qquad \int_{B_B} C_{\alpha\beta}\delta\overline{T}^{\alpha\beta} dv - \int_{S_t^4} [\chi_\alpha]\delta\overline{T}^{\alpha\beta} n_\beta d\sigma = 0,$$

satisfied by any solution $\delta\overline{T}^{\alpha\beta}$ of $\delta\overline{h}^\mu = 0$; $(\delta\overline{T}^{\alpha\beta})|_\beta = 0$ in B_0 and $[\delta\overline{T}^{\alpha\beta}]n_\beta = 0$ on $S_\chi \cap B_R$, $\delta\overline{T}^{\alpha\beta}n_\beta = 0$ on $S_\chi \cap \partial B_R$. For the hyperelastic materials with the inverted stress relation $C^{\alpha\beta} = \partial\gamma/\partial\overline{T}^{\alpha\beta}$, $\gamma = \gamma(\overline{\mathbf{T}})$ being the complementary energy, we obtain from (5.6) the known Castigliano theorem $\delta\Gamma = 0$ [Γ is an integral over $\boldsymbol{\chi}(B_R, t)$ from γ] under assumption $S_t^0 = \phi$ (or $S_t = S_t^*$).

The suitable form of the principle of complementary virtual work and that of the Castigliano formula can also be obtained if the constraints are imposed on the Cauchy stress tensor.

If the constraints are scleronomic (i.e. invariant under the group of time translations) then from integral formulae given in Sects. 3–5 we can derive the corresponding formulas for the time derivatives of the fields $\boldsymbol{\chi}$, $\mathbf{F}$, $\mathbf{T}_R$, etc.

6. Lagrange's equations and boundary conditions of the first kind

Let the constraints be given in the form (2.3), (2.4). From (5.1) and (5.2), using the known Lagrange's multipliers approach, we shall obtain the system of field equations and boundary or jump conditions for unknowns $\boldsymbol{\chi}$, $\mathbf{F}$, $\mathbf{T}_R$, $\boldsymbol{\psi}$, $\overline{\boldsymbol{\psi}}$, λ^ν, μ^ϱ, λ_μ, μ_π, where λ^ν, μ^ϱ, λ_μ, μ_π are Lagrange's multipliers for the Eqs. $(2.3)_{1,2}$ and $(2.4)_{1,2}$, respectively([4]). Assuming that the suitable regularity conditions are satisfied, we shall obtain:

1. Equations of motion in B_0:

$$(6.1) \qquad \left(T_R{}^{k\alpha} - \lambda_\nu \frac{\partial h_\nu}{\partial \chi_{k,\alpha}}\right)_{,\alpha} + \varrho_R \overline{b}^k + \lambda^\nu \frac{\partial h_\nu}{\partial \chi_k} = \varrho_R \ddot{\chi}^k, \qquad \left(\lambda^\nu \frac{\partial h_\nu}{\partial \psi^a{}_{,\alpha}}\right)_{,\alpha} - \lambda^\nu \frac{\partial h_\nu}{\partial \psi^a} = 0.$$

([4]) To make the formulas more simple, we shall take here the Eqs. $(2.4)_2$ in the form $R^\pi(\mathbf{X}, t, \mathbf{T}_R\mathbf{n}_R, \overline{\boldsymbol{\psi}}) = 0$.

2. Compatibility conditions in B_0:

$$\chi_{k,\alpha}-F_{k\alpha}+\lambda_\mu\frac{\partial h_\mu}{\partial T_R{}^{k\alpha}}-\left(\lambda_\mu\frac{\partial h_\mu}{\partial T_R{}^{k\alpha}{}_{,\alpha}}\right)_{,\beta}=0,\qquad \lambda_\mu\frac{\partial h_\mu}{\partial\bar\psi_b}-\left(\lambda_\mu\frac{\partial h^\mu}{\partial\bar\psi^b{}_{,\alpha}}\right)_{,\alpha}=0. \tag{6.2}$$

