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The self-similar problem of the unsteady motion of viscous, heat-conducting gas driven by a piston

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THE one-dimensional motion of gas is assumed to be governed by Navier-Stokes equations and induced by a piston moving according to a power-law, with the initial pressure equal to zero. The condition of self-similarity is formulated for such a motion. If this condition is satisfied the problem is reduced to that of numerical solution of a non-linear boundary-value problem for a fifth-order system of ordinary differential equations. The solution is analysed a computation example being given. In addition, a special case of heat-conducting, inviscid gas is considered.

Przyjmuje się, że jednowymiarowy ruch gazu opisany jest równaniami Naviera-Stokesa zgodnie z potęgowym prawem poruszającego się tłoka przy ciśnieniu początkowym równym zeru. Dla takiego ruchu sformułowany jest warunek samopodobieństwa, przy pomocy którego problem został sprowadzony do numerycznego rozwiązania nieliniowego problemu brzegowego, opisanego układem równań różniczkowych zwyczajnych piątego rzędu. Przeprowadzono dyskusję rozwiązania oraz dla jego ilustracji podano przykład liczbowy. Rozważono ponadto szczególny przypadek gazu nielepkiego lecz przewodzącego ciepło.

Принимается, что одномерное движение газа описывается уравнениями Навье-Стокса согласно степенному закону движущегося поршня, при начальном давлении равном нулю. Для такого движения сформулировано условие автомодельности, при помощи которого задача сведена к численному решению нелинейной краевой задачи описанной системой обыкновенных дифференциальных уравнений пятого порядка. Проведено обсуждение решения, а также для его иллюстрации приведен числовой пример. Кроме этого рассмотрен частный случай невязкого, но теплопроводного газа.

THE one-dimensional non-stationary motion of gas driven by a piston is usually considered as a classical example of propagation of wave perturbations. Self-similar problems of this class for adiabatic flow with shock waves were stated first by L. I. SEDOV [1]. Their detailed analysis was made by N. N. KOCHINA and N. S. MELNIKOVA [2, 3] and S. S. GRIGORIAN [4].

Together with the motion of gas compressed by a plane piston may be considered motions under the action of an expanding cylinder or sphere. The solution of the self-similar problem in the case of an expanding sphere was obtained by N. L. KRASHENINNIKOVA [5], who considered some variants of the power-law of expansion, one of the variants being analysed with consideration of the influence of viscosity and heat conduction.

If the density of the unperturbed gas is that of normal atmospheric conditions, the influence of viscosity and heat conduction reduces to the display of a shock wave structure and a thermal boundary layer at the surface of the piston. However, reduced density gives rise to conditions, under which the influence of dissipation extends over the entire region of motion of the gas. As a consequence, complete Navier-Stokes equations are needed for the correct description of that motion.

Below we consider the general case of the self-similar problem of a piston acting on a viscous, heat-conducting gas, using power-laws of motion of the piston and variation of viscosity and any type of symmetry of the motion.

1

Let us consider the one-dimensional non-stationary motion of a viscous heat-conducting gas, characterized by plane, axial or central symmetry. If the gas of arbitrary nature has constant coefficients of specific heat and constant Prandtl number and the viscosity and heat conductivity are related to the temperature by power-laws, the motion of the gas is governed by the Navier-Stokes equations. The form of these equations and the method of their transformation to a dimensionless form corresponding to the self-similar motion is given in Ref. [6].

Let the gas, the initial density of which, ϱ_1, is uniform, move under the action of a plane piston, or an expanding cylinder or sphere; this impermeable moving surface will be referred to in all the cases considered as the piston. Its velocity of displacement will be prescribed by the equation

$$U = ct^m, \quad c, m = \text{const}. \tag{1.1}$$

It will be assumed that there is, at any instant of time, no heat exchange between the gas and the surface of the piston. Then, the initial and boundary conditions for the solution of the piston problem can be written in the form

$$\begin{aligned} &\varrho = \varrho_1, \quad p = \varepsilon = v = 0 \quad \text{for} \quad t = 0 \quad \text{and} \quad t > 0,\ r = r_f, \\ &\varkappa\sigma^{-1}\mu\partial\varepsilon/\partial r = 0,\ v = ct^m \quad \text{for} \quad t > 0,\ r = r_p, \end{aligned} \tag{1.2}$$

where v—velocity, p—pressure, ε—internal energy, σ—Prandtl number, $\varkappa$—ratio of specific heats, μ—viscosity, r_p—coordinate of the piston and r_f—coordinate of the perturbation front. Let us observe that the existence of the perturbation front is connected with the assumption that the temperature of the gas in the unperturbed region is zero (see [8]).

