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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Symmetric forms of the equations of heat transport in a rigid conductor of heat with internal state variables
I. Analysis of the model and thermodynamic restrictions via the "main dependency relation"

W. LARECKI (WARSZAWA)

THE OBJECTIVE of this series of two papers is twofold: to analyse the phenomenological model of a rigid conductor of heat with (vector) internal state variable, and to promote the application of the "main dependency relation" (MDR) as a tool for derivation of the restrictions on constitutive functions implied by the entropy inequality as well as a tool for direct derivation of alternative symmetric systems of field equations. In this paper (Part I), the analysis of the model of a rigid conductor of heat with (vector) internal state variable is focused on two aspects, namely, on the form of the respective field equations and on the relation to other phenomenological models proposed in the literature, with the emphasis put on those models which have been succesfully adjusted to experimental data on heat transport at finite speeds. The relation to the model of a rigid conductor of heat with scalar internal state variable, called "semi-empirical temperature" is demonstrated. It is proved that, for the system of N conservation equations, consistency with the entropy inequality (in the form of first-order unilateral differential constrains) is equivalent to the requirement that the corresponding system of $N+1$ conservation equations satisfies the "main depency relation" (MDR). For the model of a rigid conductor of heat with conservative evolution equation for internal state variable, the procedure of derivation of thermodynamic restrictions via the MDR is demonstrated.

1. Introduction

THE OBJECTIVE of this series of two papers is twofold: to analyse the phenomenological model of a rigid conductor of heat with (vector) internal state variable from the point of view of relations (consistency and/or possible generalization) to some other proposed (and applied to fit the experimental data, like Maxwell – Cattaneo – Vernotte equation [1, 2, 3], or the model proposed by MORRO and RUGGERI [4, 5, 6]) phenomenological models of hyperbolic heat transport as well as from the point of view of symmetrizability (and consequently, symmmetric hyperbolicity) of the resulting field equations; and to promote the application the "main dependency relation" (MDR) introduced by FRIEDRICHS [7] which, in fact, is a natural generalization of the "entropy principle" of Extended Thermodynamics [8, 9, 10] as a tool for derivation of thermodynamic restrictions on constitutive functions. Therefore, Part I in this series is focused on the analysis of the phenomenological model of a rigid conductor of heat with

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(vector) internal state variable, on the proof of the equivalence of the "main dependency relation" (MDR) and the entropy inequality, and on application of the MDR for derivation of the restrictions on constitutive functions for the specific model of a rigid conductor of heat with (vector) internal state variable that admits conservative form of field equations.

Various types of heat conduction equations leading to finite speed of propagation of thermal waves were postulated for rigid and deformable heat conductors in the last four decades. Different formulations of continuum thermodynamics (for example, Rational Thermodynamics, Extended Thermodynamics, Extended Irreversible Thermodynamics (EIT)) proved useful in deriving the heat transport equations leading to finite wave speeds. The literature on the subject is too extensive to be quoted here and therefore we refer the interested readers to the review papers by JOU and CASSAZ–VASQUEZ [11, 12] and to the general overeview by JOSEPH and PREZIOSI [13, 14].

Some phenolmenological modeles of heat transport with finite speeds are motivated by or even directly related ([15, 16, 5, 6]) to the second sound in solids. The second sound was detected in crystalline He^4, NaF and Bi in heat pulse experiments at very low temperatures but very few quantitative data concerning second sound measurements were published in early 70-ties (extensive list of references can be found in [17]). Besides the results of the second sound measurements, there is other published interesting experimental evidence (for example, [18, 19, 20]) of the wave nature of heat propagation (or dominance of wave behaviour over diffussion) at moderate temperature ranges in materials of practical applications in technology and engineering. KAMIŃSKI [18] determined the constant τ of the Maxwell – Cattoneo – Vernotte equation for various materials with "nonhomogeneous inner structure" ("complex systems made up of solid, liquid and gas, e.g., porous-capillary bodies, cellular systems, suspensions, etc.") with the aid of the original experimental method. MITRA, KUMAR, VEDAVARZ and MOALLEMI [19] presented an experimental evidence of the wave-type heat transport in processed meat and showed that Maxwell – Cattaneo – Vernotte equation provides an accurate description, on a macroscopic level, of the heat conduction process in such biological materials. In [18, 19], the investigated materials had refined complex inner structure and, as a consequence, the observed heat transport was due to cummulative effect of different transfer mechanisms, for instance, particle-to-particle contact, free convection in closed space, radiation, etc. Even in the absence of such inner structure, the effect of finite speed of heat transport has been confirmed experimentally, namely, TZOU [20] concluded the wave nature of heat conduction from the comparison of the wave solution of the Maxwell – Cattaneo – Vernotte equation for the temperature rise induced by a propagating crack tip with the experimental results obtained by ZEHNDER and ROSAKIS [21] for 4340 steel. In [18, 19, 20], the wave features of heat transport were concluded by means of adjusting the experimental data to the simplest phenomenological model of hyperbolic heat conduction, i.e.

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Maxwell - Cattaneo - Vernotte equation in which the single numerical constant τ is the only material parameter responsible for wave features. It can be supposed that this simplest model will be insufficient to describe properly the material thermal response in wide range of thermal and mechanical conditions, especially in the case of materials with inner structure where different transport mechanisms are simultaneously involved with relative intensities dependent on thermal and mechanical conditions. The phenomenological model of hyperbolic heat conduction that can be proposed for practical use should involve relatively simple but sufficiently general constitutive relations, should enable one to analyse waves of weak and of strong discontinouity and should lead to the system of field equations for which the Cauchy problem is well-posed and numerical methods are easily applicable. One of the purely phenomenological models of heat transport with finite speed that seems to meet those requirements is the model of a rigid conductor of heat with (vector) internal state variable.