3. Kinetic boundary conditions on $S_R\cap\partial B_R$:

$$\begin{aligned}\left(T_R{}^{k\alpha}-\lambda^\nu\frac{\partial h_\nu}{\partial\chi_{k,\alpha}}\right)n_{R\alpha}&=\bar p_R{}^k+\mu^\varrho\frac{\partial R_\varrho}{\partial\chi_k}-\left(\mu^\varrho\frac{\partial R_\varrho}{\partial\chi_{k,K}}\right)_{,K},\\ -\lambda^\nu\frac{\partial h_\nu}{\partial\psi^a{}_{,\alpha}}n_{R\alpha}&=\mu^\varrho\frac{\partial R_\varrho}{\partial\psi^a}-\left(\mu^\varrho\frac{\partial R_\varrho}{\partial\psi^\alpha{}_{,K}}\right)_{,K}.\end{aligned} \tag{6.3}$$

4. Kinetic jump conditions on $S_R\cap B_R$:

$$\begin{aligned}\left[\mathbf{T}_R{}^{k\alpha}-\lambda^\nu\frac{\partial h_\nu}{\partial\chi_{k,\alpha}}\right]n_{R\alpha}&=\mu^\varrho\frac{\partial R_\varrho}{\partial\chi_k}-\left(\mu^\varrho\frac{\partial R_\varrho}{\partial\chi_{k,K}}\right)_{,K},\\ -\left[\lambda^\nu\frac{\partial h_\nu}{\partial\psi^a{}_{,\alpha}}\right]n_{R\alpha}&=\mu^\varrho\frac{\partial R_\varrho}{\partial\psi^a}-\left(\mu^\varrho\frac{\partial R_\varrho}{\partial\psi^a{}_{,K}}\right)_{,K};\qquad R_\varrho\equiv0\ \text{on}\ S_t.\end{aligned} \tag{6.4}$$

5. Kinetic edge conditions on the lines $L\cap S_{(\delta)}$, where $S^{(\delta)}\subset S_\chi$, $\delta=1,\dots,k$, are smooth surfaces on which functions R_ϱ are differentiable:

$$\sum_\delta\mu^\varrho\frac{\partial R_\theta}{\partial\chi^k{}_{,K}}n_K^{(\delta)}=0,\qquad\sum_\delta\mu^\varrho\frac{\partial R_\varrho}{\partial\psi^a{}_{,K}}n_K^{\delta}=0, \tag{6.5}$$

$\mathbf{n}^{(\delta)}$ being the unit vector normal to L and tangent to $S_{(\delta)}$.

6. Boundary conditions of compatibility on $S_R\cap\partial B_R$:

$$\lambda_\mu\frac{\partial h^\mu}{\partial T_R{}^{k\alpha}{}_{,\beta}}n_{R\beta}=0,\qquad\lambda_\mu\frac{\partial h^\mu}{\partial\bar\psi^b{}_{,\beta}}n_{R\beta}=0. \tag{6.6}$$

7. Jump conditions of compatibility on $S_R\cap B_R$:

$$\begin{aligned}\left[-\chi_k\delta^\beta{}_\alpha+\lambda_\mu\frac{\partial h^\mu}{\partial T_R{}^{k\alpha}{}_{,\beta}}\right]n_{R\beta}&=\mu_\pi\frac{\partial R_\pi}{\partial(T_R{}^{k\beta}n_{R\beta})}n_{R\beta},\\ -\left[\lambda_\mu\frac{\partial h^\mu}{\partial\bar\psi^b{}_{,\beta}}\right]n_{R\beta}&=\mu_\pi\frac{\partial R^\pi}{\partial\bar\psi^b};\qquad R^\pi\equiv0,\ [\chi_k]\equiv0\quad\text{on}\ S_R-S_t^*.\end{aligned} \tag{6.7}$$

The Eqs. (6.1)–(6.7), the Eqs. (2.3), (2.4) defining the constraints [where $(2.4)_2$ are taken in the form $R^\pi(\boldsymbol\chi,t,\mathbf{T}_R\mathbf{n}_R,\bar\psi)=0$], load relations (2.1) and a stress relation (2.2) constitute the fundamental system of equations for material continuum with constraints imposed on the deformation function $\boldsymbol\chi$ and on the first Piola-Kirchhoff stress tensor $\mathbf{T}_R$. This system has to be considered together with the Eqs. (1.3), (1.4), (3.1) which determine the reaction forces $\mathbf{r}$, $\mathbf{s}_R$ and with the Eqs. (3.4) which determine the incompatibilities $\mathbf{I}$ and jumps $\mathbf{d}$ of the determination function.