Denoting $\delta = 1+\mathrm{m}$ and following [6] we shall adopt as an argument the quantity

$$\eta = ar/r_p = a\delta r/(ct^\delta) \tag{1.3}$$

and transform the variables, which are sought for, according to the formulae

$$\begin{gathered} \varrho = \varrho_1 R(\eta), \quad v = \frac{U}{a\delta} V(\eta), \\ p = \varrho_1\left(\frac{U}{a\delta}\right)^2 P(\eta), \quad \varepsilon = \frac{1}{\varkappa-1}\left(\frac{U}{a\delta}\right)^2 N(\eta), \\ \mu = \chi\varrho_1 \frac{U}{a\delta} N(\eta), \end{gathered} \tag{1.4}$$

where

$$\chi = \frac{A}{(\varkappa-1)^n\varrho_1}\,\frac{(U/a\delta)^{2n-1}}{r_p/a}. \tag{1.5}$$

If the expression (1.1) for U, and the expression for r_p obtained from it by integration are substituted in (1.5), the condition of independence of the parameter χ on time will represent a condition of self-similarity of the motion, which has, in our case, the form

$$n = 1+(2m)^{-1}. \tag{1.6}$$

The condition (1.6) will be considered, in what follows, to be satisfied, but there are other limitations (see [2]), valid in the limiting case of adiabatic flow and also the condition of non-negative exponent n.

Taking into account all the limitations we can consider the values of n and m whithin two corresponding intervals

$$\frac{\nu-2}{2\nu} \geqslant n \geqslant 0, \quad -\frac{\nu}{\nu+2} \leqslant m \leqslant -\frac{1}{2} \text{ and } m > 0,\ n > 1. \tag{1.7}$$

If the condition of self-similarity (1.6) is satisfied, the Navier-Stokes equations reduce, by virtue of (1.4), to the form

$$\delta\eta R' - VR' - RV' - (\nu-1)RV/\eta = 0,$$

$$R[(\delta-1)V-\delta\eta V'+VV']+R'N+RN' = \frac{4}{3}\chi\left[N^n\left(V'-\frac{\nu-1}{2\eta}V\right)\right]' + 2\frac{\nu-1}{\eta}\chi N^n\left(V'-\frac{1}{\eta}V\right),$$

$$R[2(\delta-1)N-\delta\eta N'+VN']+(\varkappa-1)RN\left(V'+\frac{\nu-1}{\eta}V\right) = \chi\frac{\varkappa}{\sigma}\left[(N^nN')'+\frac{\nu-1}{\eta}N^nN'\right] \tag{1.8}$$

$$+2(\varkappa-1)\chi N^n\left[V'^2+\frac{\nu-1}{\eta^2}V^2-\frac{1}{3}\left(\frac{\nu-1}{\eta}V+V'\right)^2\right].$$

The dimensionless coefficient a in (1.3) can always be selected so that $\eta_f = ar_f/r_p = 1$. Then, instead of (1.2) we obtain the conditions

$$\begin{gathered} R(1) = 1, \quad P(1) = N(1) = V(1) = 0, \\ \varkappa\sigma^{-1}\chi N^n(a)N'(a) = 0, \quad V(a) = a\delta. \end{gathered} \tag{1.9}$$

These conditions, together with the Eqs. (1.8), ensure complete formal interpretation of the self-similar problem of a piston in a viscous heat-conducting gas. The constant dimensionless parameter χ [Eq. (1.5)] vanishes together with the coefficient of viscosity and for a finite Prandtl number σ it may be used as a criterion for the combined influence of viscosity and heat conduction.

