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Variational principles in the linear theory of mixtures

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IN the linear theory of isotropic mixtures of two elastic solids the principle of minimum potential energy, the principle of minimum complementary energy and the counterparts of the Hu-Washizu and Reissner-Hellinger variational principles are established. A reciprocity theorem is also given.

W ramach liniowej teorii izotropowych mieszanin dwóch sprężystych ciał stałych wyprowadzono zasadę minimum energii potencjalnej, zasadę minimum energii dopełniającej oraz odpowiedniki zasad wariacyjnych Hu-Washizu i Reissnera-Hellingera. Podano również twierdzenie o wzajemności.

В рамках линейной теории изотропных смесей двух упругих твердых тел выведены принцип минимума потенциальной энергии, принцип минимума дополнительной энергии, а также эквиваленты вариационных принципов Гу-Вашизу и Рейснера-Гелингера. Приведена тоже теорема взаимности.

1. Introduction

IN the last few years a number of problems have been solved in the linear theory of mixtures. In our previous papers [1] and [2] we have considered the problem of the existence and uniqueness of weak solutions in the linear theory of mixtures of two elastic solids. Here we deal with some variational theorems which rank among very important approximative methods. Thus, we establish a principle of minimum complementary energy and a principle of minimum potential energy. Variational principles which correspond to the principle of HU-WASHIZU [3] and REISSNER-HELLINGER [4] in classical elasticity are also given. In the last section we prove a reciprocity theorem. The first section deals with a brief summary of results obtained in [2].

2. Summary on boundary-value problems

The basic equations of the linear theory of isotropic mixtures of two elastic solids as given in [6] are:

— constitutive law

(2.1)
$$\sigma_{(ij)} = \{-\alpha_2+\lambda_1 e_{pp}+\lambda_3 g_{pp}\}\delta_{ij}+2\mu_1 e_{ij}+2\mu_3 g_{ij},$$
$$\pi_{(ij)} = \{\alpha_2+\lambda_4 e_{pp}+\lambda_2 g_{pp}\}\delta_{ij}+2\mu_3 e_{ij}+2\mu_2 g_{ij},$$
$$\sigma_{[ij]} = -\pi_{[ij]} = -2\lambda_5 h_{[ij]}, \qquad \pi_i = \frac{\varrho_2}{\varrho}\alpha_2 e_{pp,i}+\frac{\varrho_1}{\varrho}\alpha_2 g_{pp,i};$$

— the equations of static equilibrium

$$\sigma_{ji,j}-\pi_i+F_i = 0, \quad \pi_{ji,j}+\pi_i+G_i = 0; \tag{2.2}$$

— the geometrical equations

$$e_{ij} = \frac{1}{2}(\omega_{i,j}+\omega_{j,i}), \quad g_{ij} = \frac{1}{2}(\eta_{i,j}+\eta_{j,i}),$$
$$h_{[ij]} = \frac{1}{2}(\omega_{j,i}-\omega_{i,j}+\eta_{i,j}-\eta_{j,i}). \tag{2.3}$$

In the above $\sigma_{(ij)}$ and $\pi_{(ij)}$, $\sigma_{[ij]}$ and $\pi_{[ij]}$ represent, respectively, the symmetric and the skew symmetric parts of the partial stresses σ_{ij} and π_{ij}, π_i — the components of the diffusive force, ω_i, η_i — the components of the two displacement vectors, ϱ_1, ϱ_2 — the initial mass-densities of the two solids, F_i, G_i — the components of the two body forces and $\alpha_2, \lambda_1, \mu_1$ ... etc. — the material constants. We have denoted also

$$\varrho = \varrho_1+\varrho_2. \tag{2.4}$$

Throughout the paper an orthogonal Cartesian coordinate system is employed. As usually, a comma denotes the partial derivative and the convention of summing over repeated indices is adopted.

We consider the following boundary conditions which seem to be of practical interest [6]:

$$\omega_i = \eta_i = k_i \quad \text{on } \Gamma_1,$$
$$(\sigma_{ji}+\pi_{ji})n_j = T_i, \quad \omega_i = n_i \quad \text{on } \Gamma_2, \tag{2.5}$$
$$\overline{\Gamma}_1 \cup \Gamma_2 = \Gamma, \quad \Gamma_1 \cap \Gamma_2 = \phi,$$

where Γ is the boundary of the bounded region Ω occupied by the mixture.

We suppose that Γ is a Lipschitz boundary (see [7]) and that

$$k_i \in W_2^1(\Omega), \quad T_i \in L_2(\Gamma_2), \tag{2.6}$$

where $W_2^1(\Omega)$ is the Sobolev's space and $L_2(\Gamma_2)$ is the space of square-integrable functions on Γ_2.