The Eqs. (6.1), (6.2) will be called Lagrange's field equations of the first kind, and the Eqs. (6.3)–(6.7) will be termed Lagrange's boundary, jump or edge conditions of the first kind, respectively. The system of relations analogous to (6.1)–(6.7) but for constraints imposed on the convective stress tensor or on the Cauchy stress tensor can also be obtained using Lagrange's multipliers approach; in theses cases the Eqs. (6.1), (6.3)–(6.5) are valid in unchanged form.

7. Lagrange's equations and boundary conditions of the second kind

Now let us assume that the constraints (2.3) and (4.1) are given in an explicit form

$$\tag{7.1} \begin{aligned} &\chi^k(\mathbf{X}, t) = \Phi^k(\mathbf{X}, t, \boldsymbol{\psi}), \mathbf{X} \in B_R; \quad R_\theta(\mathbf{X}, t, \boldsymbol{\psi}) = 0,\ \mathbf{X} \in \partial B_R; \quad \boldsymbol{\psi} = \boldsymbol{\psi}(\mathbf{X}, t), \\ &\overline{T}^{\alpha\beta}(\mathbf{X}, t) = \overline{\Psi}^{\alpha\beta}(\mathbf{X}, t, \overline{\boldsymbol{\psi}}, \nabla\overline{\boldsymbol{\psi}}, \nabla^2\overline{\boldsymbol{\psi}}),\ \mathbf{X} \in B_R; \quad R^\pi(\mathbf{X}, t, \overline{\boldsymbol{\psi}}) = 0,\ \mathbf{X} \in \partial B_R, \end{aligned}$$

where Φ_k, $\overline{\Psi}^{\alpha\beta}$ are known functions and either $\boldsymbol{\psi} = (\psi^1)$ or $\boldsymbol{\psi} = (\psi^1, \psi^2)$ ([5]). As basic unknowns we shall take the vectors $\overline{\boldsymbol{\psi}}$ and $\boldsymbol{\psi} = (\psi^b)$, $b = 1, \ldots, \overline{n}$, which will be called the generalized deformations and stresses, respectively. Using (5.1) and (7.1) we shall obtain in B_R the Lagrange's equations of motion of the second kind

$$\tag{7.2} H_a^{\ \alpha}|_\alpha + h_a + f_a = \frac{d}{dt}\frac{\partial \varkappa}{\partial \dot{\psi}^a} - \frac{\partial \varkappa}{\partial \psi^a},$$

where

$$\tag{7.3} \begin{aligned} &H_a^{\ \alpha} \equiv \overline{\Psi}^{\alpha\beta}\Phi^k{}_{,\beta}\frac{\partial \Phi_k}{\partial \psi^a}, \qquad h_a \equiv -\overline{\Psi}^{\alpha\beta}\left(\Phi^k{}_{,\alpha}\frac{\partial \Phi_k}{\partial \psi^a}\right)\bigg|_\beta, \\ &f_a \equiv \varrho \overline{b}^k \frac{\partial \Phi_k}{\partial \psi^a}, \qquad \varkappa \equiv \frac{1}{2}\varrho \dot{\Phi}^k \dot{\Phi}_k, \end{aligned}$$

and the kinetic boundary conditions of the second kind on ∂B_R

$$\tag{7.4} H_a^{\ \alpha} n_\alpha = p_a,$$

where

$$\tag{7.5} p_a \equiv \overline{p}^k \frac{\partial \Phi_k}{\partial \psi^a} - \mu^\varrho \frac{\partial R_\varrho}{\partial \psi^a},$$

and where μ^ϱ are boundary Lagrange's multipliers due to the constraints $R^\varrho = 0$. From $(4.5)_1$ and (7.1), using the standard approach, we shall obtain in B_R the following relations

$$\tag{7.6} D_{\alpha\beta}\frac{\partial \overline{\Psi}^{\alpha\beta}}{\partial \overline{\psi}^b} - \left[D_{\alpha\beta}\frac{\partial \overline{\Psi}^{\alpha\beta}}{\partial \overline{\psi}^b|_\gamma} - \left(D_{\alpha\beta}\frac{\partial \overline{\Psi}^{\alpha\beta}}{\partial \overline{\psi}^b|_\gamma}\right)\bigg|_\beta\right]\bigg|_\gamma = 0,$$