Basic constitutive assumptions for modelling heat transport at finite speed by means of internal state variable, formulated by KOSIŃSKI [22], are recalled and the general form of the model of a rigid conductor of heat with (vector) internal state variable is introduced in Sec. 2.2.1. Then, two types of the additional constitutive assumptions are analysed subsequently in Secs. 2.2.2 and 2.2.3, namely, assumptions concerning the form of the evolution equation for the vector internal state variable and the assumptions concerning dependence of the heat flux vector and the internal energy on the vector internal state variable and on the temperature.

In Secs. 2.2.4, 2.2.5, 2.2.6, further constitutive assumptions corresponding to specific forms of free energy and specific forms of the source term in the evolution equation for internal state variable are investigated in the case of the model of a rigid conductor of heat with conservative form of the evolution equation for internal state variable. It is shown that this model can be interpreted as a representation of the class of phenomenological models comprising both the Maxwell - Cattaneo - Vernotte equation and the models proposed in [4, 5, 6, 23]. Within this class, various particular generalizations or corrections to those models can be easily introduced.

Recently, a phenomenological model of rigid conductor of heat with scalar internal state variable called "semi-empirical temperature" has been proposed [24 - 29]. This model is discussed in Sec. 2.3, and it is shown that the same system of field equations as suggested in Sec. 2.2.6 for the model of a rigid conductor of heat with (vector) internal state variable also describes the model with "semi-empirical temperature", if supplemented by additional involutive constraints.

In order to promote the "main dependency relation" [7] as a tool for derivation of thermodynamic restrictions, its equivalence to the entropy inequality is established. Namely, we prove in Sec. 3.2 that if the system of N conservation equations has nonsingular $N \times N$ matrix multiplying time derivatives of the

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unknows, then every Lipschitz continuous solution of this system satisfies the entropy inequality (in the form of first-order unilateral differential constraints) if and only if the MDR is satisfied by the respective system of $N+1$ conservation equations (system of N conservation equations supplemented by the conservation equation corresponding to the entropy inequality). For the proof, the results established by Kosiński [30, 31] for systems of conservation equations in normal (Cauchy) form are employed. From the point of view of thermodynamics, the MDR can be regarded as a generalization of the "entropy principle" of extended thermodynamics of MÜLLER and LIU [8, 9, 10] in the sense that it assigns an analogue of Lagrange–Liu multiplier also to the balance of entropy. It reduces to the "entropy principle" for the value of that additional multiplier equal to -1. The advantage of employing the MDR instead the "entropy principle" is that it enables one to derive equivalent (for classical solutions) alternative symmetric systems directly (see, Part II).

For the model of a rigid conductor of heat with conservative evolution equation for internal state variable, the procedure of derivation of thermodynamic restrictions on constitutive functions via the MDR is demonstrated in Sec. 3.3. According to this procedure, a family of solutions of the MDR is determined in Sec. 3.4 and the restrictions on constitutive functions are derived in Sec. 3.5.

2. Rigid conductor of heat with internal state vriables

2.1. Rigid heat conductor

By a rigid heat conductor we mean the undeformable (rigid) continuous material body which can conduct the heat. It is assumed that for an inertial observer, the body remains at rest and the materal points (particles) are identified with points of the tree-dimensional Euclidean point space E^3. For a rigid heat conductor, all balance laws of continuum mechanics are satisfied trivially except the energy balance, which assumes the following local form:

$$\varrho_0 \dot{\varepsilon} + \operatorname{div} \mathbf{q} = \varrho_0 r \,, \tag{2.1}$$

where ϱ_0 is a constant mass density, ε denotes energy density referred to the unit mass, $\mathbf{q} = [q^\alpha]$, $\alpha = 1, 2, 3$ is a heat flux and r is a heat source.

In the order to formulate the model of a rigid conductor of heat in the framework of Rational Mechanics, the Clausius–Duhem entropy inequality

$$\varrho_0 \sigma := \varrho_0 \dot{\eta} + \operatorname{div}\left(\frac{1}{\theta}\mathbf{q}\right) \geq \varrho_0 \frac{1}{\theta} r \,, \tag{2.2}$$

should be taken into account, where η is the entropy density referred to the unit mass, θ denotes the temperature and σ is the entropy production.

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In this paper, the usual summation convention over repeated lower and upper indices is employed and the notation $\partial_t := \frac{\partial}{\partial t}$, $\partial_\alpha := \frac{\partial}{\partial X^\alpha}$ is used for time and spatial derivatives, respectively. Spatial coordinates X^α correspond here to the fixed Cartesian frame, for simplicity. Dot over a letter denotes material time derivative which, in our case (rigid, unmoving body), corresponds to the partial time derivative ∂_t.

2.2. Model of a rigid conductor of heat with (vector) internal variable

2.2.1. General form of the model. The concept of constitutive modelling (in the framework of Rational Mechanics) with the aid of internal state variables is due to COLEMAN and CURTIN [32]. Constitutive modelling of heat propagation at finite speeds (thermal waves of weak and strong discontinuity) by means of internal state variables was proposed by KOSIŃSKI [22] ([1]) for the general case of deformable (inelastic) body. It can be easily reduced to the model of a rigid conductor of heat, simply by neglecting the dependence on deformation.

The following constitutive assumptions were intrduced in [22]:

- response of the material in particle X^α at time t depends on the values of the temperature $\theta(t, X^\alpha)$ and the internal state variables $\mathbf{w}(t, X^\alpha) = [w_\sigma(t, X^\alpha)]$;
- the evolution of the internal state variables $\mathbf{w}(t, X^\alpha) = [w_\sigma(t, X^\alpha)]$ during the thermodynamic processes is governed by a vector differential equation of the first order, dependent on the temperature gradient as the additional variable.

