Archives of Mechanics • Archiwum Mechaniki Stosowanej • 28, 1, pp. 63–74, Warszawa 1976

Similarity analysis for impact of rods of non-linear rate-sensitive strain-hardening materials(*)

R. SESHADRI (EDMONTON) and M. C. SINGH (WARSZAWA)

SIMILARITY solutions are determined for a semi-infinite rate-sensitive strain-hardening rod subjected to a velocity impact. The system of governing non-linear partial differential equations is transformed to a system of ordinary differential equations by means of a similarity transformation. The method of HELLUMS and CHURCHILL is utilized for obtaining a one-dimensional similarity representation. The method of collocation is used to obtain approximate solutions of the resulting system of ordinary differential equations and their auxiliary conditions. For special forms of the non-linear constitutive relationship, closed form solutions are obtained.

Określono rozwiązania podobieństwa dla półnieskończonego pręta poddanego uderzeniu prędkości. Pręt wykonany jest z materiału wrażliwego na prędkość odkształcenia ze wzmocnieniem. Układ podstawowych nieliniowych równań różniczkowych cząstkowych został przekształcony do układu równań różniczkowych zwyczajnych za pomocą transformacji podobieństwa. W celu uzyskania bezwymiarowej reprezentacji podobieństwa wykorzystano metodę HELLUMSA i CHURCHILLA. Rozwiązanie przybliżone wyprowadzonego układu równań różniczkowych zwyczajnych z warunkami pomocniczymi otrzymano metodą kollokacji. Dla szczególnych postaci nieliniowego związku konstytutywnego uzyskano rozwiązanie w postaci zamkniętej.

Определены решения подобия для полубесконечного стержня, подвергнутого действию удара скоростью. Стержень изготовлен из материала, чувствительного на скорость деформации с упрочнением. Система основных нелинейных дифференциальных уравнений в частных производных преобразована в систему обыкновенных дифференциальных уравнений при помощи преобразования подобия. С целью получения безразмерного представления подобия использован метод Хеллумса и Черчилля. Приближенное решение выведенной системы обыкновенных дифференциальных уравнений с вспомогательными условиями получено методом прогонки. Для частных видов нелинейного определяющего соотношения получены решения в замкнутом виде.

Notations

- x coordinate along the axis of the rod (this is a Lagrangian coordinate system, where x denotes the position of the particle in the initial unstrained state),
- t time,
- $\sigma(x, t)$ nominal compressive stress (force transmitted across a cross-section of the rod divided by initial cross-sectional area), compressive stress is assumed to be positive,
- $e(x, t)$ nominal compressive strain (change in length divided by the initial length of an element parallel to x axis) compressive strain is assumed to be positive,
- $v(x, t)$ particle velocity,
- ϱ mass density of the material in initial unstrained state,
- k, p, q material constants for a limited range of stress at constant temperature in an equation of state describing stationary creep phenomena,

(*) This paper is based on a part of the first author's Ph. D. thesis submitted to the Department of Mechanical Engineering, The University of Calgary, Calgary, Alberta, Canada 1973.

α exponent of time in the velocity end condition,
ξ similarity variable, non-dimensional,
X^i generalized variables,
P^1 generalized parameters,
a_r parameters of the group of transformation,
X_0^i unspecified (reference) generalized variables,
$\pi_\varkappa$ functionally dependent dimensionless products,
π_j functionally independent dimensionless products,
λ_{kl} matrix of exponents of the functionally independent dimensionless products,
δ rank of matrix of exponents (λ_{kl}),
μ total number of generalized variables (x_l).

1. Introduction

IN the recent past the problem of impact of non-linear rate-sensitive rods has been a source of keen interest among the research workers in the area of wave propagation. An effective technique used for the analysis of non-linear partial differential equations arising in such problems has been that of similarity analysis [1] which essentially deals with the reduction of the number of variables in a system of partial differential equations and their associated auxiliary conditions.

TAULBEE, COZZARELLI and DYM [2] used separation of variable technique [3] to determine the similarity variables and obtained some closed form solutions for the impact of non-linear elastic and non-linear viscous rods. SINGH and SESHADRI [4] used a group theoretic procedure [5] and obtained similarity solutions for the problem of a semi-infinite non-linear viscoplastic rod. BURNISTON and CHANG [6] considered the propagation of non-linear waves in rate-sensitive, elastoplastic material.