which are said to be Lagrange's compatibility conditions of the second kind. We also obtain the integral conditions of compatibility

$$\tag{7.7} \oint_{\partial B}\left\{\left[D_{\alpha\beta}\frac{\partial \overline{\Psi}^{\alpha\beta}}{\partial \overline{\psi}^b|_\gamma} - \left(D_{\alpha\beta}\frac{\partial \overline{\Psi}^{\alpha\beta}}{\partial \overline{\psi}^b|_{\gamma\delta}}\right)\bigg|_\delta\right]\delta\overline{\psi}^b + D_{\alpha\beta}\frac{\partial \overline{\Psi}^{\alpha\beta}}{\partial \overline{\psi}^b|_{\gamma\delta}}\delta\overline{\psi}^b|_\delta\right\} n_\gamma d\sigma = 0,$$

which has to be satisfied by any $\delta\overline{\psi}^b$ satisfying $(\partial R^\pi/\partial \psi^b)\delta\overline{\psi}^b = 0$.

([5]) Examples of constraints $(7.1)_1$ can be given by axisymmetric, $\boldsymbol{\psi} = (\psi^1, \psi^2)$, or centrosymmetric, $\boldsymbol{\psi} = (\psi^1)$, deformations. Moreover, functions $\overline{\Psi}^{\alpha\beta}$ for some of the pairs $(\alpha, \beta) = (\beta, \alpha)$ of indices can be identically equal to zero. We also denote $\Delta\overline{\boldsymbol{\psi}} \equiv (\overline{\psi}^b|_\alpha)$, $\Delta^2\overline{\boldsymbol{\psi}} \equiv (\overline{\psi}^b|_{\alpha\beta})$, where the vertical line denotes covariant differentiation in the metric $\chi^k{}_{,\alpha}\chi_{k,\beta}$.

Field equations (7.2), (7.6), the stress relation $(4.3)_1$, and equations $R_\varrho = 0$ lead to the basic system of equations for $\overline{\psi}, \psi, \mathbf{C}$ and μ^ϱ, after taking into account (2.1), (7.1), $(4.4)_1$, (7.3), (7.5).

8. Stress functions

Let the constraints for stresses be given in the form $(7.1)_2$ and let $R_\Delta^\pi \equiv 0$. Let us also assume that the body is in rest and

$$\overline{\Psi}^{\alpha\beta}|_\beta + \varrho b^\alpha \equiv 0 \tag{8.1}$$

holds for any vector $\overline{\psi}$. In this case the generalized stresses are termed stress functions. Because of $\delta\overline{\Psi}^{\alpha\beta}|_\beta \equiv 0$, ($b^\alpha$ is independent of $\overline{\psi}$), from $(4.5)_1$ we obtain

$$\oint_{\partial B} \chi^k \chi_{k,\beta}\, \delta\overline{\Psi}^{\alpha\beta} n_\alpha d\sigma = \int_B C_{\alpha\beta}\, \delta\overline{\Psi}^{\alpha\beta} dv, \tag{8.2}$$

for any $\delta\overline{\psi} \in C^1(\overline{B}_R)$ (when $R^\pi \equiv 0$ then the surface integral in $(4.5)_1$ is equal to zero). It follows that

$$C_{\alpha\beta}\frac{\partial\overline{\Psi}^{\alpha\beta}}{\partial\overline{\psi}^b} - \left[C_{\alpha\beta}\frac{\partial\overline{\Psi}^{\alpha\beta}}{\partial\overline{\psi}^b|_\gamma} - \left(C_{\alpha\beta}\frac{\partial\overline{\Psi}^{\alpha\beta}}{\partial\overline{\psi}^b|_{\gamma\delta}}\right)\Bigg|_\delta\right]\Bigg|_\gamma = 0, \tag{8.3}$$