Taking into account those assumptions and conforming to the equipresence principle of Rational Mechanics, we postulate, according to [22], the constitutive equations

$$\varepsilon = \tilde{\varepsilon}(\theta, \mathbf{w}), \quad \mathbf{q} = \tilde{q}(\theta, \mathbf{w}), \quad r = \tilde{r}(\theta, \mathbf{w}, t, X^\alpha), \quad \eta = \tilde{\eta}(\theta, \mathbf{w}) \tag{2.3}$$

and the following evolution equation for internal state variable $\mathbf{w}$

$$\dot{\mathbf{w}}(t, X^\alpha) = \mathbf{g}(\theta(t, X^\alpha), \operatorname{grad}\theta(t, X^\alpha), \mathbf{w}(t, X^\alpha)) \tag{2.4}$$

with the initial-value problem

$$\mathbf{w}(t_0, X^\alpha) = \mathbf{w}_0(X^\alpha). \tag{2.5}$$

It is assumed that the initial-value problem has a unique solution and this implies that the function $\mathbf{g}$ is Lipshitz continuous with respect to $\mathbf{w}$ and continuous with respect to the other remaining arguments.

([1]) In this paper, we restrict considerations to the models of a rigid conductor of heat developed in the framework of Rational Mechanics, and therefore we do not discuss those phenomenological models employing internal state variables which violate the axioms of Rational Mechanics, like, for example, the model proposed by BAMPI, MORRO and JOU [33], where the entropy flux vector is assumed to be different from $(1/\theta)\mathbf{q}$.

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Substitution of the constitutive functions $(2.3)_{1,2,3}$ into the balance of energy (2.1) together with the evolution equation (2.4), results in the nonlinear (for general $\mathbf{g}(\theta, \operatorname{grad}\theta, \mathbf{w})$) system of the four first-order partial differential equations for the unknowns θ, w_γ

$$\tag{2.6} \begin{aligned} &\varrho_0\partial_t\widetilde{\varepsilon}(\theta, w_\gamma) + \partial_\alpha\widetilde{q}^\alpha(\theta, w_\gamma) = \varrho_0\widetilde{r}(\theta, w_\gamma, t, X^\alpha),\\ &\partial_t w_\beta = g_\beta(\theta, \partial_\alpha\theta, w_\gamma), \qquad \alpha, \beta, r = 1, 2, 3 \end{aligned}$$

subject to unilateral first-order differential constraints

$$\tag{2.7} \varrho_0\sigma := \varrho_0\partial_t\widetilde{\eta}(\theta, w_\gamma) + \partial_\alpha\left[\frac{1}{\theta}\widetilde{q}^\alpha(\theta, w_\gamma)\right] \geq \varrho_0\frac{1}{\theta}\widetilde{r}(\theta, \mathbf{w}, t, X^\alpha),$$

obtained by substituting the constitutive functions $(2.3)_{2,4}$ into the entropy inequality (2.2).

With the aid of the free energy function $\widetilde{\Psi}_C(\theta, w_\gamma)$

$$\tag{2.8} \widetilde{\Psi}_C(\theta, \mathbf{w}) = \widetilde{\varepsilon}(\theta, \mathbf{w}) - \theta\widetilde{\eta}(\theta, \mathbf{w}),$$

it has been derived in [22] that the entropy inequality (2.7) imposed on the system (2.6) implies

$$\tag{2.9} \eta(\theta, w_\gamma) = -\frac{\partial\widetilde{\Psi}_C(\theta, w)}{\partial\theta},$$

and the following inequality holds

$$\tag{2.10} -\frac{\partial\widetilde{\Psi}(\theta, w)}{\partial w_\alpha}\partial_t w_\alpha - \frac{1}{\varrho_0}\widetilde{q}^\alpha(\theta, w_\gamma)\partial_\alpha\theta \geq 0.$$

2.2.2. Assumption leading to the quasi-linear system of field equations. The additional general assumption that the function $\mathbf{g}(\theta, \operatorname{grad}\theta, \mathbf{w})$ is linear in $\operatorname{grad}\theta$, namely

$$\tag{2.11} \mathbf{g}(\theta, \operatorname{grad}\theta, \mathbf{w}) = \mathbf{M}(\theta, \mathbf{w})\operatorname{grad}\theta + \mathbf{b}(\theta, \mathbf{w}),$$

where $\mathbf{M}(\theta, \mathbf{w})$ is a tensor function of θ and $\mathbf{w}$, makes the system (2.6) quasi-linear but not in a conservative form.

Assuming that, besides (2.9), (2.10), no other restrictions are imposed, it is apparently possible to find specific restrictions on the components ${M_\beta}^\alpha(\theta, w_\gamma)$ of the matrix representation of $\mathbf{M}(\theta, \mathbf{w})$ and on the functions $\widetilde{\varepsilon}(\theta, w_\gamma)$, $\widetilde{q}^\alpha(\theta, w_\gamma)$ such that the system (2.6), (2.11) can be transformed into equivalent quasi-linear symmetric hyperbolic system by premultiplication (left multiplication) by a nonsingular 4×4 matrix $S^{IJ}(\theta, w_\gamma)$ $I, J = 1, 2, 3, 4$. It can be done simply by rewriting the system (2.6), (2.11) in a matrix form and requiring the resulting matrices to have common nonsingular left symmetrizer $S^{IJ}(\theta, w_\gamma)$ such that symmetrized matrix multiplying $[\partial_t\theta, \partial_t w_\gamma]^T$ is positive definite. In this context, the question

