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J. J. Moreau's approach to a fundamental problem: the quasi-static evolution of a perfectly elasto-plastic body(*)

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THE paper concerns the foundations of perfectly elastic-plastic bodies based on functional analysis. This functional analysis approach is used to formulate the basic system of equations describing a typical boundary value problem for infinitesimal strains and associated flow rules. Certain existence and uniqueness theorems are proved and discussed.

Praca dotyczy podstaw sprężysto-idealnie plastycznych ciał w terminach analizy funkcjonalnej. Metody analizy funkcjonalnej wykorzystano do wyprowadzenia podstawowego układu równań opisujących typowy problem brzegowy przy założeniu nieskończenie małych odkształceń i stowarzyszonych praw płynięcia. Udowodniono i przedyskutowano pewne twierdzenia o istnieniu i jednoznaczności typowego problemu.

Работа касается основ упруго-идеально пластических тел в терминах функционального анализа. Методы функционального анализа использованы для вывода основой системы уравнений, описывающих типичную краевую задачу, при предположении бесконечно малых деформаций и ассоциированных законов течения. Доказана и обсуждена некоторая теорема существования и единственности типичной задачи.

VERY important progress has been made in the last five years, and they are mainly due to French research workers G. DUVAUT and J. L. LIONS on one hand and J. J. MOREAU on the other. DUVAUT and LIONS paid attention to the boundary value problem for partial derivative equations and much is to be learnt from their attempt to build a suitable functional framework; actually their work seems to fit better the theme that was given to me. On the other hand J. J. MOREAU sets an abstract problem that avoids some of the difficulties of functional analysis; in fact, and at least up to now, Moreau has been more concerned with the algebraic structure of the actual problem, than the functional frame would enable the boundary value problem to be completely solved. A lot of fundamental questions may be dealt with by means of this approach such as shake-down theorems, convergence of some algorithms etc., and the existence of complete solutions in the case of the finite degree of freedom. Overall Moreau's algebraic construction furnishes a fine reference framework which should lead mechanists to a much better understanding of plasticity. This is the reason why I chose to talk about it, thinking it would be very useful for people concerned with plasticity.

Let me now recall what the situation was up to 1970. The famous paper of W. T. KOITER ([6], 1960) still summed up the theoretical knowledge.

Many things were understood but few of them were actually proved: an easy uniqueness theorem for the stress-field and an upper bound for plastic work in shake-down theory. Variational principles were only half-established: it was proved that any possible solution of the given problem would minimize such or such a functional, but generally the reciprocal statements were not established. MELAN's theorem was nothing but a reasonable conjecture, the so-called "proof" of which, often repeated in literature, had no value (cf. O. DEBORDES et B. NAYROLES [3], 1975). Even the theory of limit loads needed to be established on a better mathematical basis which I gave (B. NAYROLES [17], 1970), using a new theorem

(*) I warmly thank the Polish Academy of Sciences for having invited me to give a general lecture on the theme "Boundary value problems in elastoplasticity".

of convex analysis (R. T. ROCKAFELLAR [20], 1968). The very important theoretical question was open: does there exist a solution to the evolution problem? A heuristic approach had led to some more or less plausible algorithms, the efficiency of which rested mainly on empirical skill, but no actual theory existed that could be compared to the firmly constructed theory of linear elasticity. The main reason for this situation was the lack of appropriate mathematics and the present progress is due to the development of non-linear analysis. Hence it is not due to chance that J. L. LIONS is one of the main contributors to the development of functional methods in numerical analysis, while J. J. MOREAU ([7], 1966–67) and R. T. ROCKAFELLAR ([22], 1970) have been the promotors of modern convex analysis, the former with mechanical motivations, the latter with optimization problems in mind.

In 1971, Moreau published his first big paper on plasticity [9], followed by many others ([10] to [16]). Reference [15] is a synthetic exposition written in English and can be regarded as a good introduction to the others; it is part of a collective book containing also lectures by C. CASTAING [2], G. DUVAUT [4], and myself [18] on closely connected matters. Up to the present time MOREAU has proved all the desirable theorems for the case of the finite degree of freedom such as is met in finite elements methods. Dealing with continuous media he proved the existence of a unique solution for the evolution of the stress field, using an algorithm that permits the computation of approximate solutions; the solution is reached as a strong limit of these [10]. Probably at the same time, although published a little later, G. DUVAUT and J. L. LIONS obtained a similar existence theorem ([5], 1972); but the numerical accessibility of the solution, which is reached as a weak limit, does not appear so clearly. For continuous media these three research workers presently attempt to solve the big difficulty encountered with the introduction of spaces of the L^{∞}' type.

The connection between Moreau's papers and boundary value problems may not be evident to everybody, and my own contribution to the subject of this lecture will be to make this connection clear. It will appear that the weak formulation associated with the virtual work principle is the key to this connection; it allows mechanists to distinguish continuum mechanics from the corresponding theory of partial derivative equations which they should regard only as a closely connected matter, but not as the actual basis of these mechanics. I ought to inform the reader that I shall have to propose a functional framework which seems suitable for completing the following theory. I hope this choice will appear to be a good one later on with the achievement of the theory.