Let $\mathring{W}_2^1(\Omega)$ be the closure of $D(\Omega)$ in $W_2^1(\Omega)$, $D(\Omega)$ being the space of real functions having continuous partial derivatives of all orders and compact support in Ω, and let $\mathbf{V}$ be a closed subspace of $\mathbf{W}^1(\Omega)$ such that $\mathring{\mathbf{W}}^1(\Omega) \subset \mathbf{V} \subset \mathbf{W}^1(\Omega)$, where

$$\mathbf{W}^1(\Omega) = W_2^1(\Omega)\times \ldots \times W_2^1(\Omega), \quad \text{(6 times)},$$
$$\mathring{\mathbf{W}}^1(\Omega) = \mathring{W}_2^1(\Omega)\times \ldots \times \mathring{W}_2^1(\Omega), \quad \text{(6 times)}. \tag{2.7}$$

$\mathbf{W}^1(\Omega)$ is a Hilbert space, provided with the norm

$$\|\mathbf{u}\|_{\mathbf{W}^1(\Omega)} = \Big[\sum_{i=1}^{3}(\|\omega_i\|^2_{W_2^1(\Omega)}+\|\eta_i\|^2_{W_2^1(\Omega)})\Big]^{\frac{1}{2}}, \tag{2.8}$$
$$\mathbf{u} \equiv \{\omega_1, \omega_2, \omega_3, \eta_1, \eta_2, \eta_3\}.$$

$\mathbf{V}$ is the subspace of $\mathbf{W}^1(\Omega)$ of all elements which satisfy the homogeneous boundary conditions (2.5).

In [2] the weak solution of the boundary value problem is defined to be a function $\mathbf{u} \in \mathbf{W}^1(\Omega)$, so that

$$\mathbf{u} - \hat{\mathbf{u}} \in \mathbf{V}, \quad \hat{\mathbf{u}} \equiv \{k_1, k_2, k_3, k_1, k_2, k_3\}, \tag{2.9}$$

and

$$\int_\Omega [M_{rsij}\tilde{e}_{rs}e_{ij} + P_{rsij}(\tilde{g}_{rs}e_{ij} + \tilde{e}_{rs}g_{ij}) + Q_{rsij}\tilde{g}_{rs}g_{ij} - 2\lambda_5 \tilde{h}_{[ij]}h_{[ij]}]d\Omega = \int_\Omega (F_i\tilde{\omega}_i + G_i\tilde{\eta}_i)d\Omega + \int_\Gamma T_i\omega_i d\Gamma \tag{2.10}$$

holds for each $\mathbf{v} \equiv \{\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3\} \in \mathbf{V}$.

In the above we have denoted

$$\begin{aligned} M_{rsij} &= \left[\left(\lambda_1 - \frac{\varrho_2}{\varrho}\alpha_2\right)\delta_{rs}\delta_{ij} + \frac{1}{2}\mu_1(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr})\right], \\ P_{rsij} &= \left[\left(\lambda_3 - \frac{\varrho_1}{\varrho}\alpha_2\right)\delta_{rs}\delta_{ij} + \frac{1}{2}\mu_3(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr})\right], \\ Q_{rsij} &= \left[\left(\lambda_2 + \frac{\varrho_1}{\varrho}\alpha_2\right)\delta_{rs}\delta_{ij} + \frac{1}{2}\mu_2(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr})\right]; \end{aligned} \tag{2.11}$$

$$\begin{aligned} \tilde{e}_{rs} &\equiv e_{rs}(\mathbf{v}) = \frac{1}{2}(\tilde{\omega}_{r,s} + \tilde{\omega}_{s,r}), \\ \tilde{g}_{rs} &\equiv g_{rs}(\mathbf{v}) = \frac{1}{2}(\tilde{\eta}_{r,s} + \tilde{\eta}_{s,r}), \\ \tilde{h}_{[rs]} &\equiv h_{[rs]}(\mathbf{v}) = \frac{1}{2}(\tilde{\omega}_{s,r} - \tilde{\omega}_{r,s} + \tilde{\eta}_{r,s} - \tilde{\eta}_{s,r}). \end{aligned} \tag{2.12}$$

The signification of this definition is obvious (see also [8]) in view of the principle of virtual work in the linear theory of isotropic mixtures of two elastic solids [2]:

$$\int_\Omega [M_{rsij}e_{rs}e_{ij} + 2P_{rsij}g_{rs}e_{ij} + Q_{rsij}g_{rs}g_{ij} - 2\lambda_5 h_{[ij]}h_{[ij]}]d\Omega = \int_{\Gamma_1} (\sigma_{ji} + \pi_{ji})k_i n_j d\Gamma + \int_{\Gamma_2} T_i\omega_i d\Gamma + \int_\Omega (F_i\omega_i + G_i\eta_i)d\Omega. \tag{2.13}$$

The quantity

$$A(\mathbf{u}, \mathbf{u}) \equiv \int_\Omega W(e_{rs}; g_{rs}; h_{[rs]})d\Omega, \tag{2.14}$$

where

$$2W(e_{rs}; g_{rs}; h_{[rs]}) = M_{rsij}e_{rs}e_{ij} + 2P_{rsij}g_{rs}e_{ij} + Q_{rsij}g_{rs}g_{ij} - 2\lambda_5 h_{[ij]}h_{[ij]}, \tag{2.15}$$

represents the global internal energy of the body.