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arises wheather the restrictions on $M^{\alpha\beta}(\theta, w_\gamma)$, $\widetilde{\varepsilon}(\theta, w_\gamma)$ and $\widetilde{q}^\alpha(\theta, w_\gamma)$ imposed by this procedure are related to the restrictions imposed by the inequality (2.7). LEFLOCH [34, 35] introduced the concept of the additional conservation equation ("balance of entropy") for quasi-linear systems in non-conservative form in one spatial dimension and related the consistency of the non-conservative quasi-linear systems with the additional conservation equation to the existence of the left symmetrizer of this quasi-linear system in the form of matrix representation of transposed gradient of a vector function of the unknowns (gradient with respect to these unknowns). Therefore, for the system (2.6), (2.11) with assumed dependence of the unknowns θ, w_γ on only one spatial coordinate, the answer can be expected to be affirmative provided that it will be proved, for a non-conservative first-order quasi-linear system (at least in the one-dimensional case), that the restrictions imposed by consistency with the additional conservation equation ("balance of entropy") are equivalent to the restrictions imposed by the corresponding first-order differential unilateral constraints ("entropy inequality", like (2.7)). Hence, the problem is still open even in the one-dimensional case.

2.2.3. Conservative form of field equations. The system of field equations (2.6), (2.11) takes a conservative form if the next additional assumption that $\mathbf{M}(\theta, \mathbf{w})$ is an isotropic tensor function of the temperature is introduced since, without loss of generality, we may write

$$\mathbf{M}(\theta, \mathbf{w}) = -f'(\theta)\mathbf{I}, \qquad f'(\theta) = \frac{df(\theta)}{d\theta}, \qquad f'(\theta) \neq 0 \tag{2.12}$$

and, consequently, rearrange the evolution equation for $\mathbf{w}$ to the form

$$\dot{\mathbf{w}} = -\operatorname{div}(f(\theta)\mathbf{I}) + \mathbf{b}(\theta, \mathbf{w}). \tag{2.13}$$

The evolution equation for internal state variable with the source linear in $\mathbf{w}$

$$\mathbf{b}(\theta, \mathbf{w}) = -\mathbf{N}(\theta)\mathbf{w}, \tag{2.14}$$

and $\mathbf{N}(\theta)$ positive definite, was assumed for the model of a rigid conductor of heat proposed by MORRO and RUGGERI [4] and, therefore, it can be considered as a special case of the model developed previously in [22]. It has been shown in [4] that, in the case of evolution equation of the type (2.13), the entropy inequality (2.7) implies

$$\widetilde{q}^\alpha(\theta, w_\gamma) = \varrho_0 \theta f'(\theta) \frac{\partial \widetilde{\Psi}_C(\theta, w_\gamma)}{\partial w_\alpha}, \tag{2.15}$$

and

$$-\varrho_0 \frac{\partial \widetilde{\Psi}_C(\theta, w_\gamma)}{\partial w_\alpha} \widetilde{b}^\alpha(\theta, w_\gamma) \geq 0. \tag{2.16}$$

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To this end, we note that (2.8), (2.9) imply

$$\widetilde{\varepsilon}(\theta, w_\gamma) = \widetilde{\Psi}_C(\theta, w_\gamma) - \theta \frac{\partial \widetilde{\Psi}_C(\theta, w_\gamma)}{\partial \theta} \tag{2.17}$$

and therefore, the considered model of a rigid conductor of heat is entirely determined by the free energy $\widetilde{\Psi}(\theta, \mathbf{w})$, scalar function of the temperature $f(\theta)$ and the source terms $\widetilde{r}(\theta, \mathbf{w}, X^\alpha, t)$, $\mathbf{b}(\theta, \mathbf{w})$.

2.2.4. Further constitutive assumptions

A. $\widetilde{\mathbf{q}}(\theta, \mathbf{w})$ *linear in* $\mathbf{w}$

According to (2.15), the additional constitutive assumption that $\widetilde{\mathbf{q}}(\theta, \mathbf{w})$ is a linear function of $\mathbf{w}$

$$\widetilde{\mathbf{q}}(\theta, \mathbf{w}) = \mathbf{S}(\theta)\mathbf{w}, \tag{2.18}$$

with $\mathbf{S}(\theta)$ symmetric nonsingular tensor function of $\theta > 0$, is equivalent to the postulate that free energy takes the following special form:

$$\widetilde{\Psi}_C(\theta, \mathbf{w}) = \widetilde{\Psi}_{C0}(\theta) + \frac{1}{2\theta f'(\theta)\varrho_0} \mathbf{w} \cdot \mathbf{S}(\theta)\mathbf{w}. \tag{2.19}$$

As a consequence of this assumption, we obtain from (2.8), (2.9)

$$\begin{aligned}
\widetilde{\eta}(\theta, \mathbf{w}) &= \widetilde{\eta}_0(\theta) - \frac{1}{2\varrho_0}\mathbf{w} \cdot \left[\frac{d}{d\theta}\left(\frac{1}{\theta f'(\theta)}\mathbf{S}(\theta)\right)\right]\mathbf{w},\\
\widetilde{\varepsilon}(\theta, \mathbf{w}) &= \widetilde{\varepsilon}_0(\theta) + \frac{1}{2\varrho_0}\mathbf{w} \cdot \left[\frac{1}{\theta f'(\theta)}\mathbf{S}(\theta) - \theta\frac{d}{d\theta}\left(\frac{1}{\theta f'(\theta)}\mathbf{S}(\theta)\right)\right]\mathbf{w},\\
\widetilde{\eta}_0(\theta) &= -\widetilde{\Psi}'_{C0}(\theta), \qquad \widetilde{\varepsilon}_0(\theta) = \widetilde{\Psi}_{C0}(\theta) - \theta\widetilde{\Psi}'_{C0}(\theta).
\end{aligned} \tag{2.20}$$