Notation

Ω non-empty bounded open set in R^3,

$\bar{\Omega}$ closure of Ω,

$\partial\Omega$ boundary of Ω,

$\partial_1\Omega$, $\partial_2\Omega$, F, u^0 $\partial\Omega$ is divided in two complementary parts. On $\partial_1\Omega$ is given the displacement field u^0; on $\partial_2\Omega$ is given a superficial density of forces F. We may assume, with no restriction of generality that u^0 is defined on the whole $\bar{\Omega}$ and not only on $\partial_1\Omega$,

$u+u^0$ actual unknown displacement field, u vanishes on $\partial_1\Omega$,

f given density of body forces, defined on Ω,

T $t\in[0, T]$ interval of variation for the time t,

$\mathrm{grad}_s u$ $(\mathrm{grad}_s u)_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})$,

x unknown field of strain tensors, defined on Ω

x^0 given field of strain tensors: $x^0 = \mathrm{grad}_s u^0 - \chi\theta$,

χ given field of modulus of thermal dilatation tensors,

θ given field of temperature,

s unknown field of stress tensors (symmetrical),

$\mathrm{div}\, s$ $(\mathrm{div}\, s)_i = s_{ij,j}$,

n external unit vector normal to $\partial\Omega$,

s^0 particular solution of the equilibrium Eq. (1.6),

$\tilde{s}^0$ elastic solution for stresses,

k field of stiffness tensors,

e elastic part of the strain x; $x = e+p$,

p plastic part of the strain,

$\mathbf{E}^6$ space (isomorphic to R^6) of strain tensors. At point M of $\Omega x(M)$ belongs to E^6,

$\mathbf{F}^6$ space (isomorphic to R^6) of stress tensors $s(M) \in \mathbf{F}^6$,

$a \cdot b$ any completely contracted product of two tensors or vectors: $a \cdot b = a_i b_i$,
$(a \cdot b)_i = a_{ij} b_j$,
$a \cdot b = a_{ij} b_{ij}$,
$(k \cdot e) = (k_{ijlm} e_{lm})$, and so on,

C_M domain of elasticity at point M $C_M \subset \mathbf{F}^6$,

$|a|$ $(a \cdot a)^{1/2}$,

a. e. Ω "almost everywhere in Ω", with respect to the ordinary Lebesgue's measure,

$B(x, r)$ ball with centre x and radius r, in any metric space.

1. A typical boundary value problem. Virtual work approach and duality

With the notations defined above, let us write the equations of a typical boundary value problem. Actually this one will be used only as a reference to introduce a much more general framework

$$\begin{cases} x = \mathrm{grad}_s u + x^0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial_1 \Omega; \end{cases} \tag{1.1}$$

$$\begin{cases} \mathrm{div}\, s = -f & \text{on } \Omega, \\ s \cdot n = F & \text{on } \partial_2 \Omega; \end{cases} \tag{1.2}$$

$$x = e+p, \tag{1.3}$$

$$s = Ke, \tag{1.4}$$

$$\forall M \in \Omega, \quad \dot{p}(M) \in \partial\Psi_{C(M)}(s(M)). \tag{1.5}$$

As the Eq. (1.5) concerns the derivative $\dot{p}$ this set of equations defines an evolution problem, at least if some initial conditions are added to it. The Eqs. (1.3) to (1.5) define the elastoplastic behaviour and will be studied in Sect. 2. Let us emphasize the meanings of (1.1) and (1.2).

The Eqs. (1.1) express some constraints on the kinematical variables x and u. The given tensor field x^0 summarizes all the useful information on χ, θ, u^0, the definition of x^0, including the term $-\chi\theta$, is such that the null value of x corresponds to the unstressed state when p is zero.

The Eqs. (1.2) are the classical conditions of equilibrium. They can be understood in the sense of distributions, or in what is more generally called a weak sense. Let us choose some space of "test-functions", for instance

$$V = \{\boldsymbol{v} \in (C^1_{\Omega})^3 / v = 0 \quad \text{on } \partial_1 \Omega\},$$

where C^1_{Ω} denotes the space of the restrictions to $\bar{\Omega}$ of functions that are continuously differentiable on R^3. In short, V is a space of regular displacement fields that vanish on $\partial_1 \Omega$. A classical integration by parts shows that the Eqs. (1.2) imply

$$\forall v \in V: \quad -\int_{\Omega} \mathrm{grad}_s v \cdot s + \int_{\Omega} v \cdot f + \int_{\partial_2 \Omega} v \cdot F = 0. \tag{1.6}$$

Conversely the "variational" equation (1.6) implies the Eqs. (1.2) if, at least, s is regular enough to make them meaningful. Mathematicians, who usually start from (1.2), call (1.6) the weak form of (1.2); but mechanists call it the virtual work equation and may take it as a starting point. Therefore (1.6) is not weaker than (1.2) from a mechanical point of view. We shall only use (1.6) in what follows.

However, the space V is possibly not rich enough to include the researched virtual field u in the general case. We shall replace it by some larger space U of vector fields that also vanish on $\partial_1 \Omega$ but subject to less restrictive regularity conditions. These conditions will arise from the study of the constitutive equation (1.5), in a natural way, as well as the regularity involved in the definition of the spaces X and S which are now to be introduced.

The space of the considered strain fields x, subject to some regularity conditions, will be denoted by X. This space will be chosen in such a way that

$$\mathrm{grad}_s(U) \subset X,$$

and the image space $\mathrm{grad}_s(U)$ will be denoted by I.

The space of the considered stress fields s, subject to some regularity conditions, will be denoted by S. The spaces X and S will be chosen in such a way that the integral

$$(x, s) \in X \times S \to \langle x, s \rangle = \int_{\Omega} x \cdot s,$$

exists, at least in a more or less sophisticated sense. Hence the bilinear form denoted by $\langle \cdot, \cdot \rangle$ is defined on the product $X \times S$, and places these two spaces in duality. Let us assume that, for instance:

$$[C^0_{\Omega}]^6 \subset S, \quad [C^0_{\Omega}]^6 \subset X.$$

Then we have the following property: x(resp.: s), belonging to X (resp.: S) is null if, and only if, $\langle x, s \rangle$ is zero whatever is s (resp.: x) in S (resp.: X). This property defines the duality as "separating".