Essentially in [2], we have proved the following two theorems regarding the existence and uniqueness of solutions of boundary value problems.

THEOREM 1. *The first and the mixed boundary value problems in the linear theory of isotropic mixtures of two elastic solids have a unique weak solution if*

$$W(e_{rs}; g_{rs}; h_{[rs]}) \geqslant c \sum_{r,s=1}^{3} (e_{rs}^2 + g_{rs}^2 + h_{[rs]}^2), \tag{2.16}$$

where c is a strictly positive constant.

Remark. The last inequality, which means that the quadratic form $W(e_{rs}; g_{rs}; h_{[rs]})$ is positive definite, leads to the following restrictions on the material constants:

$$\begin{gathered} \lambda_1 + \frac{2}{3}\mu_1 - \frac{\varrho_2}{\varrho}\alpha_2 > 0, \quad \lambda_2 + \frac{2}{3}\mu_2 + \frac{\varrho_1}{\varrho}\alpha_2 > 0, \\ \left(\lambda_3 + \frac{2}{3}\mu_3 - \frac{\varrho_1}{\varrho}\alpha_2\right)^2 < \left(\lambda_1 + \frac{2}{3}\mu_1 - \frac{\varrho_2}{\varrho}\alpha_2\right)\left(\lambda_2 + \frac{2}{3}\mu_2 + \frac{\varrho_1}{\varrho}\alpha_2\right), \\ \mu_1 > 0,\ \mu_2 > 0, \quad \mu_3^2 < \mu_1\mu_2,\ \lambda_5 < 0. \end{gathered} \tag{2.17}$$

The above inequalities have also been obtained in [6] as conditions for the uniqueness of classical solutions.

THEOREM 2. *Let the condition* (2.16) *hold. Then the second boundary value problem* ($\Gamma = \Gamma_2, \Gamma_1 = \phi$) *in the linear theory of isotropic mixtures of two elastic solids has a unique weak solution if, and only if,*

$$\begin{gathered} \int_{\Omega} (F_i + G_i)\, d\Omega + \int_{\Gamma} T_i\, d\Gamma = 0, \\ \int_{\Omega} \varepsilon_{ijk} x_j (F_k + G_k)\, d\Omega + \int_{\Gamma} \varepsilon_{ijk} x_j T_k\, d\Gamma = 0. \end{gathered} \tag{2.18}$$

The weak solution belongs to the function space defined by

$$\mathbf{V}_p = \Big\{\mathbf{v} \in \mathbf{V};\ \sum_{i=1}^{6} p_i^2(\mathbf{v}) = 0,\ p_i(\mathbf{v}) = \int_{\Omega} \omega_i\, d\Omega,\ (i = 1, 2, 3),\ p_j(\mathbf{v}) = \int_{\Omega} \varepsilon_{(j-3)kl}\, \omega_{l,k}\, d\Omega,\ j = 4, 5, 6\Big\}. \tag{2.19}$$

The conditions (2.18) *express the total equilibrium of external forces.*

3. The principle of minimum potential energy and the principle of minimum complementary energy

Since the quantity

$$A(\mathbf{v}, \mathbf{u}) = \int_{\Omega} [M_{rslj}\tilde{e}_{rs} e_{ij} + P_{rsij}(\tilde{g}_{rs} e_{ij} + \tilde{e}_{rs} g_{ij}) + Q_{rsij}\tilde{g}_{rs} g_{ij} - 2\lambda_5 \tilde{h}_{[ij]} h_{[ij]}]\, d\Omega \tag{3.1}$$

is a symmetric bilinear form we can apply the general theory developed in [7] in order to establish the principle of minimum potential energy. Thus, we define the quadratic functional on **V** by

$$\Phi(\mathbf{v}) = A(\mathbf{v}, \mathbf{v}) - 2\{f(\mathbf{v}) + g(\mathbf{v}) - A(\mathbf{v}, \hat{\mathbf{u}})\}, \tag{3.2}$$

where

$$f(\mathbf{v}) = \int_{\Omega} (F_i \tilde{\omega}_i + G_i \tilde{\eta}_i) d\Omega, \quad g(\mathbf{v}) = \int_{\Gamma} T_i \tilde{\omega}_i d\Gamma, \tag{3.3}$$

and consider $\mathring{\mathbf{u}} \in \mathbf{W}^1(\Omega)$ to be a weak solution of the boundary value problem. Taking into account the definition of the weak solution we have

$$\mathring{\mathbf{u}} - \hat{\mathbf{u}} \equiv \mathbf{w} \in \mathbf{V}, \tag{3.4}$$

and

$$A(\mathbf{v}, \mathbf{w}) = f(\mathbf{v}) + g(\mathbf{v}) - A(\mathbf{v}, \hat{\mathbf{u}}), \quad \mathbf{v} \in \mathbf{V}. \tag{3.5}$$