In this paper, similarity solutions are obtained for non-linear rate-sensitive strain-hardening rods subjected to velocity impact. The constitutive equation considered describes the stationary creep behaviour of materials for a certain range of stress. For special types of non-linearity, closed form solutions are obtained, however, for the general problem, approximate solutions are obtained using the method of collocation where the errors are minimized using maximum norm criteria [7].

For specific initial and boundary value problems in engineering, it is desirable to employ a procedure for determining a similarity representation which takes into account the auxiliary conditions at the outset of the analysis. HELLUMS and CHURCHILL technique [5] is such a procedure, which can be used to obtain either a normalized representation or a similarity representation for a given physical problem. This procedure is essentially an extension of Birkhoff's method of search for symmetric solutions [8] and is based on group-theoretic concepts. The routine selection of mass, length and time as fundamental dimensions is implied in the procedure.

MORAN [8, 9] sought a generalization of the technique by making use of a multiparameter group of dimensional transformations. In contrast to Moran's development in which reference is made to a very general representation of the problem, the theory discussed herein is applied to a representation involving a given set of partial differential equations and associated conditions arising from the physical considerations. More concretely, a similarity representation is determined for the problem of impact of rate-sensitive/strain-hardening rods that are subjected to a power law time variation in stress. The resulting system of ordinary differential equations is solved by collocation where the equations-residuals are minimized by making use of maximum norm criteria. For special values of the exponents the results obtained by collocation are compared with numerical solutions. Despite the nature of approximation in the construction of the trial functions, there is a good agreement in the results.

2. Group theoretic analysis of Hellums and Churchill procedure

Consider the following system of partial differential equations and conditions describing a certain class of physical problems with m independent variables and n dependent variables:

$$\phi_\gamma\left(x^1, \ldots, x^m; y_1, \ldots, y_n; \frac{\partial y_1}{\partial x^1}, \ldots, \frac{\partial^k y_n}{\partial (x^m)^k}\right) = 0, \quad \gamma = 1, \ldots, N \tag{2.1}$$

subjected to auxiliary conditions [7]:

$$\beta_u\left(\frac{\partial^s y_1}{\partial (x^1)^s}, \ldots, \frac{\partial y_n}{\partial x^m}, y_1, \ldots, y_n, x^1, \ldots, x^m\right) = B_u(\sigma^1, \ldots, \sigma^t), \tag{2.2}$$

on $\Sigma_u: \{x^i = b_u^i(\sigma^1, \ldots, \sigma^t)\}$, $t \leqslant m$, for σ^q in the region of validity, where $\sigma^q (q = 1, \ldots, t)$ locate the boundaries. Replacing $x^1, \ldots, x^m$; $y_1, \ldots, y_n$ by $X^1, \ldots, X^\mu$ respectively ($\mu = m+n$) and noting the fact that, in addition to variables, there can be physical parameters entering the problem, a generalization of Hellums-Churchill procedure is obtained by using a multiparameter group of dimensional transformations (Γ_D) pertaining to the form used in the theory of generalized dimensional analysis, i.e.,

$$\Gamma_D: \begin{bmatrix} \bar{X}^i = a_1^{\beta_{i1}} \ldots a_r^{\beta_{ir}} X^i \\ \bar{P}^l = a_1^{c_{l1}} \ldots a_r^{c_{lr}} P^l \end{bmatrix}, \quad \begin{matrix} i = 1, \ldots, \mu \\ l = 1, \ldots, p \end{matrix} \tag{2.3}$$

where X^i are the variables, P^l are the parameters, $\{a_1, \ldots, a_r\}$ are the parameters of the transformations. As the dimensional group of transformations (Γ_D) is not general, not every similarity representation can be deduced, i.e., the similarity transformation is obtained using an *assumed class of transformations*.

The dimensional matrix for Γ_D can be written in the following form:

$$\begin{array}{c|c} & X^1 \ldots X^\mu, \; P^1 \ldots P^p \\ \hline a_1 & \beta_{11} \ldots \beta_{\mu 1}, c_{11}, \ldots, c_{p1} \\ a_2 & \beta_{12} \ldots \beta_{\mu 2}, c_{12}, \ldots, c_{p2} \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_r & \beta_{1r} \ldots \beta_{\mu r}, c_{1r}, \ldots, c_{pr} \end{array}$$

Let the rank of the dimensional matrix be r and the rank of matrix $[c_{la}]$ $(l = 1, \ldots, p;$ $a = 1, \ldots, r)$ be s, where $s \leqslant r$.