In this case, the inequality (2.16) implies

$$-\varrho_0 \frac{1}{\theta f'(\theta)} \mathbf{w} \cdot \mathbf{S}(\theta)\widetilde{\mathbf{b}}(\theta, \mathbf{w}) \geq 0, \tag{2.21}$$

and, for the source term linear in $\mathbf{w}$ (2.14), it gives the following condition:

$$\varrho_0 \frac{1}{\theta f'(\theta)} \mathbf{w} \cdot \mathbf{S}(\theta)\mathbf{N}(\theta)\mathbf{w} \geq 0, \tag{2.22}$$

which requires $\mathbf{S}(\theta)\mathbf{N}(\theta)$ to be positive semi-definite for $f'(\theta) > 0$ or negative semi-definite for $f'(\theta) < 0$. This additional constitutive assumption (2.18), together with the assumption (2.14), was introduced in [4] and motivated by the requirement that, in stationary conditions defined as $\dot{\mathbf{q}} = \mathbf{0}$ and $\dot{\theta} = 0$, the evolution equation (2.13), (2.14) should coincide with Fourier's law assumed in the following form:

$$\mathbf{q} = -\mathbf{K}\,\mathrm{grad}\,\theta, \tag{2.23}$$

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where $\mathbf{K}$ is positive definite heat conductivity tensor which may be generalized to be temperature-dependent $\mathbf{K} = \mathbf{K}(\theta)$. In fact, taking the evolution equation (2.13), (2.14) for $\dot{\mathbf{w}} = \mathbf{0}$ (in view of (2.18), this corresponds to $\dot{\mathbf{q}} = \mathbf{0}$, $\dot{\theta} = 0$) and comparing with (2.23), one obtains

$$\mathbf{q}(\theta, \mathbf{w}) = \mathbf{S}(\theta)\mathbf{w}, \qquad \mathbf{S}(\theta) = \frac{1}{f'(\theta)}\mathbf{K}(\theta)\mathbf{N}(\theta). \tag{2.24}$$

With the additional assumptions (2.18), (2.14), the model considered is completely determined by the constitutive functions dependent only on the temperature $f(\theta)$, $\widetilde{\Psi}_{C0}(\theta)$, $\mathbf{S}(\theta)$, $\mathbf{N}(\theta)$, except the source term in the balance of energy. The corresponding system of equations, in view of (2.18), (2.20), (2.14), assumes the following form:

$$\begin{aligned} &\varrho_0 \partial_t \left\{ \widetilde{\Psi}_{C0}(\theta) - \theta \widetilde{\Psi}'_{C0}(\theta) + \frac{1}{2}\mathbf{w} \cdot \left[\frac{1}{f'(\theta)}\mathbf{S}(\theta) - \theta \frac{d}{d\theta}\left(\frac{1}{\theta f'(\theta)}\mathbf{S}(\theta) \right) \right] \cdot \mathbf{w} \right\} \\ &\qquad\qquad + \operatorname{div}(\mathbf{S}(\theta)\mathbf{w}) = \varrho_0 \widetilde{r}(\theta, \mathbf{w}), \\ &\partial_t \mathbf{w} + \operatorname{div}(f(\theta)\mathbf{I}) = -\mathbf{N}(\theta)\mathbf{w}, \end{aligned} \tag{2.25}$$

and is consistent with the entropy inequality

$$\begin{aligned} &\varrho_0 \partial_t \left\{ -\widetilde{\Psi}'_{C0}(\theta) - \frac{1}{2}\mathbf{w} \cdot \left[\frac{d}{d\theta}\left(\frac{1}{\theta f'(\theta)}\mathbf{S}(\theta) \right) \right] \mathbf{w} \right\} \\ &\qquad\qquad + \operatorname{div}\left[\frac{1}{\theta}(\mathbf{S}(\theta)\mathbf{w} \right] \geq \varrho_0 \frac{1}{\theta} \widetilde{r}(\theta, \mathbf{w}, t, X^\alpha) \end{aligned} \tag{2.26}$$

provided that the inequality (2.22) holds for all $\mathbf{w}$ and $\theta > 0$.

In the case of "thermally" isotropic body, the relations (2.18) – (2.20) simplify according to the well known representation theorems for tensor functions

$$\mathbf{S}(\theta) = \alpha(\theta)\mathbf{I}, \qquad \mathbf{N}(\theta) = \nu(\theta)\mathbf{I}, \qquad \mathbf{K}(\theta) = \kappa(\theta)\mathbf{I}. \tag{2.27}$$

B. $\widetilde{\varepsilon}(\theta, \mathbf{w}) = \widetilde{\varepsilon}_0(\theta)$

The next simplifying assumption introduced in [4] together with the assumptions (2.14), (2.18) is that internal energy $\widetilde{\varepsilon}(\theta, \mathbf{w})$ does not depend on the internal state variable $\mathbf{w}$

$$\widetilde{\varepsilon}(\theta, \mathbf{w}) = \widetilde{\varepsilon}_0(\theta). \tag{2.28}$$

According to $(2.20)_2$, the following equation for $\mathbf{S}(\theta)$ results as a consequence of this assumption:

$$\mathbf{S}(\theta) = \theta^2 f(\theta) \frac{d}{d\theta}\left[\frac{1}{\theta f'(\theta)}\mathbf{S}(\theta) \right]. \tag{2.29}$$

In [4], the following solution of (2.29) was found

$$\mathbf{S}(\theta) = \theta^2 f(\theta)\mathbf{B}, \tag{2.30}$$

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where $\mathbf{B}$ is a positive definite constant tensor, and, according to (2.24), it relates the source factor $\mathbf{N}(\theta)$ to the temperature-dependent heat conductivity $\mathbf{K}(\theta)$

$$\mathbf{N}(\theta) = \theta^2[f'(\theta)]^2\mathbf{K}^{-1}(\theta)\mathbf{B}. \tag{2.31}$$