Then, the equality

$$A(\mathbf{v}, \mathbf{u}) = A(\mathbf{u}, \mathbf{v}) \tag{3.6}$$

and (3.4) and (3.5) imply that

$$\Phi(\mathbf{v}) = A(\mathbf{v}, \mathbf{v}) - 2A(\mathbf{v}, \mathbf{w}) = A(\mathbf{v}-\mathbf{w}, \mathbf{v}-\mathbf{w}) - A(\mathbf{w}, \mathbf{w}). \tag{3.7}$$

Now (2.16) implies that $\Phi(\mathbf{v})$ attains its minimum on $\mathbf{V}$ if, and only if, [7]

$$\mathbf{v} = \mathring{\mathbf{u}} - \hat{\mathbf{u}} + \mathbf{p}, \quad \mathbf{p} \in \mathbf{P}, \tag{3.8}$$

where [2]

$$\mathbf{P} = \{\mathbf{v} \in \mathbf{V};\, \tilde{\omega}_k = \tilde{\eta}_k = a_k + \varepsilon_{klm} b_l x_m,\ a_k, b_k = \text{const}\}. \tag{3.9}$$

Since

$$\begin{gathered} \Phi(\mathbf{u} - \hat{\mathbf{u}}) = A(\mathbf{u}, \mathbf{u}) - 2[f(\mathbf{u}) + g(\mathbf{u})] - A(\hat{\mathbf{u}}, \hat{\mathbf{u}}) + 2[f(\hat{\mathbf{u}}) + g(\hat{\mathbf{u}})], \\ \mathbf{u} \in \mathbf{W}^1(\Omega), \quad \mathbf{u} - \hat{\mathbf{u}} \in \mathbf{V}, \end{gathered} \tag{3.10}$$

the functional

$$\begin{aligned} \mathscr{L}(\mathbf{u}) &= \frac{1}{2} A(\mathbf{u}, \mathbf{u}) - f(\mathbf{u}) - g(\mathbf{u}) \\ &= \frac{1}{2} \int_{\Omega} \Big[\Big(\lambda_1 - \frac{\varrho_2}{\varrho}\alpha_2\Big) e_{pp}^2 + \Big(\lambda_2 + \frac{\varrho_1}{\varrho}\alpha_2\Big) g_{pp}^2 + 2\Big(\lambda_3 - \frac{\varrho_1}{\varrho}\alpha_2\Big) e_{pp} g_{pp} \\ &\quad + \mu_1 e_{ij} e_{ij} + \mu_2 g_{ij} g_{ij} + 2\mu_3 e_{ij} g_{ij} \Big] d\Omega - \int_{\Omega} (F_i \omega_i + G_i \eta_i) d\Omega - \int_{\Gamma} T_i \omega_i d\Gamma, \end{aligned} \tag{3.11}$$

attains its minimum on the set

$$\hat{\mathbf{u}} \oplus \mathbf{V}, \tag{3.12}$$

if, and only if,

$$\mathbf{u} = \mathring{\mathbf{u}} + \mathbf{p}, \quad \mathbf{p} \in \mathbf{P}. \tag{3.13}$$

This is the principle of minimum potential energy in the linear theory of isotropic mixtures of two elastic solids. We can also formulate this principle as follows [7]:

The quadratic functional (3.11) attains its minimum on the set

$$\hat{\mathbf{u}} \oplus \mathbf{V}_p \tag{3.14}$$

if, and only if,

$$\mathbf{u} = \mathring{\mathbf{u}}, \tag{3.15}$$

where $\mathring{\mathbf{u}}$ is the weak solution being unique in $\mathbf{V}_p$.

In order to obtain the principle of minimum complementary energy we shall use the method of orthogonal projections in Hilbert spaces [8].