2

To solve the boundary-value problem it is necessary to describe in a more accurate manner the behaviour of the integral curves in the neighbourhood of the singular point $\eta = 1$, that is in the neighbourhood of the perturbation front. As was observed in Ref. [6], the form of the expansion in the neighbourhood of that point and the number of terms necessary to obtain a "free" parameter, depends on the physical characteristics of the gas, or, more rigorously speaking, on the power exponent n and the quantity

$$\zeta = \frac{3\varkappa}{4\sigma}. \tag{2.1}$$

Let us consider the case of a piston, the velocity of which varies according to the law (1.1) and $m = 5$. By setting $n = 1.1$ we can ensure the satisfaction of the self-similarity condition (1.6). The case considered is that of uniform initial density ($\varrho_1 = \text{const}$) and, if we set $\zeta = 3/2$ (that is $\varkappa = 2\sigma$), then, according to [6], use can be made, for small values of $z = 1-\eta$, of the following asymptotic representations

$$\begin{gathered} V = A_v z^{10/11} + B_v z^{15/11}, \quad N = 2A_v z^{10/11} + B_N z^{20/11}, \\ R = 1 + (A_v/6) z^{10/11} + (B_v/6) z^{15/11}, \\ A_v = \frac{1}{2}\left(\frac{33}{10\chi}\right)^{10/11}, \quad B_N = -\frac{5}{21}(\varkappa-1)A_v^2. \end{gathered} \tag{2.2}$$

The Eqs. (1.8) will be satisfied in an approximate manner by the Eqs. (2.2) for any value of B_v, so that there is some arbitrariness which is necessary to satisfy the boundary conditions (1.9) at both ends of the interval $a \leqslant \eta \leqslant 1$ considered.

In practice the computation is started from a certain arbitrary value $B_v^{(0)}$, which is introduced into the formulae (2.2), thus enabling us to determine the values of the sought-for variables at the original point $\eta_0 = 1-\varepsilon$ ($\varepsilon \ll 1$). Numerical integration of the Eqs. (1.8) is performed by negative steps, starting from the point $\eta = \eta_0$ and ending at that point $\eta = a^{(0)}$ where the condition $V(a^{(0)}) = a^{(0)}\delta$ is satisfied. Then, a simple algorithm enables us to vary the value of B_v in such a manner, that the second of the conditions (1.9) established at the point $\eta = a$ should be satisfied with a prescribed accuracy. The value of a is determined at the same time.

On the basis of the above method and following the standard Runge-Kutta procedure, computations were performed for a plane piston ($\nu = 1$), for the values $\varkappa = 1.4$, $\sigma = 0.7$, $m = 5$, $n = 1.1$. Let us emphasize once again the fact that the values of m and n are interrelated by the self-similarity condition (1.6) and the value of n selected enables us to ensure satisfactory approximation to the real law of viscosity variation at a certain, sufficiently large temperature interval. It is also worth while observing that the kindred variant of the self-similarity problem of the piston, for $m = 1$, $n = 3/2$, was considered by V. V. SYCHEV and N. C. AVANESOVA [7].

The patterns of variation of velocity, density and temperature in the region between the surface of the piston and the perturbation front are shown in Figs. 1 to 3, respectively. Each of the diagrams shows curves found by numerical integration of the Eqs. (1.10). The figures express the values of the parameter χ. In the case of $\chi = 0$ the computation was carried out using equations obtained from (1.8), but the external boundary of the flow region is not a perturbation front but a shock wave with a relevant modification of the boundary conditions (see [1]). As is seen from the diagrams, the weakening of the influence of viscosity and heat conduction, that is the reduction in χ, is manifested first by a narrower perturbation region. Second, as χ decreases, the profiles of the gas-dynamic parameters approach discontinuous profiles of adiabatic motion. Nevertheless, even for a very small

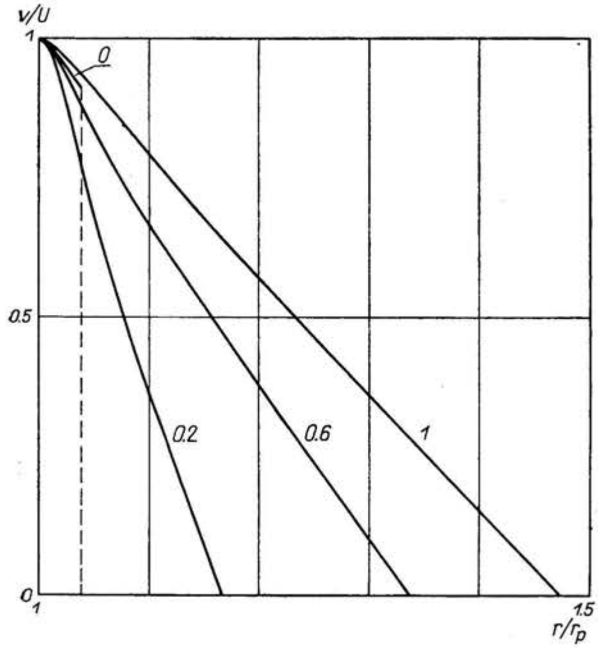

Fig. 1.