It follows from (2.19), (2.20), (2.15) that (2.30), (2.31) additionally imply

$$\begin{aligned} \widetilde{\Psi}_{C0}(\theta,\mathbf{w}) &= \widetilde{\Psi}_{C0}(\theta) + \frac{1}{2\varrho_0}\theta\mathbf{w}\cdot\mathbf{B}\mathbf{w}, \\ \widetilde{\eta}(\theta,\mathbf{w}) &= -\frac{\partial\widetilde{\Psi}(\theta,\mathbf{w})}{\partial\theta} = \widetilde{\eta}_0(\theta) - \frac{1}{2\varrho_0}\mathbf{w}\cdot\mathbf{B}\mathbf{w}, \\ \widetilde{\eta}_0(\theta) &= -\widetilde{\Psi}'_{C0}(\theta), \qquad \widetilde{\varepsilon}(\theta,\mathbf{w}) = \widetilde{\varepsilon}(\theta) = \widetilde{\Psi}_{C0}(\theta) - \theta\widetilde{\Psi}'_{C0}(\theta), \end{aligned} \tag{2.32}$$

$$\widetilde{\mathbf{q}}^{\alpha}(\theta, w^{\gamma}) = \varrho_0\theta f'(\theta)\frac{\partial\widetilde{\Psi}(\theta,\mathbf{w}_{\gamma})}{\partial w_{\alpha}} = \theta^2 f'(\theta)\mathbf{B}\mathbf{w}, \tag{2.33}$$

and the system of field equations (2.25) further simplifies

$$\begin{aligned} \varrho_0\partial_t\widetilde{\varepsilon}_0(\theta) + \operatorname{div}\varrho_0\theta^2 f'(\theta)\mathbf{B}\mathbf{w}) &= \varrho_0\widetilde{r}(\theta,\mathbf{w}), \\ \partial_t\mathbf{w} + \operatorname{div}(f(\theta)\mathbf{I}) = -\mathbf{N}(\theta)\mathbf{w} &= -\theta^2[f'(\theta)]^2\mathbf{K}^{-1}(\theta)\mathbf{B}\mathbf{w}. \end{aligned} \tag{2.34}$$

In the isotropic case (2.27), Eqs.(2.30), (2.31) yield

$$\alpha(\theta) = c_2\theta^2 f(\theta), \qquad c_2 = \text{const}, \tag{2.35}$$

$$\nu(\theta) = c_2\theta^2[f'(\theta)]^2[\kappa(\theta)]^{-1} \tag{2.36}$$

and the corrsponding field equations are

$$\begin{aligned} \varrho_0\partial_t\widetilde{\varepsilon}_0(\theta) + \operatorname{div}(c_2 f'(\theta)\mathbf{w}) &= \varrho_0\widetilde{r}(\theta,|\mathbf{w}|), \\ \partial_t\mathbf{w} + \operatorname{div}(f(\theta)\mathbf{I}) = -\nu(\theta)\mathbf{w} &= -c_2\frac{\theta^2[f'(\theta)]^2}{\kappa(\theta)}\mathbf{w}. \end{aligned} \tag{2.37}$$

In [4], the systems (2.34) and (2.37) were transformed, with the aid of (2.18), (2.24), (2.36), to the corresponding equivalent systems with respect to $\theta, \mathbf{q}$. Then in [5, 6], the transformed system (2.37) was alternatively derived as a special (linear in $\mathbf{q}$) version of the extended thermodynamics of a rigid conductor of heat, and successively applied for phenomenological modelling of some observed features of the second sound propagation in dielectric crystals.

2.2.5. Comparison with the Maxwell–Cattoneo–Vernotte equation. Assumption that $\widetilde{q}(\theta,\mathbf{w})$ is linear in $\mathbf{w}$ together with the assumption that the source term $\mathbf{b}(\theta,\mathbf{w})$ is also linear in $\mathbf{w}$, make it possible to express the evolution equation for $\mathbf{w}$ equivalently as the evolution equation for the heat flux vector $\mathbf{q}$, and to replace the corresponding constitutive functions (2.19), (2.20) and by $\widehat{\varepsilon}(\theta,\mathbf{q}) = \widetilde{\varepsilon}(\theta,\mathbf{S}^{-1}(\theta)\mathbf{q})$,

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$\widehat{\eta}(\theta, \mathbf{q}) = \widetilde{\eta}(\theta, \mathbf{S}^{-1}(\theta)\mathbf{q})$, and $\widehat{\psi}_C(\theta, \mathbf{q}) = \widetilde{\Psi}_C(\theta, \mathbf{S}^{-1}(\theta)\mathbf{q})$. Substitution of (2.18) into (2.13), (2.14) yields the following evolution equation for the heat flux vector

$$(\mathbf{S}^{-1}(\theta)\mathbf{q})^{\cdot} + \operatorname{div}(f(\theta)\mathbf{I}) = -\mathbf{N}(\theta)\mathbf{S}^{-1}(\theta)\mathbf{q}. \tag{2.38}$$