If we suppose that the determinant

$$\Delta = \begin{vmatrix} 3\lambda_1+2\mu_1 & 3\lambda_3+2\mu_3 \\ 3\lambda_4+2\mu_3 & 3\lambda_2+2\mu_2 \end{vmatrix} \tag{3.16}$$

is different from zero, the inequalities $(2.17)_{6,7}$ assure the reversibility of the constitutive equations $(2.1)_{1,2,3}$ such that we can obtain:

$$\begin{aligned} e_{rs} &= K_{rsij}\hat{\sigma}_{(ij)}+L_{rsij}\hat{\pi}_{(ij)}, \\ g_{rs} &= L_{rsij}\hat{\sigma}_{(ij)}+N_{rsij}\hat{\pi}_{(ij)}, \\ h_{[rs]} &= -\frac{1}{2\lambda_5}\sigma_{[rs]} = \frac{1}{2\lambda_5}\pi_{[rs]}, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} K_{rsij} &= k_1\delta_{rs}\delta_{ij}+k_2(\delta_{ir}\delta_{js}+\delta_{is}\delta_{jr}), \\ L_{rsij} &= l_1\delta_{rs}\delta_{ij}+l_2(\delta_{ir}\delta_{js}+\delta_{is}\delta_{jr}), \\ N_{rsij} &= n_1\delta_{rs}\delta_{ij}+n_2(\delta_{ir}\delta_{js}+\delta_{is}\delta_{jr}), \end{aligned} \tag{3.18}$$

and

$$\hat{\sigma}_{(ij)} = \sigma_{(ij)}+\alpha_2\delta_{ij}, \quad \hat{\pi}_{(ij)} = \pi_{(ij)}-\alpha_2\delta_{ij}. \tag{3.19}$$

The density of internal energy on unit volume is given by [5]:

$$H = \frac{1}{2}[\sigma_{(ij)}e_{ij}+\pi_{(ij)}g_{ij}+\sigma_{[ij]}h_{[ij]}+\pi\theta], \tag{3.20}$$

where

$$\pi = \alpha_2\left(\frac{\varrho_2}{\varrho}e_{pp}+\frac{\varrho_1}{\varrho}g_{pp}\right), \quad \theta = g_{pp}-e_{pp}. \tag{3.21}$$

Taking into account (3.17), we have

$$H = \frac{1}{2}\Big[K^*_{rsij}\hat{\sigma}_{(rs)}\hat{\sigma}_{(ij)}+2L^*_{rsij}\hat{\sigma}_{(rs)}\hat{\pi}_{(ij)}+N^*_{rsij}\hat{\pi}_{(rs)}\hat{\pi}_{(ij)} -\frac{1}{2\lambda_5}\sigma_{[ij]}\sigma_{[ij]}+\alpha_2(g_{pp}-e_{pp})\Big], \tag{3.22}$$

where K^*_{rsij} etc. represent the quantities K_{rsij} etc. to which are added additional terms which come from the product $\pi\theta$.

In view of boundary conditions, we obtain

$$\int_\Omega (g_{pp}-e_{pp})\,d\Omega = 0 \tag{3.23}$$

such that the global internal energy of the body is given by

$$(3.24)\qquad A(\mathbf{u},\mathbf{u}) = \frac{1}{2}\int\limits_{\Omega}\Big[K^{*}_{rsij}\hat{\sigma}_{(ij)}\hat{\sigma}_{(rs)} + 2L^{*}_{rsij}\hat{\sigma}_{(rs)}\hat{\pi}_{(ij)} + N^{*}_{rsij}\hat{\pi}_{(rs)}\hat{\pi}_{(ij)} - \frac{1}{2\lambda_5}\sigma_{[ij]}\sigma_{[ij]}\Big]d\Omega .$$

We construct the Hilbert space $\mathscr{H}$ of the stress field

$$(3.25)\qquad \mathbf{S} \equiv \{\hat{\sigma}_{(ij)}, \hat{\pi}_{(ij)}, \sigma_{[ij]}\}, \qquad \hat{\sigma}_{(ij)}, \hat{\pi}_{(ij)}, \sigma_{[ij]} \in L_2(\Omega),$$

defining the scalar product

$$(3.26)\qquad (\mathbf{S}', \mathbf{S}'') = \int\limits_{\Omega}\Big[K^{*}_{rsij}\hat{\sigma}'_{(rs)}\hat{\sigma}''_{(ij)} + L^{*}_{rsij}(\hat{\sigma}'_{(rs)}\hat{\pi}''_{(ij)} + \hat{\sigma}''_{(rs)}\hat{\pi}'_{(ij)}) + N^{*}_{rsij}\hat{\pi}'_{(rs)}\hat{\pi}''_{(ij)} - \frac{1}{2\lambda_5}\sigma'_{[ij]}\sigma''_{[ij]}\Big]d\Omega.$$

The inequality (2.16) and (3.6) imply that all the axioms of the scalar product are satisfied.