Defining a set of non-dimensional variables, we have

$$\tilde{X}^i = \frac{X^i}{X_0^i}, \tag{2.4}$$

X_0^i are initially unspecified quantities corresponding to X^i and are introduced with the stipulation that $X_0^i > 0$. By definition, X_0^i has the same dimensions as X^i, so that

$$\bar{X}_0^i = a_1^{\beta_{i1}} \ldots a_r^{\beta_{ir}} X_0^i. \tag{2.5}$$

The X_0^i will be determined subsequently in order to produce a description of the problem in terms of *minimum possible number of parameters*.

On introducing the Eq. (2.4) into the Eqs. (2.1) and (2.2), a non-dimensional representation for the system of differential equations and conditions can be written in terms of the non-dimensional variables $\tilde{X}^i$ and the set C of non-dimensional products of the form

$$\pi = [\{(X_0^1)^{\lambda_1} \ldots (X_0^\mu)^{\lambda\mu}\}\{(P^1)^{\delta_1} \ldots (P^p)^{\delta_p}\}]. \tag{2.6}$$

Since the π are absolutely invariant, the λ's and δ's satisfy

$$\sum_{i=1}^{\mu} \lambda_i \beta_{i\alpha} + \sum_{l=1}^{p} \delta_l c_{l\alpha} = 0, \quad \alpha = 1, 2, \ldots, r. \tag{2.7}$$

When Hellums-Churchill procedure is applied to differential equations and auxiliary conditions arising out of physical requirements of a continuum theory, the equations and conditions are dimensionally homogeneous [3], and as such imply the existence of a one-dimensional representation.

By setting $\lambda_i = 0$ in the Eq. (2.6), a subset C_1 of non-dimensional products purely in terms of the parameters can be obtained as

$$\pi = [(P^1)^{\delta_1} \ldots (P^p)^{\delta_p}]. \tag{2.8}$$

The other subset C_2 consists of all the remaining products of C. Suppose that there are ξ such members denoted by

$$\pi_k = [\{X_0^1)^{\lambda_{k1}} \ldots (X_0^\mu)^{\lambda_{k\mu}}\}\{(P^1)^{\delta_{k1}} \ldots (P^p)^{\delta_{kp}}\}], \quad k = 1, \ldots, \xi. \tag{2.9}$$

Let the rank of the matrix of exponents $[\lambda_{ki}]$ be equal to $\hat{\sigma}$ $(\hat{\sigma} \leqslant \mu)$. Thus, there is a subset $C_2^{\hat{\sigma}}$ which contains $\hat{\sigma}$ functionally independent rows of the exponents of the Eq. (2.9), so that

$$\pi_j = [(\{(X_0^1)^{\lambda_{j1}} \ldots (X^{\lambda_{j\mu}})\}\{(P^1)^{\delta_{j1}} \ldots (P^p)^{\delta_{jp}}\}]; \quad j = 1, \ldots, \hat{\sigma}. \tag{2.10}$$

In view of the functional independence of π_j,

$$\lambda_{ki} = \sum_{j=1}^{\hat{\sigma}} A_{kj} \lambda_{ji}, \quad k = 1, \ldots, \xi; \; i = 1, \ldots, \mu, \tag{2.11}$$

where A_{kj} are real numbers.

On substituting the Eq. (2.11) into the Eq. (2.9) and rearranging, we have

$$(2.12)_1 \qquad \pi_k = [\{(\pi_1)^{A_{k1}} \ldots (\pi_{\hat{\sigma}})^{A_{k\hat{\sigma}}}\}(\pi^k)].$$

where

$$(2.12)_2 \qquad \pi^k = \{(P^1)^{\Delta_{k1}} \ldots (P^p)^{\Delta_{kp}}\}$$

and

$$(2.12)_3 \qquad \Delta_{kl} = \delta_{kl} - \sum_{j=1}^{\hat{\sigma}} A_{kj}\delta_{jl}.$$

Thus, the products π_k can be expressed in terms of $\hat{\sigma}$ functionally independent members of $\hat{C}_2$, π_j; $j = 1, \ldots, \hat{\sigma}$ and a non-dimensional parameter-product π^k $(k = 1, \ldots, \xi)$. π^k is appended to the subset C_1 and subsequently a functionally independent set $\tilde{C}_1$ is obtained.

A representation for the system of equations and conditions can now be obtained in terms of

(i) the non-dimensional variables $\tilde{X}^i$ $(i = 1, \ldots, \mu)$,

(ii) the non-dimensional functionally independent products π_j $(j = 1, \ldots, \hat{\sigma})$,

and

(iii) the functionally independent set of parameter products $\tilde{C}_1$.