There is no loss of generality in assuming that S contains at least one particular solution s^0 of the Eq. (1.6).

Let us consider now the associated homogeneous equation, obtained for $f = 0$ and $F = 0$. The set of its solutions which belong to S is

$$J = \{s \in S / \forall x \in \mathrm{grad}_s (V) \, \langle x, s \rangle = 0\}.$$

This linear subspace of S is the polar space of $\mathrm{grad}_s (V)$, hence denoted by

$$J = [\mathrm{grad}_s\,(V)]^0.$$

The space of displacement fields U is to be constructed in such a way that I be the polar set of J

$$I = J^0 = [\mathrm{grad}_s\,(V)]^{00}.$$

As is well known, I is nothing but the closure of $\mathrm{grad}_s(V)$ with respect to the topologies which are "consistent with the duality"(1). The reader is referred to [17] for some more details.

Now the Eqs. (1.1) and (1.6) can be written down shortly

$$x = x^0 + I, \tag{1.7}$$

$$s = s^0 + J, \tag{1.8}$$

where I and J are mutually polar sets.

2. Constitutive equations and functional setting

The Eq. (1.3) expresses the strain e as the sum of an elastic term e and of a plastic term p respectively related to the stress s by the Eq. (1.4) and (1.5). Some more explanations should be given about these equations.

At each point M of Ω is given the stiffness $k(M)$, a positive symmetric tensor; we assume that the field $M \to k(M)$ is measurable. The following assumption is usually made in the linear theory of elasticity and we shall keep it for our purpose:

ASSUMPTION 1. *There exist two positive numbers m_1, m_2 such that*

$$\forall M \in \Omega,\ \forall a \in E^6: \quad m_1|a|^2 < a \cdot k(M) \cdot a < m_2|a|^2.$$

The field of inverse tensors k^{-1} possesses the same properties.

Now we may call "global stiffness" the operator K defined on the space of tensor fields by

$$(Ke)(M) = k(M) \cdot e(M).$$

Assumption 1 ensures that K is a one to one mapping of the functional space $[L^\alpha(\Omega)]^6$ into itself, whatever is α in $[1, +\infty]$. Let us now consider the plastic part of strain.

At every point M of Ω is given the "domain of elasticity" C_M, a subset of $\mathbf{F}^6$ depending on M(2). In the present study we do not pay attention to the particular criterion that defines C_M, such as the Mises or Tresca's criteria; we merely assume that *C_M is a closed convex set, the interior of which contains the origin.*

The law of standard plasticity (i.e. satisfying Hill's principle) is currently formulated in these words: The stress tensor $s(M)$ belongs to C_M, and only two cases can happen; either $s(M)$ belongs to the interior of C_M and then the plastic rate $p(M) = 0$, or $s(M)$ belongs to the boundary and then the plastic strain rate $\dot{p}(M)$ may be any outward normal vector

(1) These topologies are those for which the linear forms of the type

$$x \in X \to \langle x, s \rangle \in R, \quad s \in S,$$

are all continuous linear forms defined on X.

(2) This dependence obviously means that the considered material is non-homogeneous.

to C_M at the point $s(M)$ (Fig. 1). One generally assumes that the boundary of C_M consists of a finite number of smooth surfaces, hence the outward normal vectors are easily defined at every point of this boundary. This formulation is usually called "the theory of generalized plastic potential".

HILL's principle is known to give an equivalent formulation by writing

$$(2.1)\qquad \begin{aligned} &s(M) \in C_M \\ \forall \sigma \in C_M:\quad &\dot{p}(M)\cdot[s(M)-\sigma] \geqslant 0, \end{aligned}$$

which is the most efficient way to formulate the plasticity law. In fact it is free of any regularity assumption but the convexity of C_M. Inequalities of this sort are very often met with in convex analysis; hence a special concept has been elaborated to formulate them.

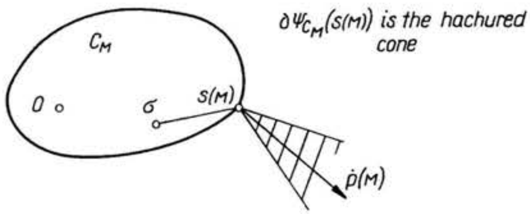

FIG. 1.

Conditions (2.1) express $\dot{p}(M)$ as an element of the "subdifferential" set denoted by $\partial\Psi_{C_M}(s(M))$. A shorter way to write (2.1) is

$$(2.2)\qquad \dot{p}(M) \in \partial\Psi_{C_M}(s(M)).$$

This subdifferential set $\partial\Psi_{C_M}(s(M))$ is nothing but the cone of outward normal vectors to C_M at the point $s(M)$. If $s(M)$ belongs to the interior of C_M, then this cone is reduced to the single element 0. If $s(M)$ belongs to the boundary, this cone contains other elements besides 0. On the other hand, if (M) does not belong to C_M, this cone is empty. Let us now put

$$C = \{s \in S/\text{a.e.}\Omega\ s(M) \in C_M\}.$$

Using the subdifferential notation in the same way as in the above we can write

$$(2.3)\qquad \dot{p} \in \partial\Psi_C(s) \Leftrightarrow \begin{cases} s \in C, \\ \forall\sigma \in C, \quad \langle \dot{p}, s-\sigma\rangle = 0. \end{cases}$$