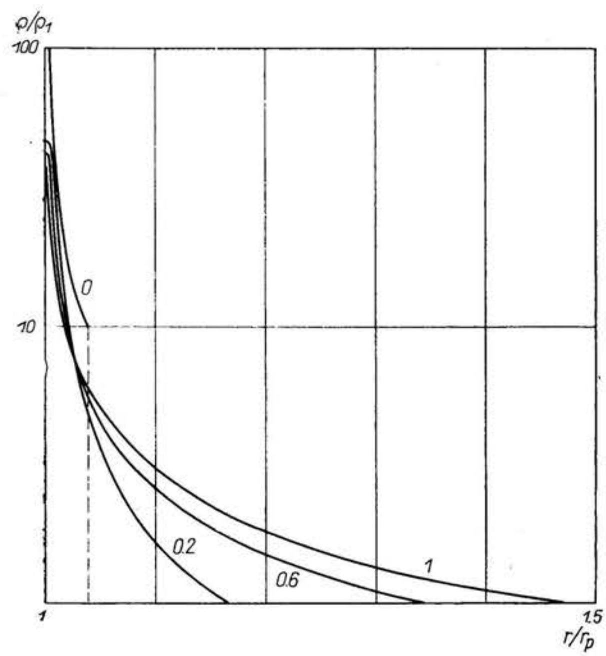

Fig. 2.

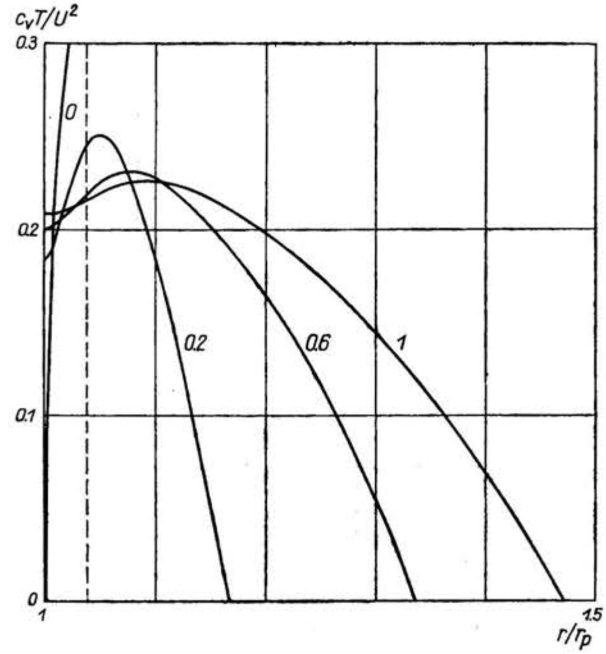

FIG. 3.

value of χ, different from zero, the perturbation front is a surface of weak discontinuity, not strong. In addition, for any $\chi > 0$, the temperature and the density at the surface of the piston are finite and different from zero, whereas for the "adiabatic" case the temperature at the surface is equal to zero and the density is infinite.

3

From the above it follows that at the limit $\chi \to 0$ there is a transition from a continuous motion to a motion with a strong discontinuity (shock wave). The case in which the Prandtl number tends to zero with the parameter χ and

$$\lim_{\chi\to 0,\, \sigma\to 0}\left(\frac{\varkappa\chi}{\sigma}\right) = \chi_T = \text{const} \tag{3.1}$$

is of particular interest. If it is borne in mind that the quantity χ is proportional to the coefficient of viscosity μ and the Prandtl number is equal, according to a definition, to $\sigma = \mu c_p/\lambda$, it is clear that the condition (3.1) is satisfied, if the gas is inviscid but heat-conducting. Then, the parameter χ_T is, of course, in direct proportion to the coefficient of heat conduction of the gas λ.