and that the following integral condition

$$\oint_{\partial B}\left\{\left[C_{\alpha\beta}\frac{\partial\overline{\Psi}^{\alpha\beta}}{\partial\overline{\psi}^b|_\gamma} - \left(C_{\alpha\beta}\frac{\partial\overline{\Psi}^{\alpha\beta}}{\partial\overline{\psi}^b|_{\gamma\delta}}\right)\Bigg|_\delta\right]\delta\overline{\psi}^b + C_{\alpha\beta}\frac{\partial\overline{\Psi}^{\alpha\beta}}{\partial\overline{\psi}^b|_{\gamma\delta}}\,\delta\overline{\psi}^b|_\delta\right\} n_\gamma d\sigma = \oint_{\partial B}\chi^k\chi_{k,\beta}\,\delta\overline{\Psi}^{\alpha\beta} n_\alpha d\sigma \tag{8.4}$$

holds for each $\delta\overline{\psi}$. The Eqs. (8.3) hold only if an identity (8.1) is valid. From (8.3) and (7.6) we conclude that $C_{\alpha\beta} = \chi^k{}_{,\alpha}\chi_{k,\beta}$ is the solution of (8.3); Eq. (8.3) will be termed the compatibility equation for the stress function $\overline{\psi}$, and with the stress relation $(4.3)_1$, written now in the form $\overline{\Psi}^{\alpha\beta} = \mathfrak{h}^{\alpha\beta}(\mathbf{X}, \mathbf{C}^{(r)})$, constitute the basic system of the field equations for $C_{\alpha\beta}$ and $\overline{\psi}^b$. The unknown equilibrium configuration of the body [being described by the vector ψ, cf. (7.1)], is now determined not by the Eq. (7.2) but by the relation $\Phi^k{}_{,\alpha}\Phi_{k,\beta} = C_{\alpha\beta}$ and (7.1).

9. Examples of constraints imposed on stresses

Many special cases of constraints imposed on the deformation function were given in [1, 2]; here we shall confine ourselves to constraints imposed on stresses. This problem is not a new one. The known applications of the Castigliano formula and Castigliano theorem in the problems of linear elasticity (cf. [3], pp. 222–225, 337–355, 396, 454) are very special cases of continuum mechanics with constraints imposed on stresses. In this Section we are to deal only with simple but non-linear problems.

As the first example let us discuss the constraints of the form

$$\mathbf{T}_R = 0 \quad \text{in} \quad B_R \tag{9.1}$$

assuming, that there are no constraints imposed on deformations. It is a body in which there are no interactions between its particles. Local deformation $\mathbf{F}$ for each particle of the continuum is determined now by (9.1) and (2.2). From $\delta \mathbf{T}_R = 0$ follows that (3.5) is an identity (we assume $S_{t}^{*} = \phi$ since there are no surface constraints). Because there is no constraints imposed on deformations, we have $\mathbf{r} = \mathbf{0}, \mathbf{s}_R = \mathbf{0}$ from (3.2) and then $\bar{\mathbf{b}} = \mathbf{b}, \bar{\mathbf{p}}_R = \mathbf{p}$ from (3.1) (system of body and surface forces coincide with the system of external loads). Using (9.1), (1.3) and (2.1) we shall obtain

$$\beta^k(\boldsymbol{\chi}) = \ddot{\chi}^k \quad \text{in } B_0, \quad \pi_{R}{}^k(\boldsymbol{\chi}) = 0 \quad \text{on } \partial B_R. \tag{9.2}$$

Let the body loads $\bar{\mathbf{b}}$ be continuous in each $B_a \subset B_R$ (cf. Sect. 1); then $S_t^0 = \cup (B_R \cap \partial B_a)$, and each part B_a of the body moves independently of any other part, according to $(9.2)_1$. From $(9.2)_2$ it follows that the body under consideration is unable to carry any surface loads.