By integration over Ω it can easily be seen that if (2.2) is satisfied almost everywhere in Ω then (2.3) is satisfied; but the reciprocal statement is much more difficult to establish. R. T. ROCKAFELLAR ([20], 1968) gave a set of assumptions which ensures the implication (2.3) $\Rightarrow$ (2.2). They involve, at first, the measurability of the multivalued mapping:

$$M \in \Omega \to C_M \subset \mathbf{F}^6,$$

and some conditions of "properness" which are always satisfied in mechanical practical cases. Rockafellar's theory needs, in additon, that X and S be "decomposable" spaces, an assumption which is satisfied by space such as $[L^\alpha(\Omega)]^6$ but not by $[C^0_{\overline{\Omega}}]^6$.

As we shall comply with these assumptions the equivalence

$$[\text{a.e. } \Omega\, \dot{p}(M) \in \partial\Psi_{C_M}(s(M))] \Leftrightarrow \dot{p} \in \partial\Psi_C(s),$$

will hold; this is precisely the equation we had numbered as (1.5) above.

Now the "abstract problem" we referred to in the introduction is formulated by a set of equations which are already written

$$x \in x^0 + I, \quad s \in s^0 + J, \quad x = e + p,$$
$$s = Ke, \quad \dot{p} \in \partial\Psi_C(s),$$

where I and J are two mutually polar linear subspaces of the spaces X and S placed in duality. At this stage we seem far away from the boundary value problem initially formulated as a reference. But this set of equations holds for every perfectly elastoplastic mechanical system, a continuous solid or a discrete structure, under the usual assumption of infinitesimal displacements. This generality is the main advantage of the abstract approach.

Up to now Moreau worked under the assumption that K^{-1} is a mapping of S onto X, which is the usual case for discrete structures, X and S being then of the same finite dimension. When X and S are function spaces we shall have to give up this assumption, otherwise the abstract problem would seem of no practical meaning.

Let us now introduce:

ASSUMPTION 2. *The convex sets C_M are uniformly lower-bounded and upper-bounded by constant balls, i.e.*:

$$\exists r_1 > 0, \quad \exists r_2 > 0, \quad \forall M \in \Omega: \quad B(0, r_1) \subset C_M \subset B(0, r_2).$$

This assumption seems to forbid the use of criteria of the Von Mises type since they give a cylinder for C_M. But we shall see in Sect. 5 that it can be replaced by another assumption which comes back to the same if one uses a more sophistocated functional framework. Let us keep assumption 2 for the sake of simplicity.

The right-hand side inclusion implies that every measurable stress field, whose value belongs to C_M, for almost every M belongs to $[L^\infty(\Omega)]^6$; thus we shall choose this function space as S. Then the left-hand side inclusion involves that C possesses a non-empty interior with respect to the norm topology of S:

$$B(0, r_1) \subset C.$$

By some classical properties of convex sets in topological linear spaces this ensures that $\partial\Psi_C(s)$ is not reduced to $\{0\}$ when s belongs to the boundary of C, and, obviously, this is necessary for the existence of a non-zero plastic strain rate $\dot{p}$. Such a $\dot{p}$ is expected as an element of the topological dual space of S; hence, we are induced to choose

$$X = [L^{\infty\prime}(\Omega)]^6, \qquad S = [L^\infty(\Omega)]^6. \tag{2.4}$$

In view of the properties of K previously noticed the elastic strain e is an element of the space

$$E = K^{-1}(S) = [L^\infty(\Omega)]^6.$$

The choice of X and S has been made in a somewhat compulsory way, and now we meet some difficulties. These will now be explained.

To define J is easy since we can use the space V of test-functions and put

(2.5) $$J = [\mathrm{grad}_s(V)]^0.$$

Then I is defined as the polar set of J

(2.6) $$I = J^0,$$

and I is the closure of $\mathrm{grad}_s(V)$ with respect to the topologies which are compatible with the duality. Observe that such is not the case for the norm topology of X, and that, consequently, the space I will probably strictly embed the closure of $\mathrm{grad}_s(V)$ with respect to this norm topology.

What is then the corresponding space U, i.e. the space of displacement fields satisfying

$$I = \mathrm{grad}_s(U)$$

and what are the regularity properties to expect of its elements? In what way can they be approximated by regular fields?

All these questions are probably still open and answers are to be given by specialists of functional analysis. But we may consider that the choices (2.4) to (2.6) define an abstract problem closely connected with the contemplated boundary value problem.

Another difficulty arises: which meaning can we ascribe to the derivatives $\dot{x}, \dot{p}, \dot{e}, \dot{s}$? Indeed the functions $t \to x(t)$ and others take their values in the nonreflexive Banach spaces X and S. Nevertheless we can define the function $t \to \dot{x}(t)$ as a function which takes its values in X such that the usual relation holds

(2.7) $$\forall t: \quad x(t) = x(0) + \int_0^t \dot{x}(\theta)\,d\theta.$$

The reader is referred to H. BREZIS [1]; this is a clear monograph about monotone operators which contains a simple and precise appendix devoted to such questions as derivation and integration in function spaces. Following the notation used by this author we shall sum up the previous lines and write the final functional choice

(2.8) $$x, p \in W^{1,1}\big((0,T),X\big), \qquad s \in W^{1,1}\big((0,T),S\big), \qquad e \in W^{1,1}\big((0,T),E\big).$$

Due to the definition of the integral that appears in (2.7) and to I and J being closed with respect to the norm topologies, we have the following equivalences

(2.9) $$\begin{aligned} &\forall t \in (0,T): \quad z(t) - z(0) \in I \Leftrightarrow \text{a.e. } (0,T): \quad \dot{z}(t) \in I, \\ &\forall t \in (0,T): \quad \tau(t) - \tau(0) \in J \Leftrightarrow \text{a.e. } (0,T): \quad \dot{\tau}(t) \in J. \end{aligned}$$

Another assumption will be the following

ASSUMPTION 3. *There exists at least one function*

$$t \to s^0(t),$$

belonging to $W^{1,1}\big((0,T),S\big)$ *which is a particular solution of* (1.6).