The functionally independent products π_j are determined before seeking a minimum description for the problem. In practice this can be done by choosing $\hat{\sigma}$ members out of the π_k $(k = 1, \ldots, \xi)$ that correspond to σ linearly independent rows of the matrix $[\lambda_{kl}]$. This is a direct consequence of the determination of the rank $\hat{\sigma}$ of this matrix. The remaining $\pi_k(k = \hat{\sigma}+1, \ldots, \xi)$ can be expressed as a function of the π_j through the Eqs. (2.12).

For a minimum description of the problem,

$$(2.13) \qquad \pi_j = 1, \quad j = 1, \ldots, \hat{\sigma}.$$

As a consequence, the representation is simplified considerably. Depending on whether $\hat{\sigma} \leqslant \mu$, two distinct possibilities arise:

Case 1. *Normalized representation*

In this case, $\hat{\sigma} = \mu$ and the reference quantities can be fixed in terms of the system parameters, so that Eq. (2.13) is satisfied.

Case 2. *Similarity representation*

When $\hat{\sigma} < \mu$, a normalized representation cannot evolve. Designating

$$(2.14) \qquad \varrho = \mu - \hat{\sigma}$$

it is seen that an invariant representation evolves. In practice, this is obtained by suitably eliminating the reference variables that remain after setting $\pi_j = 1$. With a larger number of reference variables remaining arbitrary, the process of their elimination becomes manipulative and increasingly difficult. Thus, for $\hat{\sigma} < \mu$, the system of equations and conditions is invariant under a ϱ-parameter group of transformations.

3. Basic equations for the velocity impact of rate-sensitive strain-hardening rods

The governing equations of motion for small deformations, within the framework of uniaxial theory of thin rods are

$$\text{(3.1)} \qquad \frac{\partial \sigma}{\partial x} = -\varrho \frac{\partial v}{\partial t}, \quad \frac{\partial e}{\partial t} = -\frac{\partial v}{\partial x}, \quad \frac{\partial e}{\partial t} = k\sigma^p e^q,$$

where x is a Lagrangian coordinate and σ and e are nominal compressive stress and nominal compressive strain, respectively.

The Eq. $(3.1)_3$ is a one-dimensional constitutive relationship which expresses the dependence of strain rate on stress and strain. k, p and q are material constants that are valid for a limited range of stress at constant temperature [10]. For structural steel, $k = 1/531.000$; $p = 10$, $q = -2$. The relationship $(3.1)_3$ describes the stationary creep behaviour of materials. The exponent q is negative, zero or positive according as creep is primary, secondary or tertiary, respectively.

The system of equations (3.1) is quasi-linear, parabolic and coupled, however, for $q = 0$ the system uncouples.

The auxiliary conditions are

$$\text{(3.2)}_1 \qquad v(0, t) = v_c t^\alpha, \quad t > 0,$$

$$v_c > 0; \ \alpha \text{ is a parameter.}$$

When $\alpha = 0$, we have the familiar constant velocity impact. When α is allowed to take on negative values, the physically interesting case of an applied volocity which is infinitely large at $t = 0$, is accounted for. Further,

$$\text{(3.2)}_2 \qquad v(x, 0) = \sigma(x, 0) = e(x, 0) = 0, \quad x \geqslant 0.$$

Based on physical considerations, stress, strain and the particle velocity tend to zero as x goes to infinity, i.e.

$$\text{(3.2)}_3 \qquad \sigma(x \to \infty, t) = e(x \to \infty, t) = v(x \to \infty, t) = 0.$$

In the transformed system, the Eqs. $(3.2)_2$ and $(3.2)_3$ coalesce into one set of conditions in the similarity coordinate.

4. Similarity analysis

The variables appearing in the description of the problem are rendered non-dimensional by introducing arbitrary reference quantities as follows:

$$\text{(4.1)} \qquad \bar{v} = \frac{v}{v_B}, \quad \bar{\sigma} = \frac{\sigma}{\sigma_0}, \quad \bar{e} = \frac{e}{e_0}, \quad \bar{x} = \frac{x}{x_0} \text{ and } \bar{t} = \frac{t}{t_0},$$

where v_B, σ_0, e_0, x_0 and t_0 are the reference quantities with the same dimensions as their respective variables.