Assuming that the heat conduction is related with the temperature by a power-law of the same type as hitherto assumed for the viscosity, that is

$$\lambda = c_v A_T \varepsilon^n, \tag{3.2}$$

we can generalize, in a formal manner, many former results to the limiting case considered. In particular, the conditions (1.6) and (1.7) are invariable. The independent variable (1.3) may also remain in the former form. Next, from the (1.8) we obtain, by setting $\chi = 0$, $\varkappa\chi/\sigma = \chi_T$,

$$
\begin{aligned}
&\delta\eta R' - VR' - RV' - (\nu-1)RV/\eta = 0,\\
&R[(\delta-1)V - \delta\eta V' + VV'] + RN' + R'N = 0,\\
&R[2(\delta-1)N - \delta\eta N' + VN'] + (\varkappa-1)RN[V' + (\nu-1)V/\eta]\\
&\qquad = \chi_T[(N^n N')' + (\nu-1)N^n N'/\eta].
\end{aligned} \tag{3.3}
$$

By preserving the condition of zero heat transfer at the surface of the piston, we obtain, instead of (1.9),

$$
\begin{aligned}
&R(1) = 1, \quad P(1) = N(1) = V(1) = 0,\\
&\chi_T N^n(a)N'(a) = 0, \quad V(a) = a\delta.
\end{aligned} \tag{3.4}
$$

Despite the external identity with the Eqs. (1.8), the Eqs. (3.3) with the boundary conditions (3.4) have no continuous solution similar to that described in Sect. 2. This can easily be explained by taking into consideration the fact that the gas considered is inviscid, but can also be confirmed by a purely formal analysis of the behaviour of the solution in the neighbourhood of the point $\eta = 1$ in the case of $\sigma \to 0$, in agreement with the general method [6].

A qualitative study of propagation of perturbations in a heat-conducting gas having zero temperature of the unperturbed volume was given in the book [8]. The result of that study reduces to a statement of necessity of occurrence of a strong discontinuity at a certain point $\eta = \eta_s$, $a < \eta_s < 1$ and due to a heat conduction the temperature of transition through the discontinuity should vary in a continuous manner. More accurate studies of motions with isothermal discontinuities were carried out by I. O. BEZHAEV [9] and V. E. NEUVAZHAEV [10], who gave also examples of computation.

The conditions at the isothermal discontinuity have, in dimensional symbols, the form

$$
\begin{aligned}
&\varrho^+(v^+ - c_s) = \varrho^-(v^- - c_s),\\
&\varrho^+(v^+ - c_s)^2 + (\varkappa-1)\varrho^+\varepsilon = \varrho^-(v^- - c_s)^2 + (\varkappa-1)\varrho^-\varepsilon,\\
&\frac{1}{2}\varrho^+(v^+ - c_s)v^{+2} + (\varkappa-1)\varrho^+v^+\varepsilon - (\lambda/c_v)(\partial\varepsilon/\partial r)^+\\
&\qquad = \frac{1}{2}\varrho^-(v^- - c_s)v^{-2} + (\varkappa-1)\varrho^-v^-\varepsilon - (\lambda/c_v)(\partial\varepsilon/\partial r)^-,
\end{aligned} \tag{3.5}
$$

the symbol c_s denoting the velocity of displacement of the discontinuity surface and the upper indices + and − denote values behind and before the discontinuity, respectively.

By virtue of self-similarity at the surface of discontinuity we have η_s = const and from the formula (1.3) it follows that

$$
r_s = \frac{c}{a\delta}\eta_s t^\delta, \quad c_s = \frac{dr_s}{dt} = \frac{U}{a\delta}\delta\eta_s,
$$

so that the dimensionless equivalent of the velocity c_s is a quantity $\delta\eta_s$. Making use of the result obtained, the relations (3.5) can be written in a dimensionless form. On solving them for the values behind the discontinuity we find

$$R^+ = R^-(V^- - \eta_s\delta)^2/N, \qquad V^+ = \eta_s\delta + N/(V^- - \eta_s\delta),$$

$$(N^+)' = (N^-)' - \frac{\varkappa - 1}{2\chi_T N^n} R^-(V^- - \eta_s\delta)^2\left[1 - \frac{N^2}{(V^- - \eta_s\delta)^4}\right]. \tag{3.6}$$

The introduction of the discontinuity surface at the point $\eta = \eta_s$ enables us to obtain the solution of the Eqs. (3.3) satisfying all the conditions (3.4) for $\eta = 1$ and the condition $V(a) = \delta a$. In addition, the unknown value of η_s is the "free" parameter, by variation of which we can also satisfy the second condition at $\eta = a$, that is we can solve in a complete manner the boundary-value problem which has been stated.