Let $\mathscr{H}_1 \subset \mathscr{H}$ be the subset of all $\mathbf{S} \in \mathscr{H}$ to which $\mathbf{u} \in \mathbf{V}$ exists such that using (2.3), the Eqs. (2.1) hold, and let $\mathscr{H}_2 \subset \mathscr{H}$ be the subset of all $\mathbf{S} \in \mathscr{H}$ such that for each $\mathbf{v} \in \mathbf{V}$,

$$(3.27)\qquad \int\limits_{\Omega}[\hat{\sigma}_{(ij)}e_{ij} + \hat{\pi}_{(ij)}g_{ij} + \sigma_{[ij]}h_{[ij]} + \pi\theta]\,d\Omega = 0.$$

Let $\mathbf{S}' \in \mathscr{H}_1$, $\mathbf{S}'' \in \mathscr{H}_2$. Then from (3.26) and (3.20) we obtain

$$(3.28)\qquad (\mathbf{S}', \mathbf{S}'') = \int\limits_{\Omega}[\hat{\sigma}''_{(ij)}e'_{ij} + \hat{\pi}''_{(ij)}g'_{ij} + \hat{\sigma}''_{[ij]}h'_{[ij]} + \pi''\theta']\,d\Omega,$$

and there exists $\mathbf{u}' \in \mathbf{V}$ such that

$$(3.29)\qquad \begin{gathered} e'_{rs} = \frac{1}{2}(\omega'_{r,s} + \omega'_{s,r}), \qquad g'_{rs} = \frac{1}{2}(\eta'_{r,s} + \eta'_{s,r}), \\ h'_{[rs]} = \frac{1}{2}(\omega'_{s,r} - \omega'_{r,s} + \eta'_{r,s} - \eta'_{s,r}). \end{gathered}$$

The definition of $\mathscr{H}_2$ involves that

$$(3.30)\qquad (\mathbf{S}', \mathbf{S}'') = 0,$$

so that $\mathscr{H}_1$ and $\mathscr{H}_2$ are orthogonal.

Let $\mathbf{S} \in \mathscr{H}$ be an arbitrary stress field for which (2.10) holds for each $\mathbf{v} \in \mathbf{V}$ and let $\mathring{\mathbf{S}}$ be the stress field which corresponds to the weak solution $\mathring{\mathbf{u}}$, by means of (2.1) and (2.3), i.e. $\mathring{\mathbf{S}} \equiv \mathbf{S}(\mathring{\mathbf{u}})$.

The definition (3.9) of the set $\mathbf{P}$ implies

$$(3.31)\qquad \mathbf{S}(\mathring{\mathbf{u}} + \mathbf{p}) = \mathbf{S}(\mathring{\mathbf{u}}), \qquad \mathbf{p} \in \mathbf{P}.$$

From (3.4), we have

$$(3.32)\qquad \mathring{\mathbf{S}} = \mathbf{S}(\hat{\mathbf{u}}) + \mathbf{S}(\mathbf{w}), \qquad \mathbf{S}(\mathbf{w}) \in \mathscr{H}_1 .$$

It is clear that $\mathring{\mathbf{S}}$ satisfies (2.10) such that $\mathbf{S}-\mathring{\mathbf{S}} \in \mathscr{H}_2$. Hence, we can obtain

$$(3.33)\qquad \|\mathbf{S}-\mathbf{S}(\hat{\mathbf{u}})\|^2_{\mathscr{H}} = \|\mathbf{S}-\mathring{\mathbf{S}}+\mathbf{S}(\mathbf{w})\|^2_{\mathscr{H}} = \|\mathbf{S}-\mathring{\mathbf{S}}\|^2_{\mathscr{H}}+\|\mathbf{S}(\mathbf{w})\|^2_{\mathscr{H}}.$$

Now it is obvious that the functional

$$(3.34)\qquad \mathscr{K}(\mathbf{S}) = \frac{1}{2}\{\|\mathbf{S}-\mathbf{S}(\hat{\mathbf{u}})\|^2_{\mathscr{H}}-\|\mathbf{S}(\hat{\mathbf{u}})\|^2_{\mathscr{H}}\} = \frac{1}{2}(\mathbf{S},\mathbf{S})-(\mathbf{S},\mathbf{S}(\hat{\mathbf{u}}))$$

attains its minimum on the set of $\mathbf{S}$ which satisfy (2.10) if, and only if, $\mathbf{S} = \mathring{\mathbf{S}}$.