As a second simple example let us take the hyperelastic body, which in the reference configuration constitutes a plate $\Pi \times (-h, h)$, where Π is a region on the plane $x^3 = 0$ and $-h \leqslant X^3 \leqslant h$, cf. Fig. 2. Let the constraints $(4.1)_1$ be given in the form

$$\bar{T}^{KL} = \bar{\psi}^{KL}(\mathbf{X}, t), \quad \bar{T}^{\alpha 3} = 0, \tag{9.3}$$

where $\bar{\psi}^{KL} \in C^1(B_R \times R)$ are arbitrary differentiable functions [we have $\bar{\boldsymbol{\psi}} = (\bar{\psi}^{11}, \dots, \bar{\psi}^{22})$], and let us assume that there are no constraints for deformations. A continuous body with

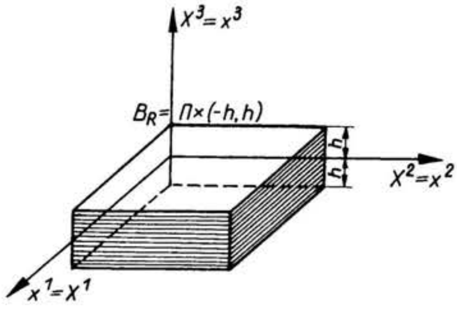

FIG. 2.

constraints (9.3) can be interpreted as multilayered body (each layer is represented by a material surface $X^3 = \text{const}$), in which there are no interactions between layers. Since there are no constraints for deformations, there are no reaction forces: $\mathbf{r} = 0$, $\mathbf{s}_R = 0$, and $\mathbf{b} = \bar{\mathbf{b}}$, $\mathbf{p}_R = \bar{\mathbf{p}}_R$ [cf. (3.1)]. The Eqs. (1.5) are given by

$$\begin{gathered} \bar{\psi}^{KL}\|_L + \left\{ {}^{3}_{3M} \right\} \bar{\psi}^{KM} + \varrho \bar{b}^k X^K{}_{,k} = \varrho \ddot{\chi}^k X^K{}_{,k}, \\ \left\{ {}^{3}_{KL} \right\} \bar{\psi}^{KL} + \varrho \bar{b}^k X^3{}_{,k} = \varrho \ddot{\chi}^k X^3{}_{,k}; \quad \left\{ {}^{\alpha}_{\beta\gamma} \right\} \equiv \chi^k{}_{,\beta\gamma} X^\alpha{}_{,k}, \end{gathered} \tag{9.4}$$

where the double vertical line denotes covariant differentiation in the metric $\chi^k{}_{,K} \chi_{k,L}$ [i.e. on the surface $\boldsymbol{\chi}(\Pi, X^3, t)$ for given X^3, t]. From (7.6) we obtain $D_{KL} = 0$ i.e.

$$C_{KL} = \chi^k{}_{,K} \chi_{k,L}, \quad D_{\alpha 3} = C_{\alpha 3} - \chi^k{}_{,\alpha} \chi_{k,3}. \tag{9.5}$$

The stress relation $(4.3)_1$ for hyperelastic materials has the form $\bar{T}^{\alpha\beta} = 2\bar{\varrho}\partial\sigma(\mathbf{C})/\partial C_{\alpha\beta}$; $\bar{\varrho} \equiv \varrho_R(\det \mathbf{C})^{0.5}$; hence we obtain

$$\bar{\psi}^{KL} = 2\bar{\varrho}\frac{\partial\sigma(\mathbf{C})}{\partial C_{KL}}, \quad \frac{\partial\sigma(\mathbf{C})}{\partial C_{\alpha 3}} = 0. \tag{9.6}$$

Substituting the right-hand side of $(9.5)_1$ into (9.6) we obtain $C_{\alpha 3} = f_{\alpha 3}(\mathbf{X}; \chi^k{}_{,M}\chi_{k,N})$ and then $\bar{\psi}^{KL} = f^{KL}(\mathbf{X}; \chi^k{}_{,M}\chi_{k,N})$. Substituting $\bar{\psi}^{KL} = f^{KL}(\mathbf{X}; \chi^k{}_{,M}\chi_{k,N})$ into the Eqs. (9.4) we arrive at the system of equations for $\chi^k(\mathbf{X}, t)$. We can observe, that if X^3 = const is a discontinuity surface of body loads $\bar{\mathbf{b}}$, then X^3 = const is a part of the surface S_t (cf. Sect. 1); surface S_t^* is an empty set because we have not postulated any constraints for stresses on S_t, i.e. $S_t = S_t^0$. When the solution $\boldsymbol{\chi}(\mathbf{X}, t)$ of the boundary value problem is known, then we can calculate strain incompatibilities from $(9.5)_2$: $D_{\alpha 3} = f_{\alpha 3}(\mathbf{X}; \chi^k{}_{,M}\chi_{k,N})$.