Furthermore, the given function

$$t \to s^0(t),$$

belongs to $W^{1,1}\left((0, T), E\right)$.

The first part of this assumption does not restrict generality. The second part involves that u^0 is regular enough; besides χ is practically a bounded field and the temperature field is always very regular; hence Assumption 3 will always be satisfied in actual cases.

Let me draw now the outline of Moreau's technique.

3. Technique of resolution

Introducing the new unknowns

$$z = x - x^0, \qquad \tau = s - s^0,$$

we can write down the considered system

$$z \in I, \qquad \tau \in J,$$

$$p = K^{-1}(\tau + s^0) + x^0 + z, \tag{3.1}$$

$$\dot{p} \in \partial\psi_C(\tau + s^0). \tag{3.2}$$

The first step will be the elimination of p and z. The following equality is easily proved

$$\forall \tau \in S: \quad \partial\psi_C(\tau + s^0) = \partial\psi_{C-s^0}(\tau). \tag{3.3}$$

As τ must belong to $C - s^0$ and to J the set

$$\Gamma' = (C - s^0) \cap J$$

is introduced in a natural way, and we have the inclusion (cf. [7], Ch. 10)

$$\forall \tau \in S: \quad \partial\Psi_{\Gamma'}(\tau) \supset \partial\Psi_{C-s^0}(\tau) + \partial\Psi_J(\tau),$$

which implies

$$\forall \tau \in S: \quad \partial\Psi_{\Gamma'}(\tau) \subset \partial\Psi_{C-s^0}(\tau) + I. \tag{3.4}$$

Now $-\dot{z}$ belongs to I because of (2.9); thus (3.1) and (3.2) imply, through (3.3) and (3.4):

$$-K^{-1}(\dot{\tau} + \dot{s}^0) + \dot{x}^0 \in \partial\Psi_{\Gamma'}(\tau), \tag{3.5}$$

where only the unknown τ appears.

Now the question is: assume τ is a solution of (3.5); does there exist $t \to z(t) \in I$ and p such that (p, z, τ) is a solution of (3.1) and (3.2)? Let the first member of (3.5) be denoted by γ and let us fix the time t. The reader is referred to Fig. 2, drawn for the case of dimension 2; then I and J can be drawn as orthogonal spaces. The derivative $\dot{p}$ is an outward normal vector to $C - s^0$ at the point τ; any vector γ such that

$$\zeta = \dot{p} - \gamma \in I,$$

is obviously normal to $\Gamma' = J \cap (C - s^0)$, at the point τ: this is exactly the meaning of (20). Conversely, let γ be an outward normal vector to Γ' at the point τ; does there exist any ζ belonging to I such that $\gamma+\zeta$ is an outward normal vector to $C-s^0$ at the point τ? In other words if we call

$$Z_\gamma = \{\zeta \in I/\zeta+\gamma \in \partial\Psi_{C-s^0}(\tau)\},$$

is Z_γ empty or not? At last the non-emptiness of Z_γ for any γ belonging to $\partial\Psi_{\gamma'}(\tau)$ is equivalent to the equality

$$\partial\Psi_{\gamma'}(\tau) = I+\partial\Psi_{C-s^0}(\tau). \tag{3.6}$$

In the case represented in Fig. 2 this equality is obvious. But Fig. 3 shows a case where it is wrong: Γ' is reduced to the single point τ and any vector γ is normal to Γ'; Z_γ is empty

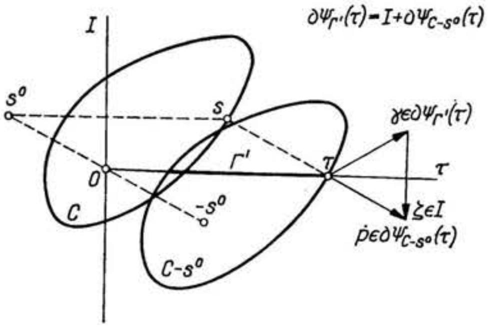

FIG. 2.

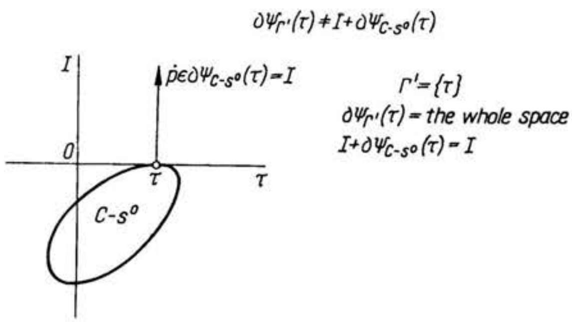

FIG. 3.

if γ does not belong to I. These remarks are classically extended to the general case and (3.6) may be classically ensured by the following

ASSUMPTION 4. *J meet the interior of $C-s^0$; in other words s^0+J meets the interior of C.*