Taking into account that

$$(3.35)\qquad \hat{e}_{ij} \equiv e_{ij}(\hat{\mathbf{u}}) = g_{ij}(\hat{\mathbf{u}}) = \frac{1}{2}(k_{i,j}+k_{j,i}) \equiv k_{ij},$$

$$\hat{h}_{[ij]} \equiv h_{[ij]}(\hat{\mathbf{u}}) = 0, \qquad \hat{\theta} = \theta(\hat{\mathbf{u}}) = 0,$$

we have

$$(3.36)\qquad (\mathbf{S},\mathbf{S}(\hat{\mathbf{u}})) = \int_\Omega [\hat{\sigma}_{(ij)}+\hat{\pi}_{(ij)}]k_{ij}\,d\Omega = \int_\Omega [\sigma_{(ij)}+\pi_{(ij)}]k_{ij}\,d\Omega.$$

Now, the principle of minimum complementary energy can be stated as below:

The quadratic functional

$$(3.37)\qquad \mathscr{K}(\mathbf{S}) = \int_\Omega \Big[K^*_{rsij}\hat{\sigma}_{(rs)}\hat{\sigma}_{(ij)}+2L^*_{rsij}\hat{\sigma}_{(rs)}\hat{\pi}_{(ij)}+N^*_{rsij}\hat{\pi}_{(rs)}\hat{\pi}_{(ij)} - \frac{1}{2\lambda_5}\sigma_{[ij]}\sigma_{[ij]}\Big]d\Omega - \int_\Omega [\sigma_{(ij)}+\pi_{(ij)}]k_{ij}\,d\Omega,$$

attains its minimum on the set $\mathbf{S} \in \mathscr{H}$ which satisfy the equations of equilibrium (2.2) and the boundary conditions (2.5) in the sense of (2.13) if, and only if, $\mathbf{S} = \mathring{\mathbf{S}}$, where $\mathring{\mathbf{S}} \equiv \mathbf{S}(\mathring{\mathbf{u}})$, $\mathring{\mathbf{u}}$ being the weak solution.

We consider the weak solution $\mathring{\mathbf{u}}$ such that $\mathbf{S}(\mathring{\mathbf{u}})$ satisfies the equilibrium equations (2.2) in the sense of $\mathbf{L}_2(\Omega)$ and the boundary conditions in the sense of traces and take $\mathbf{S}$ to satisfy the same conditions. By applying the principle of virtual work to the field $\mathbf{S}$ in $\hat{\mathbf{u}}$, we obtain

$$(3.38)\qquad \int_\Omega [\sigma_{(ij)}+\pi_{(ij)}]k_{ij}\,d\Omega = \int_\Omega (F_i+G_i)k_i\,d\Omega+\int_{\Gamma_1}(\sigma_{ji}+\pi_{ji})k_in_j\,d\Gamma+\int_{\Gamma_2}T_i\omega_i\,d\Gamma.$$

If we omit the integrals not depending on $\mathbf{S}$, the principle of minimum complementary energy can be stated in the following form:

The quadratic functional

$$(3.39)\qquad \hat{\mathscr{K}}(\mathbf{S}) = \int_\Omega \Big[K^*_{rsij}\hat{\sigma}_{(rs)}\hat{\sigma}_{(ij)}+2L^*_{rsij}\hat{\sigma}_{(rs)}\hat{\pi}_{(ij)}+N^*_{rsij}\hat{\pi}_{(rs)}\hat{\pi}_{(ij)} - \frac{1}{2\lambda_5}\sigma_{[ij]}\sigma_{[ij]}\Big]d\Omega - \int_{\Gamma_1}(\sigma_{ji}+\pi_{ji})k_in_j\,d\Gamma,$$

attains its minimum on the set of statically admissible stress fields $\mathbf{S} \in \mathscr{H}$ (which satisfy the equilibrium equations in the sense of $\mathbf{L}_2(\Omega)$ and the boundary conditions in the sense of traces) if, and only if, $\mathbf{S} = \mathring{\mathbf{S}}$.

4. Other variational principles

We shall use the method of Lagrange multipliers in order to obtain the counterparts of HU-WASHIZU [3] and REISSNER-HELLINGER [4] variational principles.

Starting from the functional (3.11) we consider a new functional of the form

$$(4.1)\quad \mathscr{H}(\omega_i, \eta_i, e_{ij}, g_{ij}, h_{[ij]}, \lambda_{ij}, \mu_{ij}, \nu_{ij}, \xi_i, \zeta_i) = \int_\Omega [W(e_{ij}, g_{ij}, h_{[ij]}] - F_i\omega_i - G_i\eta_i] d\Omega$$

$$- \int_{\Gamma_2} T_i \omega_i d\Gamma + \int_\Omega \Big\{ \lambda_{ij}[\omega_{(i,j)} - e_{ij}] + \mu_{ij}[\eta_{(i,j)} - g_{ij}] + \nu_{ij}[\eta_{[i,j]} + \omega_{[j,i]} - h_{[ij]}]$$

$$+ \pi(\theta - \eta_{i,i} + \omega_{i,i}) \Big\} d\Omega - \int_{\Gamma_1} [\xi_i(\omega_i - k_i) + \zeta_i(\eta_i - k_i)] d\Gamma.$$