This fact leads to the question, under which further additional constitutive assumptions the evolution equation for the heat flux (2.38), coincides with (or can be reduced to) the Maxwell - Cattoneo - Vernotte equation

$$\tau\dot{\mathbf{q}} + \mathbf{q} = -\kappa \operatorname{grad}\theta, \tag{2.39}$$

where $\tau > 0$ is constant thermal relaxation time and $\kappa > 0$ is constant heat conductivity, or with its generalization proposed by PAO and BANERJEE [23]

$$\mathbf{T}(\theta)\dot{\mathbf{q}} + \mathbf{q} = -\mathbf{K}(\theta) \operatorname{grad}\theta, \tag{2.40}$$

where $\mathbf{T}(\theta)$ is a temperature-dependent thermal relaxation tensor and $\mathbf{K}(\theta)$ is a temperature-dependent heat conductivity tensor. For arbitrary positive definite $\mathbf{S}(\theta)$ and for arbitrary $f(\theta)$, the evolution equation (2.38), when rearranged to the respective form similar to (2.40)

$$\begin{aligned}\mathbf{S}(\theta)\mathbf{N}^{-1}(\theta)\mathbf{S}^{-1}(\theta)\dot{\mathbf{q}} + \left\{\mathbf{I} - \mathbf{S}(\theta)\mathbf{N}^{-1}(\theta)\mathbf{S}^{-1}(\theta)\left[\frac{d}{d\theta}\mathbf{S}(\theta)\right]\mathbf{S}^{-1}(\theta)\dot{\theta}\right\}\mathbf{q} \\ = -\mathbf{S}(\theta)\mathbf{N}^{-1}f'(\theta) \operatorname{grad}\theta,\end{aligned} \tag{2.41}$$

contains a term proportional to $\dot{\theta}$. It takes the form (2.40) if

$$\mathbf{S}(\theta) = \mathbf{S}_0\,, \qquad \alpha(\theta) = \alpha_0\,, \tag{2.42}$$

and the following identification holds

$$\mathbf{T}(\theta) = \mathbf{S}_0\mathbf{N}^{-1}(\theta)\mathbf{S}_0^{-1}\,, \qquad \mathbf{K}(\theta) = f'(\theta)\mathbf{S}_0\mathbf{N}^{-1}(\theta). \tag{2.43}$$

In this case, the evolution equation (2.41) for the heat flux vector takes the form

$$\frac{1}{f'(\theta)}\mathbf{K}(\theta)\mathbf{S}_0^{-1}\dot{\mathbf{q}} + \mathbf{q} = -\mathbf{K}(\theta) \operatorname{grad}\theta, \tag{2.44}$$

while the corresponding evolution equations for the internal state variable is

$$\dot{\mathbf{w}} + \operatorname{div}(f(\theta)\mathbf{I}) = -f'(\theta)\mathbf{K}^{-1}(\theta)\mathbf{S}_0\mathbf{w} = -\mathbf{N}(\theta)\mathbf{w}. \tag{2.45}$$

The model is completely determined by prescribing $f(\theta), \widetilde{\Psi}_{C0}(\theta)$, positive definite constant tensor $\mathbf{S}_0$, the source factor $\mathbf{N}(\theta)$ related to $\mathbf{S}_0$, $\mathbf{K}(\theta)$ and $f'(\theta)$ and the

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source term $\widetilde{r}(\theta, \mathbf{w}, t, X^\alpha)$, and, according to (2.42), (2.43), (2.18), (2.19),

$$
\begin{aligned}
\widetilde{\Psi}_{C0}(\theta, \mathbf{w}) &= \widetilde{\Psi}_{C0}(\theta) + \frac{1}{2\varrho_0\theta f'(\theta)}\mathbf{w}\cdot\mathbf{S}_0\mathbf{w},\\
\widetilde{\eta}(\theta, \mathbf{w}) &= \widetilde{\eta}_0(\theta) - \frac{1}{2\varrho_0}\frac{d}{d\theta}\left(\frac{1}{\theta f'(\theta)}\right)\mathbf{w}\cdot\mathbf{S}_0\mathbf{w},\\
\widetilde{\varepsilon}(\theta, \mathbf{w}) &= \widetilde{\varepsilon}_0(\theta) + \frac{1}{2\varrho_0}\left[\frac{1}{\theta f'(\theta)} - \theta\frac{d}{d\theta}\left(\frac{1}{\theta f'(\theta)}\right)\right]\mathbf{w}\cdot\mathbf{S}_0\mathbf{w},\\
\widetilde{q}^\alpha(\theta, w_\gamma) &= \varrho_0\theta f'(\theta)\frac{\partial\widetilde{\Psi}_C(\theta, w_\gamma)}{\partial w_\alpha} = S_0^{\alpha\beta}w_\beta.
\end{aligned}
\tag{2.46}
$$

It follows from (2.44) that the model of a rigid conductor of heat with vector internal state variable subject to additional constitutive assumptions (2.14), (2.18), (2.42) is not able to reproduce the generalization of the Maxwell-Cattaneo-Vernotte equation (2.40) since the relations (2.43) imply $\mathbf{T}(\theta) = \frac{1}{f'(\theta)}\mathbf{K}(\theta)\mathbf{S}_0^{-1}$, thus admitting only such temperature-dependent thermal relaxation tensors $\mathbf{T}(\theta)$ which are related to $\mathbf{K}(\theta)$ by scalar function of θ and positive definite constant tensor. The thermodynamic restrictions imposed on $\mathbf{T}(\theta)$ and $\mathbf{K}(\theta)$ by Clausius-Duhem entropy inequality were investigated by COLEMAN, FABRIZIO and OWEN [16, 17] and it was shown that $\mathbf{K}(\theta)$ should be positive definite and $\mathbf{K}^{-1}(\theta)\mathbf{T}(\theta)$ should be symmetric. The model (2.45), (2.46), (2.43) satisfies those conditions provided that $\mathbf{K}(\theta) = f'(\theta)\mathbf{S}_0\mathbf{N}^{-1}(\theta)$ is positive definite (see also (2.22)) while $\mathbf{K}^{-1}(\theta)\mathbf{T}(\theta) = \frac{1}{f'(\theta)}\mathbf{S}_0^{-1}$ is symmetric since $\mathbf{S}_0$ is positive definite.