This assumption is closely connected to the theory of limit loads (cf. comments; Sect. 6). We can now write the obvious

PROPOSITION. *If τ is a solution of* (3.5) *there exists a multivalued mapping*

$$t \in [0, T] \to Z(t) \subset I,$$

with

$$\forall t \in [0, T], \quad Z(t) \neq \phi,$$

such that, for any "selector" ζ, i.e. a funtion

$$t \in [0, T] \to \zeta(t) \in Z(t),$$

one has

$$\zeta(t) - K^{-1}(\dot{\tau} + \dot{s}^0) + \dot{x}^0 \in \partial\Psi_{C-s^0}(\tau).$$

Now the end of the proof can be divided in two parts:

Part A. Assume that τ is a solution of (3.6); prove there exists a selector ζ belonging to $L^1((0, T), X)$. Then

$$\begin{cases} z(t) = z(0) + \int\limits_0^t \zeta(\theta) d\theta, \\ p = z - K^{-1}(\tau + s^0) + x^0, \\ \tau \end{cases}$$

is a solution of the problem.

Part B. Prove that there exists at least one solution τ of (3.6).

Moreau established part A of the proof in the case of the finite degree of freedom: X is a finite dimensional space ([15 and 16]). His reasoning is based on the recent theory of "multivalued measurable mappings", a theory mainly due to C. CASTAING ([2]) and R. T. ROCKAFELLAR ([21]). The first stage is to establish the measurability of Z which involves the existence of a least one measurable selector ζ. The second stage is to prove

$$\int\limits_0^t ||\zeta(t)|| dt < +\infty,$$

which can be done by means of some inequality that easily follows from Assumption 4.

The unwieldy structure of $X = [L^{\infty\prime}(\Omega)]^6$ does not allow the same way to be followed in the case of continuous solids.

Part B of the proof was the core of Moreau's papers. He developed some new mathematical tools in order to solve the Eq. (3.5). First let us change the variable to obtain a simpler form of this equation.

We observe that all quantities that appear in (3.5) belong either to E or to S because of Assumption 3. K is a linear, symmetric, positive mapping of E onto S and each point e of E can be represented by the corresponding point Ke of S and conversely. Moreover, these two spaces can be regarded as pre-Hilbertian spaces with respect to the respective scalar products

$$(e_1, e_2) \in E \times E \quad \to \quad (e_1 || e_2) = \langle e_1, Ke_2 \rangle,$$
$$(s_1, s_2) \in S \times S \quad \to \quad (s_1 | s_2) = \langle K^{-1} s_1, s_2 \rangle,$$

and K is an "isomorphism" of these two pre-Hilbertian spaces. That means the mapping K preserves the value of the scalar product

$$(e_1 || e_2) = (Ke_1 | Ke_2).$$

Hence we can identify E with S and consider only this last space, a procedure which classically comes back to replace K and K^{-1} by 1 and the bilinear form $\langle \cdot, \cdot \rangle$ by $(\cdot | \cdot)$ in all formulae that concern only vectors of E or S.

Thus (3.5) may be written

$$-(\dot{\tau}+\dot{s}^0)+\dot{x}^0 \in \partial\Psi_{\Gamma'}(\tau),$$

which is equivalent to:

$$\begin{cases} \tau \in \Gamma'. \\ \forall \tau' \in \Gamma', \quad (-\dot{\tau}-\dot{s}^0+\dot{x}^0|\tau-\tau') \geqslant 0. \end{cases}$$

In the same way $I \cap E$ and J are a pair of linear subspaces of $\hat{S}$, mutually polar with respect to the bilinear form $(\cdot|\cdot)$.

Now we can complete S and obtain the Hilbert space denoted by S. Due to Assumption 1 this space is nothing but the well known $[L^2(\Omega)]^6$. The completion of $I \cap E$ (i.e. its closure in $\hat{S}$) will be denoted by $\hat{I}$, and its polar space by $\hat{J}$, this latter embeds the completion of J.

One classicalily prove that C is closed in $\hat{S}$. The complete proof is given in DUVAUT et LIONS [5], Chapter 3.

Another proof can be given by the mean of ROCKAFFELLAR's thorem, already quoted at the page 120.

Now we can introduce the "elastic solution" $\tilde{s}^0$ which is the only point of the intersection $(s^0+\hat{I}) \cap (x^0+\hat{J})$.

Figure 4 represents $\hat{S}$ as the $(\hat{I}, \hat{J})$ plane. Every point of $\hat{S}$ can be projected onto these two orthogonal complementary subspaces, hence we can put

$$\tilde{s}^0 = \sigma^0+\tau^0 \quad \text{with} \begin{cases} \sigma^0 = \mathrm{proj}_{\hat{I}}\, \tilde{s}^0 = \mathrm{proj}_{\hat{I}}\, s^0, \\ \tau^0 = \mathrm{proj}_{\hat{J}}\, \tilde{s}^0 = \mathrm{proj}_{\hat{J}}\, x^0. \end{cases}$$

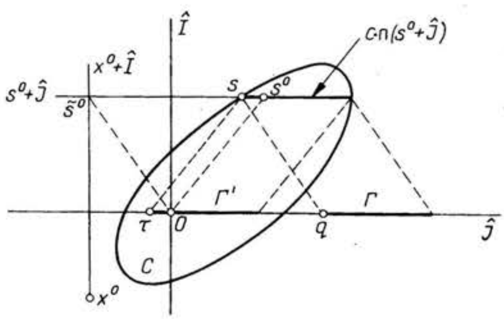

FIG. 4.