Using the Eq. (4.1), the non-dimensional products that factoredout of the Eq. (3.1) and conditions (3.2) can now be written as

$$\pi_1^e = \frac{\varrho v_B x_0}{\sigma_0 t_0}, \quad \pi_2^e = \frac{v_B t_0}{e_0 x_0}, \quad \pi_3^e = k\sigma_0^p e_0^{q-1} t_0, \tag{4.2}$$
$$\pi_1^b = \frac{v_c t_0^\alpha}{v_B},$$

where the superscripts e and b refer to the products factored out from the equations and the boundary conditions, respectively. The rank, $\hat{\sigma}$, of the exponents of these products is equal to 4. Thus, all of the products are functionally independent.

The functional form of the solution can now be written as

$$\bar{v} = \frac{v}{v_B} = f(\bar{x}, \bar{t}; \pi_1^e, \pi_2^e, \pi_3^e; \pi_1^b),$$
$$\bar{\sigma} = \frac{\sigma}{\sigma_B} = g(\bar{x}, \bar{t}; \pi_1^e, \pi_2^e, \pi_3^e; \pi_1^b), \tag{4.3}$$
$$\bar{e} = \frac{e}{e_0} = h(\bar{x}, \bar{t}; \pi_1^e, \pi_2^e, \pi_3^e; \pi_1^b).$$

The products $\pi_1^e, \pi_2^e, \pi_3^e$, and π_1^b contain arbitrary reference quantities which need to be specified. One way of doing this is to set the non-dimensional products equal to unity, i.e.

$$\pi_1^e = \pi_2^e = \pi_3^e = \pi_1^b = 1. \tag{4.4}$$

Thus, the representation is considerably simplified. Since the total number of variables, μ, is 5 and the rank, $\hat{\sigma}$, is equal to 4, the number of reference quantities remaining arbitrary is equal to one. Accordingly,

$$v_B = v_c t_0^\alpha,$$
$$\sigma_0 = \varrho^{\frac{q-1}{q-p-1}} v_c^{\frac{2(q-1)}{q-p-1}} k^{\frac{1}{q-p-1}} t_0^{\frac{2\alpha(q-1)+1}{q-p-1}}, \tag{4.5}$$
$$e_0 = \varrho^{\frac{p}{p-q+1}} v_c^{\frac{2p}{p-q+1}} k^{\frac{1}{p-q+1}} t_0^{\frac{2p\alpha+1}{p-q+1}},$$
$$x_0 = \varrho^{\frac{p}{q-p-1}} v_c^{\frac{p+q-1}{q-p-1}} k^{\frac{1}{q-p-1}} t_0^{\frac{p-q-\alpha(p+q-1)}{p-q+1}}$$

Since the arbitrary reference quantity t_0 does not occur in the original description of the problem, it can be eliminated by suitably combining the remaining arguments in the Eq. (4.5).

Thus, the similarity representation can be written as

$$v = v_c t^\alpha f(\xi),$$
$$\sigma = \varrho^{\frac{q-1}{q-p-1}} v_c^{\frac{2(q-1)}{q-p-1}} k^{\frac{1}{q-p-1}} t^{\frac{2\alpha(q-1)+1}{q-p-1}} g(\xi), \tag{4.6$_{1-3}$}$$
$$e = \varrho^{\frac{p}{p-q+1}} v_c^{\frac{2p}{p-q+1}} k^{\frac{1}{p-q+1}} t^{\frac{2p\alpha+1}{p-q+1}} h(\xi),$$

where

$$\xi = \frac{x}{k^{\frac{1}{q-p-1}} \varrho^{\frac{p}{q-p-1}} v_c^{\frac{p+q-1}{q-p-1}} t^{\frac{p-q-\alpha(p+q-1)}{p-q+1}}} \tag{4.6$_4$}$$

is the similarity variable and f, g and h are unknown functions of the similarity variable ξ. These unknown functions are to be determined by solving the resulting similarity representation.

Substituting the Eqs. (4.6) into the system of governing equations (3.1) a system of ordinary differential equations can be obtained in the following form:

$$\frac{dg}{d\xi} = -\alpha f + \gamma\xi \frac{df}{d\xi}, \quad \beta h - \gamma\xi \frac{dh}{d\xi} = -\frac{df}{d\xi}, \quad \beta h - \gamma\xi\frac{dh}{d\xi} = g^p h^q, \tag{4.7}$$

where

$$\beta = \frac{2\alpha p+1}{p-q+1}, \quad \text{and} \quad \gamma = \frac{p-q-\alpha(p+q-1)}{p-q+1}.$$