From the necessary conditions for $\delta\mathscr{H} = 0$ it is obvious that $\lambda_{ij}, \mu_{ij}, \nu_{ij}$, have the sense of $\sigma_{(ij)}, \pi_{(ij)}, \sigma_{[ij]}$, respectively, and

$$(4.2)\quad \xi_i = (-\sigma_{ij} + \delta_{ij}\pi)n_j, \qquad \zeta_i = (-\pi_{ij} - \delta_{ij}\pi)n_j.$$

Now we can establish the following variational principle: The condition $\delta\mathscr{H}(\omega_i, \eta_i, e_{ij}, g_{ij}, h_{[ij]}) = 0$, where

$$(4.3)\quad \mathscr{H}(\omega_i, \eta_i, e_{ij}, g_{ij}, h_{[ij]}) = \int_\Omega [W(e_{ij}; g_{ij}; h_{[ij]}) - \sigma_{(ij)} e_{ij} - \pi_{(ij)} g_{ij}$$

$$- \sigma_{[ij]} h_{[ij]} - \pi\theta + \sigma_{ij}\omega_{j,i} + \pi_{ij}\eta_{j,i} + \pi(\eta_{k,k} - \omega_{k,k}) - F_i\omega_i - G_i\eta_i] d\Omega$$

$$- \int_{\Gamma_2} T_i \omega_i d\Gamma - \int_{\Gamma_1} \{[\sigma_{ij}(\omega_j - k_j) + \pi_{ij}(\eta_j - k_j)]n_i + \pi(\eta_j - \omega_j)n_j\} d\Gamma,$$

yields the following Euler's conditions in Ω and the boundary conditions, respectively: the equilibrium equations (2.2), the geometrical equations (2.3), the constitutive equations (2.1) and the boundary conditions (2.5).

Taking into account (3.22) we add to the functional (3.37), by means of Lagrange multipliers, the equilibrium equations (2.2) and the boundary conditions (2.5) and obtain the following variational principle:

The condition $\delta\mathscr{R}(\sigma_{(ij)}, \pi_{(ij)}, \sigma_{[ij]}, \omega_i, \eta_i) = 0$, where

$$(4.4)\quad \mathscr{R}(\sigma_{(ij)}, \pi_{(ij)}, \sigma_{[ij]}, \omega_i, \eta_i) = \int_\Omega [H(\sigma_{(ij)}; \pi_{(ij)}; \sigma_{[ij]}) - (\sigma_{(ij)} e_{ij} + \pi_{(ij)} g_{ij} + \sigma_{[ij]} h_{[ij]}$$

$$+ \pi\theta + F_i\omega_i + G_i\eta_i] d\Omega + \int_{\Gamma_2} T_i\omega_i d\Gamma + \int_{\Gamma_1} \{[\sigma_{ij}(\omega_j - k_j) + \pi_{ij}(\eta_j - k_j)]n_i + \pi(\eta_j - \omega_j)n_j\} d\Gamma,$$

yields as Euler's conditions the equilibrium equations (2.2), the boundary conditions (2.5) and the Eqs. (3.17) where $e_{rs}, g_{rs}, h_{[rs]}$ are replaced by (2.3).

5. A reciprocity theorem

Consider the body subjected to two different systems of elastic loads:

$$\mathcal{T}^{(\alpha)} \equiv \{F_i^{(\alpha)}, G_i^{(\alpha)}, k_i^{(\alpha)}, T_i^{(\alpha)}\}, \quad \alpha = 1, 2, \tag{5.1}$$

and let $\mathcal{C}^{(\alpha)}$, $\alpha = 1, 2$, be two distinct elastic configurations of the body

$$\mathcal{C}^{(\alpha)} \equiv \{\omega_i^{(\alpha)}, \eta_i^{(\alpha)}\}. \tag{5.2}$$

We have noticed already that

$$(\mathbf{S}^1, \mathbf{S}^2) = (\mathbf{S}^2, \mathbf{S}^1), \tag{5.3}$$

the above product being defined in (3.28).

Using the Green-Gauss theorem and taking into account the equilibrium equations (2.2) and the boundary conditions (2.5) it is easily seen that the relation (5.3) reduces to

(5.4)

$$\int_\Gamma T_i^{(2)} k_i^{(1)} d\Gamma + \int_\Omega (F_i^{(2)} \omega_i^{(1)} + G_i^{(2)} \eta_i^{(1)}) d\Omega = \int_\Gamma T_i^{(1)} k_i^{(2)} d\Gamma + \int_\Omega (F_i^{(1)} \omega_i^{(2)} + G_i^{(1)} \eta_i^{(2)}) d\Omega,$$

which represents the required reciprocity theorem.

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