Let τ be replaced by the new unknown q:

$$q = s-\tilde{s}^0 = \tau+(s^0-\tilde{s}^0) \in \hat{J},$$

q is the part of the stress field which is properly due to plasticity effects. In the same way let us put

$$\Gamma = \Gamma'+s^0-\tilde{s}^0;$$

now the Eq. (3.5) takes the standard form

$$-\dot{q} \in \partial\Psi_\Gamma(q), \tag{3.7}$$

characterizing what Moreau calls the "sweeping process of the point q by the given moving set Γ".

4. The sweeping process

First let me give a mechanical interpretation of the Eq. (3.7), although it stands far away from our context. The Eq. (3.7) may be regarded as the equation of the quasi-statical evolution of a system the configuration of which is represented by the point q (for instance q is a material point in three-dimensional space). This system is only submitted to the reaction of the frictionless unilateral constraint defined by Γ. When q is interior s to Γ this force is zero; when q touches the boundary it is submitted to a force which is an invard normal vector to Γ, exactly as in the case of a frictionless contact. This image is what tends to suggest the name of "sweeping process".

The uniqueness of the solution, if any, corresponding to some given initial condition, is easily deduced from the following implications:

$$-\dot{q}_1 \in \partial\Psi_\Gamma(q_1) \Rightarrow (-\dot{q}_1|q_1-q_2) \geqslant 0,$$

$$-\dot{q}_2 \in \partial\Psi_\Gamma(q_2) \Rightarrow (-\dot{q}_2|q_2-q_1) \geqslant 0.$$

Hence

$$(\dot{q}_2-\dot{q}_1|q_2-q_1) \leqslant 0, \tag{4.1}$$

and finally

$$\frac{d}{dt}\|q_1-q_2\|^2 \leqslant 0.$$

Thus:

$$q_1(0)-q_2(0) = 0 \Rightarrow At \geqslant 0, \quad q_1(t)-q_2(t) = 0.$$

According to a general terminology (cf. H. BREZIS [1]) $\partial\Psi_\Gamma$ is a "monotone mapping", a property which implies (4.1) by the definition of this concept. Concerning plasticity this uniqueness was established many years ago.

The question of existence is not so easily settled. In [9] MOREAU established this existence in two different ways, the former used some regularization technique, the latter used the so-called "catching-up algorithm" which is to be described now. Consider a family of subdivisions of the interval $[0, T]$:

$$\mathscr{C}_i = \{0 = t_i^0, t_i^1, \ldots, t_i^n, \ldots, t_i^{Ni} = T\}, \quad i \in \{1, 2, \ldots\},$$

such that

$$\lim_{i\to\infty} \max_{1\leqslant n\leqslant N_i} (t_i^n - t_i^{n-1}) = 0.$$

For each i construct the finite sequence

$$q_i^0 = q(0) \quad \text{(given initial condition)},$$

$$(4.2) \qquad q_i^{n+1} = \mathrm{proj}_{\Gamma(t_i^{n+1})} q_i^n,$$

$$q_i^{N_i},$$

which is uniquely defined; consider the associated piecewise linear function of t:

$$t_i^n \leqslant t \leqslant t_i^{n+1} \to q_i(t) = q_i^n + \frac{t - t_i^n}{t_i^{n+1} - t_i^n}(q_i^{n+1} - q_i^n).$$

Notice that (4.2) is equivalent to

$$q_i^n - q_i^{n+1} \in \partial\Psi_{\Gamma(t_i^{n+1})}(q_i^{n+1}).$$

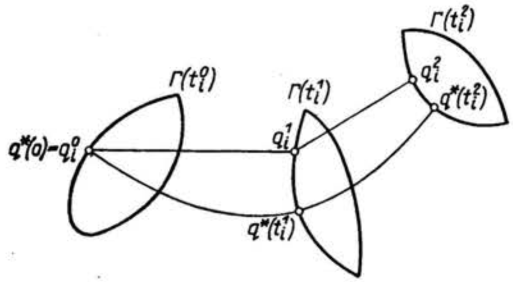

FIG. 5.

Hence this appears as an algorithm of time discretization of the implicit type (cf. Fig. 5); observe that the corresponding explicit type would be untractable as $\partial\Psi_\Gamma$ is not singlevalued.

In [9] Moreau proves the

THEOREM 1. *If Γ is of bounded variation the sequence of functions q_i converges to some q^* in $W^{1,1}((0, T), S)$ and q^* is the unique solution of* (3.7) *for given* $q(0)$.

Before explaining what is a moving set of bounded variation let me emphasize the power of this result which furnishes an algorithm the convergence of which is proved to be very strong.

What is a moving set of bounded variation? A great part of Moreau's work is precisely devoted to the extension of this concept to multivalued mappings, a familiar one for functions. At once he considers the "Hausdorff distance" $D(A, B)$ of any two subsets in some metric space in which the distance is denoted by d:

$$D(A, B) = \max\,[\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(b, A)].$$

These upper bounds are to be taken in $[0, +\infty]$ in such a way that $D(A, \phi) = 0$. It can easily be seen that D is symmetrical, non-negative

$$D(A, B) = D(B, A) \geqslant 0,$$

and that the triangle inequality holds

$$D(A, C) \leqslant D(A, B) + D(B, C).$$

At last $D(A, B)$ is zero if, and only if, A and B have the same closure.