The transformed set of auxiliary conditions are:

$$f(0) = 1, \quad f(\xi \to \infty) = 0. \tag{4.8$_1$}$$

Evaluating g and h at $\xi = 0$ by making use of the Eq. (4.7) we get

$$h(0) = \frac{p-q+1}{2\alpha p+1}\left(-\frac{df}{d\xi}\right)_{\xi=0}, \tag{4.8$_2$}$$

and

$$g(0) = \left(\frac{2\alpha p+1}{p-q+1}\right)^{1/p} (h(0))^{\frac{1-q}{p}}. \tag{4.8$_3$}$$

Moreover,

$$g(\xi \to \infty) = 0 \tag{4.8$_4$}$$

and

$$h(\xi \to \infty) = 0. \tag{4.8$_5$}$$

5. Solution of the similarity representation

(i) When $p = 1$, $q = 0$, $\alpha \neq 0$, the system of the Eqs. (4.7) uncouples and can be written as

$$\frac{d^2 f}{d\xi^2} + \frac{1}{2}\xi\frac{df}{d\xi} - \alpha f = 0. \tag{5.1}$$

The solution of the Eq. (5.1) satisfying the boundary conditions $(4.8)_1$ can be written as [2]

$$v(x, t) = v_c t^\alpha \frac{2^{\alpha+1/2}}{\sqrt{\pi}} \Gamma(\alpha+1) e^{-\xi^2/8}\, U\left(2\alpha + \frac{1}{2}, \frac{\xi}{\sqrt{2}}\right), \tag{5.2}$$

where $U(2\alpha+1/h, \xi/\sqrt{2})$ is the parabolic cylinder function. For values $\alpha = -1/2, 0, 1/2, 1, \ldots$ etc., the solution is

$$v(x, t) = v_c t^\alpha 2^{2\alpha}\Gamma(\alpha+1) i^{2\alpha} \text{ erfc } (\xi/2), \tag{5.3}$$

where $i^{2\alpha}$ erfc $(\xi/2)$ is the repeated integral of the error function.

(ii) When $p \neq 0$, $q = 0$, $\alpha = 0$; the system of the Eqs. (4.7) can be written as

$$\frac{df}{d\xi} = -g^p, \quad \frac{dg}{d\xi} = \left(\frac{p}{p+1}\right)\xi\frac{df}{d\xi}. \tag{5.4$_{1,2}$}$$

Integrating the Eqs. (5.4), the following results [4] can be obtained on using the conditions $(4.8)_1$

$$v(x, t) = v_c\left\{1 - \int_0^\xi \frac{d\gamma}{(C+\beta\gamma^2)^{p/p(-1)}}\right\},$$

(5.5)$_{1,2}$

$$\sigma(x, t) = \frac{(\varrho v_c^2)^{1/(p+1)}}{(kt)^{1/(p+1)}}\left\{\frac{1}{(C+\beta\xi^2)^{1/(p-1)}}\right\},$$

where

$$\xi = \left(\frac{k\varrho p}{v_c^{1-p}}\right)^{1/(p+1)} \frac{x}{t^{p/(p+1)}},$$

(5.5)$_{3-5}$

$$\beta = \frac{p(p-1)}{2(p+1)},$$

$$C = \left\{\frac{4\beta}{\pi}\,\frac{\Gamma(p/p-1)}{\Gamma((p/p-1)-1/2)}\right\}^{\frac{1}{1-2p/(p-1)}}.$$

(iii) For the case when $p \neq 0$, $q \neq 0$, $\alpha = 0$, the similarity representation is solved using the method of collocation where the equation-residuals are minimized based on a maximum norm criterion.

The following trial functions which satisfy the boundary conditions are chosen:

$$f(\xi) = e^{-\xi} + c_1(e^{-\xi} - e^{-3\xi}),$$

(5.6)

$$g(\xi) = (p-q+1)^{-q/p}(1-2c_1)\{e^{-\xi^2} + c_2(e^{-\xi^2} - e^{-2\xi^2})\},$$

$$h(\xi) = (p-q+1)(1-2c_1)\{c^{-2\xi} + c_3(e^{-2\xi} - e^{-3\xi})\},$$

c_1, c_2, and c_3 are unspecified coefficients to be determined based on a criterion which would minimize the equation-residuals R_j $(j = 1, 2, 3)$ which are obtained by substituting the trial functions (5.6) into the Eqs. (4.7). Collocation is performed, so that

$$R_j(c_1, c_2, c_3, \xi_j) = 0, \quad j = 1, 2, 3, \tag{5.7}$$

where the ξ_j are chosen such that

$$\text{Max}\,|Rj| = \text{Minimum over } 0 < \xi < \infty . \tag{5.8}$$

The non-linear algebraic system of the Eqs. (5.7) is solved by generalized Newton-Raphson procedure and the matrix inversion performed by Gauss elimination method.