Now let the constraints (9.3) be interpreted as certain "*a priori*" assumptions, which do not follow from the physical structure of the body. If in the suitable norm we have $||D_{\alpha\beta}|| \ll ||C_{\alpha\beta}||$, then the solution of the problem under consideration differs only slightly from the solution without assumptions (9.3), provided that the problem is stable; $||D_{\alpha\beta}||$ can be interpreted as an error of numerical calculations, for example. The state of stress postulated by (9.3) can be applied to formulate theory of elastic membranes, and the condition $||D_{\alpha\beta}|| \leqslant \delta||C_{\alpha\beta}||$, δ being known positive constant small with respect to unity, can be used to determine the allowable thickness $2h$ of this membrane (membrane is treated here as the three dimensional body) in each particular motion.

As a next example let us take the hyperelastic body which in the reference configuration constitutes a prismatic rod $\Pi\times(0, l)$, where Π is a region on the plane $x^3 = 0$ (it is a projection of an arbitrary cross-section in the reference configuration); $(X^K)\in\Pi$, $K = 1, 2$; $X^3 \in (0, l)$, cf. Fig. 3. Let the body be made of the congruence of material fibres $X^1 =$

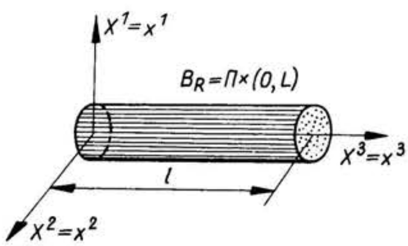

FIG. 3.

= const, X^2 = const, which do not interact (or their interactions can be neglected). This assumption enables to introduce the constraints for stresses in the form

$$\bar{T}^{33} = \bar{\psi}(\mathbf{X}, t), \quad \bar{T}^{K\alpha} = 0, \tag{9.7}$$

where $\bar{\psi}(\mathbf{X}, t)$ is an arbitrary differentiable function. The body under consideration can be also treated as a model of a red with uniaxial state of stress (9.7). Assuming that there are no constraints for deformations we have $\mathbf{s}_R = 0$, $\mathbf{r} = 0$ and $\mathbf{b} = \bar{\mathbf{b}}$, $\mathbf{p}_R = \bar{\mathbf{p}}_R$. Analogously as in the latter example we obtain the equations of motion

$$\begin{aligned} &\bar{\psi}_{,3} + \left\{ {\beta \atop \beta 3} \right\}\bar{\psi} + \left\{ {3 \atop 33} \right\}\bar{\psi} + \varrho\bar{b}^k X^3{}_{,k} = \varrho\ddot{\chi}^k X^3{}_{,k}, \\ &\left\{ {K \atop 33} \right\}\bar{\psi} + \varrho\bar{b}^k X^K{}_{,k} = \varrho\ddot{\chi}^k X^K{}_{,k}, \end{aligned} \tag{9.8}$$

the strain incompatibilities

$$D_{K\alpha} = C_{K\alpha} - \chi^k{}_{,K}\chi_{k,\alpha}, \quad D_{33} = 0 \text{ i.e. } C_{33} = \chi^k{}_{,3}\chi_{k,3}, \tag{9.9}$$

and the constitutive equations

$$\bar{\psi} = 2\bar{\varrho}\frac{\partial\sigma(\mathbf{C})}{\partial C_{33}}, \quad \frac{\partial\sigma(\mathbf{C})}{\partial C_{K\alpha}} = 0, \quad \bar{\varrho} \equiv \frac{\varrho_R}{\sqrt{\det\mathbf{C}}}. \tag{9.10}$$

From (9.10) aσd $(9.9)_3$ we obtain $C_{K\alpha} = f_{K\alpha}(\mathbf{X}; \chi^k{}_{,3}\chi_{k,3})$, $\bar{\psi} = f(\mathbf{X}; \chi_{k,3}\chi_{k,3})$; substituting the right-hand side of the latter relation into (9.8) we arrive at the system of equations for a vector $\boldsymbol{\chi}$. After solution of the corresponding boundary value problem we can calculate the strain incompatibilities from $(9.9)_1$. If the Eqs. (9.7) represent "*a priori*" hypothesis on the uniaxial state of stress, then $||D_{K\alpha}|| \leqslant \delta||C_{\alpha\beta}||$ can be used as a criterion of applicability of this hypothesis in each particular problem, δ being the known positive number, small with respect to unity.