In the case of a plane piston ($\nu = 1$) and for the same values of m, n and $\varkappa$ as in Sect. 2, the computational results are shown in a diagrammatic form in Figs. 4 to 6. The figures marking particular curves express the values of the parameter of heat conduction χ_T. Similarly to the case of combined action of viscosity and heat conduction, the reduced influence of heat condution manifested by a decrease in the parameter η_T, leads to a narrower perturbation region. As regards profiles of gas-dynamic parameters it is of particular interest to study the velocity profile (Fig. 4) and the temperature profile (Fig. 6). It is obvious that if we do not take into account the region before the shock wave, the character of the velocity profile in heat-conducting gas approaches more closely that of the "adiabatic" profile than the previous profile Fig. 1. On the contrary, the temperature profiles of a heat-conducting gas resemble more closely the profiles shown in Fig. 3 taking into considera-

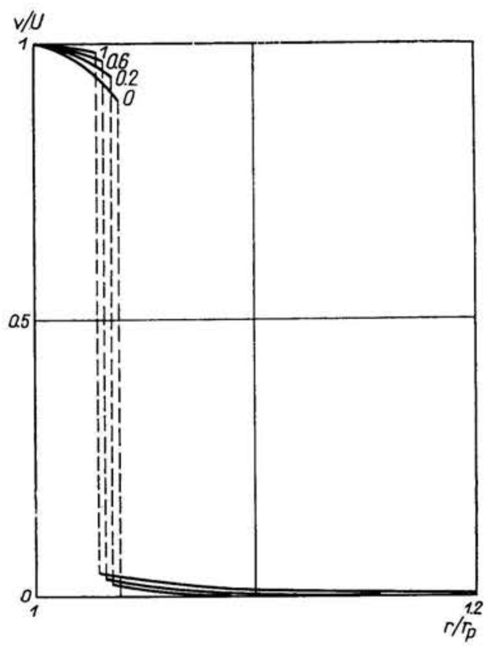

FIG. 4.

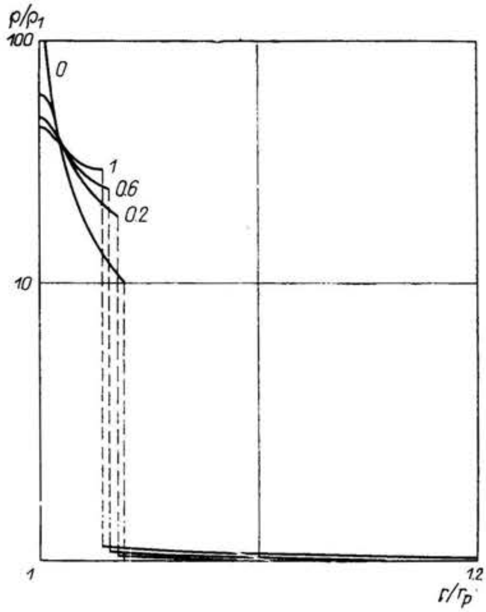

FIG. 5.

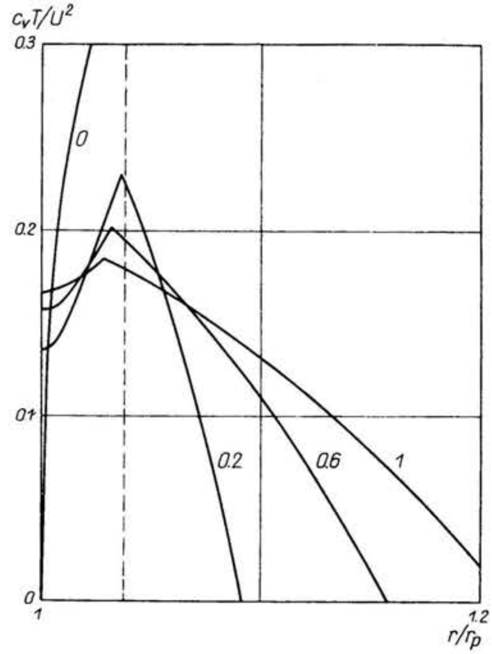

FIG. 6.

tion the simultaneous influence of viscosity and heat conduction. Thus, it may be stated that the variation of velocity of a viscous heat-conducting gas driven by a piston is determined above all by the mechanism of viscosity, while a principal role in the variation of the temperature is played by the mechanism of molecular heat conduction.

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