Now the variation on $[0, T]$ interval of the moving set Γ is defined by

$$\mathrm{Var}\,(\Gamma, [0, T]) = \sup_{i\in I} \sum_{n=1}^{Ni} D(\Gamma(t_i^{n-1}), \Gamma(t_i^n)),$$

where

$$(\mathscr{C}_i), \quad i \in I,$$

denotes the family of all subdivisions of $[0, T]$ such as

$$\mathscr{C}_i = \{0 = t_i^0, t_i^1, \ldots, t_i^{Ni} = T\}.$$

Developing this tool in connection with convex analysis Moreau obtained the previous theorem. Moreover, he proved the following theorem the terms of which are adapted to our text:

THEOREM 2. *Under Assumptions* 3 *and* 4 Γ *is of bounded variation and thus there exists a unique solution of the Eq.* (3.7) *for given* $q(0)$.

Now it can be seen that q possesses a null velocity $\dot{q}$ when Γ is expanding, thus the motion of this set plays no actual role when Γ is expanding. Hence Moreau thought the hypothesis of Theorem 1 should be improved and introduced the notion of "retraction of a moving set", which only takes into account the variation of Γ when this set is not expanding. One only considers the first upper bound that appears in the definition of Hausdorff's distance

$$E(A, B) = \sup_{a\in A} d(a, B)$$

which may be called the "excess of A upon B". Now the "retraction" of Γ during the interval $[0, T]$ is defined by

$$\mathrm{Ret}\,(\Gamma, [0, T]) = \sup_{i\in I} \sum_{n=1}^{Ni} E(\Gamma(t_i^{n-1}), \Gamma(t_i^n)).$$

E satisfies the triangle inequality but it is not symmetrical; hence Moreau's theory is quite sophisticated and many difficulties are to be solved. Eventually Moreau succeeded in obtaining a theorem similar to Theorem 1, and this result seems an optimal one. The reader is referred to [10, 11 and 12].

5. Dealing with criteria of the von Mises' type

They only concern the deviatoric part of the stress tensor. Let us regard $\mathbf{F}^6$ as the Cartesian product:

$$\mathbf{F}^6 = \mathbf{F}^1 \times \mathbf{F}^5,$$

where $\mathbf{F}^1$ is the space of spherical tensors, $\mathbf{F}^5$ the one of deviators. Then the criteria give a convex closed set C_{dM} in $\mathbf{F}^5$ and the domain of elasticity at point M is the cylinder

$$C_M = F^1 \times C_{dM}.$$

Assumption 2 is to be replaced by

ASSUMPTION 2'. *C_{dM} is uniformly upper-bounded and lower-bounded by constant balls* $B(0, r_2)$, $B(0, r_1)$.

Now X and S may also be regarded as Cartesian products:

$$X = X_s \times X_d, \quad S = S_s \times S_d,$$

where X_s and S_s are spaces of spherical tensor fields, X_d and S_d spaces of deviator fields. As

$$\langle x_s, s_d \rangle = \langle x_d, s_s \rangle = 0,$$

the formerly used bilinear form $\langle \cdot, \cdot \rangle$ can be written

$$x = (x_s, x_d),\ y = (y_s, y_d) \rightarrow \langle x, s \rangle = \langle x_s, s_s \rangle + \langle x_d, s_d \rangle.$$

Choose now the functional spaces:

$$X_s = S_s = L^2(\Omega),$$

$$X_d = [L^{\infty\prime}(\Omega)]^5, \quad X_s = [L^{\infty}(\Omega)]^5,$$

and denote the domain of elasticity of deviator fields

$$C_d = \{s_d \in S_d / \text{a.e.}\Omega\ s_d(M) \in C_{dM}\}.$$

Then C is the cylinder $S_s \times C_d$ the interior of which is non-empty with respect to the product topology. Hence the same technique can be used.

6. Some comments

6.1. About Assumptions 1 and 2

Assumption 1 could be weakened, one could only assume $k(M)$ to be positive almost everywhere on Ω. In the same way K^{-1} would define a pre-Hilbertian structure on S and a completed space $\hat{S}$, but this one would probably be different from $[L^2(\Omega)]^6$.

In the same way Assumption 2 may be weakened in order to consider a more general type of domains of elasticity, S could be chosen as the linear space generated by all the measurable fields, the values of which belong to C_M almost everywhere, the convex set C would define a semi-norm on S and so on ... But the whole corresponding functional analysis would have to be contructed and this would not be slight work.

6.2. About Assumption 4 and limit loads

Let us consider the load (f, F) in equilibrium with s^0, its limit factor is classically defined by

$$\lambda = \max\{\mu \geqslant 0 / \exists \tau \in J\ \mu s^0 + \tau \in C\}.$$

Besides Assumption 4 may be written

$$\exists \tau \in J, \quad \exists r > 0, \quad B(s^0 + \tau, r) \subset C,$$

which implies

$$s^0\left(1+\frac{r}{\|s^0\|}\right)+\tau\subset C,$$

where $\|\cdot\|$ denotes the norm in S and finally,

$$\lambda \geqslant 1+\frac{r}{\|s^0\|} > 1.$$

Conversely assume that λ is larger than 1. There exists μ such that

$$1<\mu<\lambda$$

$$\exists \tau\in J, \quad \mu s^0+\tau\in C,$$

and, as 0 belongs to the interior $\mathring{C}$ of the convex set C,

$$s^0+\frac{\tau}{\mu}\in \mathring{C}$$

which shows that Assumption 4 is satisfied. Eventually we obtain the

PROPOSITION. *Assumption 4 is satisfied if, and only if, the load is never a limit one, i.e. of its limit factor is always larger than* 1.

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