The aim of the above examples was to illustrate the idea of constraints imposed on stresses and to point out, that by postulating such constraints we make the body more "slender". In [5] the concept of ideal constraints for stresses and deformations is applied to the theory of shells and to the finite element formulation of continuum mechanics.

Finally, we shall discuss the problem of constraints (imposed on stresses) which are determined on the surface S_t^* only. This surface, oriented in the reference configuration by a unit normal vector $\mathbf{n}_R$, is not a material surface and it enables us to define a pair of moving surfaces: $\boldsymbol{\chi}^{(+)}(\mathbf{X}, t)$ and $\boldsymbol{\chi}^{(-)}(\mathbf{X}, t)$, $\mathbf{X} \in S_t^*$, where $\boldsymbol{\chi}^{(+)}$, $\boldsymbol{\chi}^{(-)}$ are boundary values of $\boldsymbol{\chi}$ on S_t^*. At the same time we assume that there exist two systems of contact forces defined on S_t^*: the system $\mathbf{T}_R\mathbf{n}_R$ which is acting on the body across $\boldsymbol{\chi}^{(+)}$ and the system $-\mathbf{T}_R\mathbf{n}_R$ which is acting on the body across $\boldsymbol{\chi}^{(-)}$, cf. Sect. 1.

Equations of the form $(2.4)_2$, $(4.1)_2$ or $(4.2)_2$ represent the constraints imposed on the system of forces $\mathbf{T}_R\mathbf{n}_R$, and, at the same time, on the system of forces $-\mathbf{T}_R\mathbf{n}_R$. As an example let us take a plate which in a reference configuration is give on Fig. 2. As a surface S_t^* let us take the part of coordinate plane $X^3 = 0$ inside the plate. Let us also assume that there in no tangent interaction between the upper $(0 < X^3 < h)$ and lower $(-h < X^3 < 0)$ parts of the plate. This statement can be expressed in the form of constraints $\bar{T}^{K3} = 0$; $K = 1, 2$; on S_t^*. Because of $\delta\bar{T}^{K3} = 0$, the condition $(4.5)_1$ (in which $\mathbf{D} \equiv 0$) leads to $d_3 = [\chi_k\chi^k{}_{,3}] = 0$, and we arrive at boundary value problems for upper and lower parts of the plate with the continuity condition $(\chi_k\chi^k{}_{,3})^{(+)} = (\chi_k\chi^k{}_{,3})^{(-)}$ on S_t^*([6]). After solving these boundary value problems we can calculate the jump $d_K \equiv [\chi_k\chi^k{}_{,K}]$ on S_t^*. In some cases the constraints of the form $(2.4)_2$ can be greated as certain "*a priori*" hypothesis which enables to replace the continuity conditions $\boldsymbol{\chi}^{(+)} = \boldsymbol{\chi}^{(-)}$ on S^*, by a suitable distributions of contact forces on S_t^*.

Other examples of constraints for stresses are given in [4–5].

References

1. Cz. Woźniak, *Non-linear mechanics of constrained material continua* I, Arch. Mech. Stos., **26**, 1, 105–118, 1974.

([6]) This condition has a clear physical meaning in the theory of small deformations only.

2. Cz. Woźniak, *Constrained continuous media*, I, II, III, Bull. Acad. Polon. Sci. Serie Sci. Techn., 21, 3–4, 109–116, 167–173, 175–182, 1973.
3. Л. С. Лейбензон, *Собрание трудов*, том I, *Теорич упругости*, изд АНСССР, Москва 1951.
4. J. Utkin, *On the technical theory of thin elastic rods*, Bull. Acad. Polon. Sci., Série Sci. Techn., 23, 1, 1975.
5. Z. Naniewicz, *Elastic continua with discretized states of stress and strain*, Bull. Acad. Polon. Sci., Série Sci. Techn., in press.

UNIVERSITY OF WARSAW
INSTITUTE OF MECHANICS.

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