In Fig. 1 the result obtained for $p = 10$ and $q = 0$ by two approaches, those of collocation technique and Runge-Kutta routine are shown for comparison. In Figs. 2 to 5, the results are shown for different values of p and q as obtained by the collocation technique.

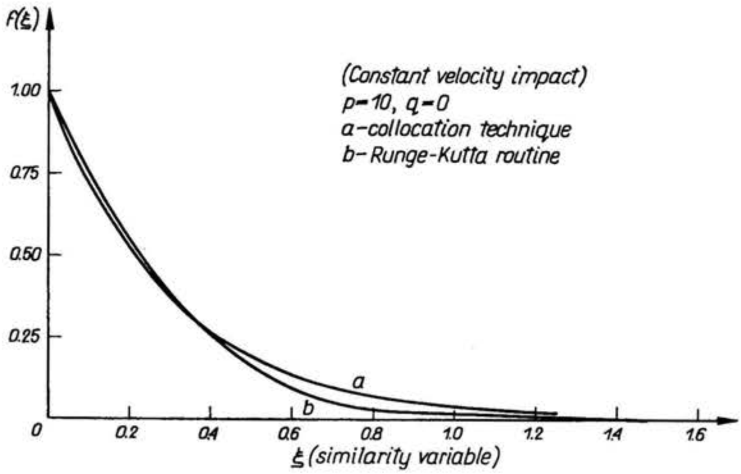

FIG. 1. $f(\xi)\,vs\cdot\xi$; a comparison of the profiles obtained by collocation and Runge-Kutta technique; $p = 10$, $q = 0$; constant velocity impact.

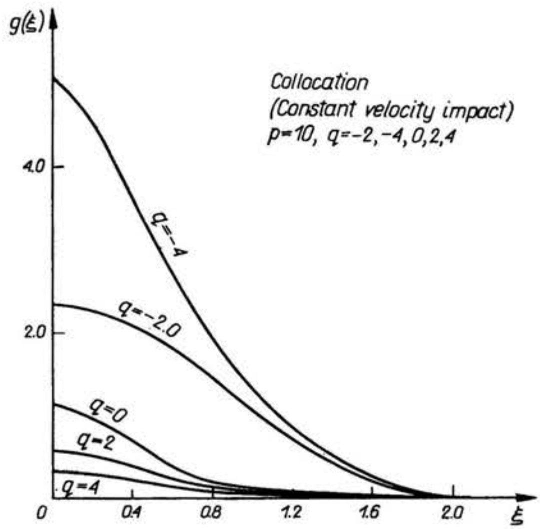

FIG. 2. Plot of $g(\xi)\,vs.\,\xi$; $p = 10$, $q = 4, -2, 0, 2, 4$; constant velocity impact.

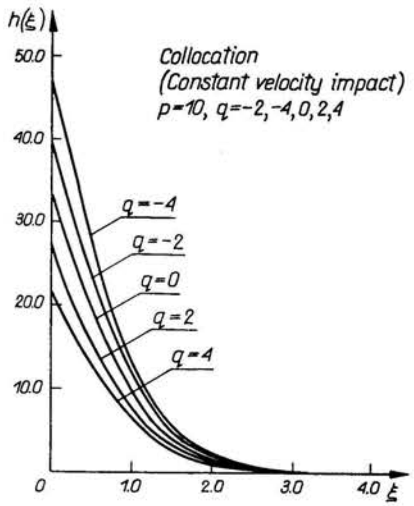

FIG. 3. Plot of $h(\xi)$ *vs.* ξ; $p = 10$, $q = -4, -2, 0, 2, 4$, constant velocity impact.

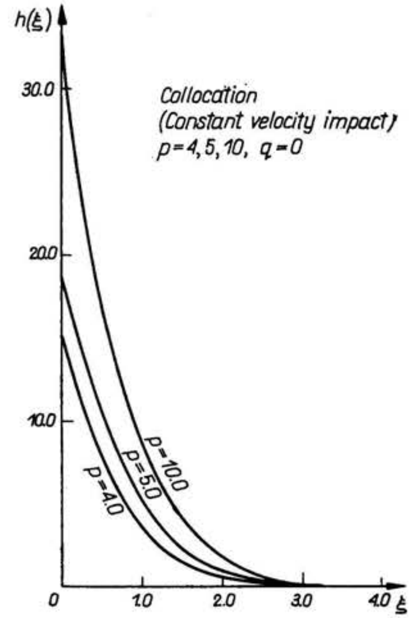

FIG. 4. Plot of $h(\xi)$ *vs.* ξ; $q = 0$; $p = 4, 5, 10$, constant velocity impact.

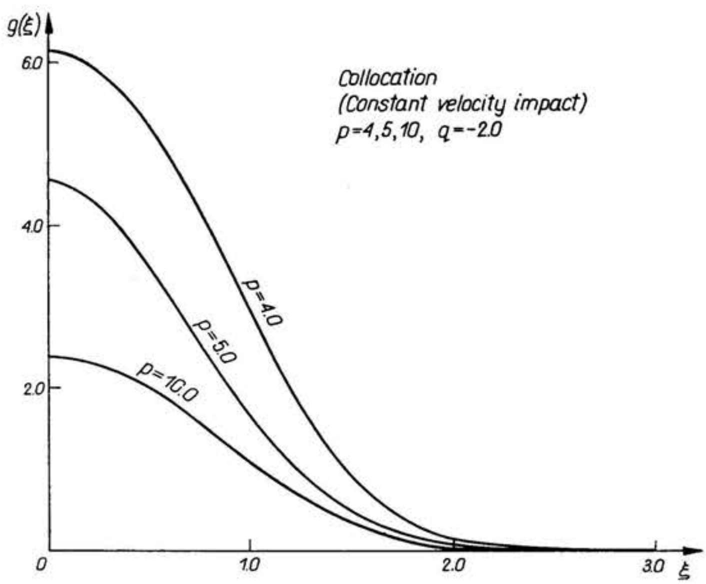

FIG. 5. Plot of $g(\xi)$ *vs.* ξ; $q = -2$, $p = 4, 5, 10$, constant velocity impact.

6. Discussion of the results

For $p = 1$, $q = 0$, $\alpha \neq 0$, that is, linear dependence of strain rate on stress, the closed form solution is the same as that obtained by TAULBEE *et al* [2]. For $p \neq 0$, $q = 0$, $\alpha = 0$, non-linear dependence of strain rate on stress, and constant velocity impact at the end $x = 0$, the solution compares with that obtained in [4].

The general problem where $p \neq 0$, $q \neq 0$, $\alpha = 0$ corresponding to the dependence of stress on strain rate as well as strain-hardening effects, the method of collocation gives good results in spite of a one term approximation for the trial function.

References

1. A. G. HANSEN, *Similarity analysis of boundary value problems in engineering*, Prentice Hall 1964.
2. D. B. TAULBEE, F. A. COZZARELLI, and C. L. DYM, *Similarity solutions to some non-linear impact problems*, Int. J. Non-linear Mech., **6**, 1971.
3. S. J. KLINE, *Similitude and approximation theory*, McGraw Hill Book Co., 1965.
4. M. C. SINGH, and R. SESHADRI, *Similarity solution for the problem of an impact on a non-linear viscoplastic rod*, Int. J. Mech. Sci., **13**, 1971.
5. J. D. HELLUMS, and S. W. CHURCHILL, *Simplification of mathematical description of boundary and initial value problems*, A. I. Ch. E. Journal.
6. E. E. BURNISTON and T. S. CHANG, *Non-linear waves in a rate-sensitive elastoplastic material*, Int. J. Engg., Sci., **10**, 1972.
7. W. F. AMES, *Non-linear partial differential equations in engineering*, Academic Press, New York 1965.
8. M. J. MORAN, *A unification of dimensional analysis and similarity analysis via group theory*, Ph. D. Dissertation, Univ. of Wisconsin, Madison, Wisconsin 1967.
9. M. J. MORAN, and R. A. GAGGIOLI, A.I.A.A.J., 6, 2014, 1968.
10. J. P. LUBAHN, and R. P. FELGAR, *Plasticity and creep of metals*, John Wiley and Sons, Inc., 1961.
11. R. SESHADRI and M. C. SINGH, *Group theoretic method of similarity analysis applied to non-linear impact problems*, Proc. of Symposium "Symmetry, Similarity and Group Theoretic Methods in Mechanics", Calgary, Alberta, August 1974.

EBA ENGINEERING CONSULTANTS
EDMONTON, ALBERTA, CANADA
and
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH,
POLISH ACADEMY OF SCIENCES, WARSZAWA.

Received January 25, 1975.