The Maxwell-Cattaneo-Vernotte equation (2.39) is recovered if we take the isotropic case (2.27) and put

$$
f(\theta) = \theta, \quad \nu(\theta) = \frac{1}{\tau} = \text{const}, \quad \alpha(\theta) = \alpha_0, \quad \kappa(\theta) = \alpha_0\tau = \kappa = \text{const}.
\tag{2.47}
$$

Substitution of (2.47) into (2.19), (2.20) yields

$$
\begin{aligned}
\widetilde{\Psi}_C(\theta, |\mathbf{w}|) &= \widetilde{\Psi}_{C0}(\theta) + \frac{\kappa}{2\varrho_0\theta\tau}|\mathbf{w}|^2,\\
\widetilde{\eta}(\theta, |\mathbf{w}|) &= \widetilde{\eta}_0(\theta) - \frac{\kappa}{2\varrho_0\theta^2\tau}|\mathbf{w}|^2,\\
\widetilde{\varepsilon}(\theta, |\mathbf{w}|) &= \widetilde{\varepsilon}_0(\theta) + \frac{\kappa}{\varrho_0\theta\tau}|\mathbf{w}|^2,\\
\widetilde{\eta}_0(\theta) &= -\widetilde{\Psi}'_{C0}(\theta), \qquad \widetilde{\varepsilon}_0(\theta) = \widetilde{\Psi}_{C0}(\theta) - \theta\widetilde{\Psi}'_{C0}(\theta),\\
\widetilde{q}^\alpha(\theta, \mathbf{w}) &= \varrho_0\theta\frac{\partial\widetilde{\Psi}_C(\theta, |\mathbf{w}|)}{\partial w_\alpha} = \frac{\kappa}{\tau}w^\alpha,
\end{aligned}
\tag{2.48}
$$

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and the system of field equations with respect to θ, $\mathbf{w}$ for a rigid conductor of heat described by the Maxwell-Cattaneo-Vernotte equation (2.39) can be obtained by substituting (2.47), $(2.48)_{3,5}$ into (2.25), (2.24)

$$\begin{aligned} \varrho_0 \partial_t \left[\varepsilon_0(\theta) + \frac{\kappa}{\varrho_0 \theta \tau}|\mathbf{w}|^2\right] + \operatorname{div}\left(\frac{\kappa}{\tau}\mathbf{w}\right) &= \varrho_0 \widetilde{r}(\theta, \mathbf{w}, t, X^\alpha), \\ \partial_t \mathbf{w} + \operatorname{div}(\theta \mathbf{I}) &= -\frac{1}{\tau}\mathbf{w}. \end{aligned} \tag{2.49}$$

This description of a rigid conductor of heat governed by the Maxwell-Cattaneo-Vernotte equation in terms of internal state variable was obtained by KOSIŃSKI [22] as a special case of the general model of a deformable (inelastic) conductor of heat with internal state variables. Substituting $(2.48)_5$ into $(2.48)_{2,3}$ we may express internal energy $\widetilde{\varepsilon}(\theta, |\mathbf{w}|)$ and entropy $\widetilde{\eta}(\theta, |\mathbf{w}|)$ as $\widehat{\varepsilon}(\theta, |\mathbf{q}|)$ and $\widehat{\eta}(\theta, |\mathbf{q}|)$, respectively, and then calculate

$$d\widehat{\eta} = \frac{1}{\theta}\frac{\partial \widehat{\eta}}{\partial \theta}d\theta + \frac{\tau}{2\varrho_0 \kappa \theta^2} q_\beta dq^\beta = \frac{1}{\theta}d\widehat{\varepsilon} - \frac{\tau}{2\varrho_0 \kappa \theta^2} q_\beta dq^\beta, \tag{2.50}$$

what agrees with the result of the analysis of the Maxwell-Cattaneo-Vernotte equation performed in the framework of EIT by JOU and CASSAS-VAZQUEZ [36]. It follows from $(2.48)_3$, $(2.46)_3$ that the model of a rigid conductor of heat with vector internal state variables satisfying additional assumptions (2.14), (2.18) and subject to the requirement $\widetilde{\varepsilon}(\theta, \mathbf{w}) = \widetilde{\varepsilon}_0(\theta)$ (discussed in Sec. 2.2.4.B) cannot reproduce the Maxwell-Cattaneo-Vernotte equation (2.39), (2.48), (2.49) but it can reproduce a special form of the generalized Maxwell-Cattaneo-Vernotte equation (2.40), (2.43) for $f(\theta) = -c/\theta$ (c – const) and, consequently, in this case the relation $\mathbf{T}(\theta) = \dfrac{\theta^2}{c}\mathbf{K}(\theta)\mathbf{S}_0^{-1}$ must be satisfied. This special form of the generalized Maxwell-Cattaneo-Vernotte equation was discussed in [16, 17, 23]. Hence, the model of a rigid conductor with vector internal state variable (2.32)-(2.34) based on all three assumptions (2.13), (2.14), (2.18), (2.28), developed in [4, 5, 6], cannot be considered as a "generalization" of the Maxwell-Cattaneo-Vernotte equation but should be understood as the "alternative" to the generalized Maxwell-Cattaneo-Vernotte equation (2.40).

2.2.6. Generalization of the the Maxwell–Cattoneo–Vernotte equation and other specific models through the model with internal state variables. Further analysis will be focused on the model described by the equation $(2.6)_1$ corresponding to the balance of energy, the evolution equation for the internal state variable $\mathbf{w}$ in the form (2.13) and the inequality (2.7). In order to emphasize the relation to the model of a rigid conductor of heat with "semi-empirical temperature" discussed in the next section and the resemblance to the Maxwell-Cattaneo-Vernotte equation, we denote

$$f_1(\theta) := -\tau f(\theta), \qquad \mathbf{c}(\theta, \mathbf{w}) = \tau \mathbf{b}(\theta, \mathbf{w}), \qquad \tau = \text{const} \tag{2.51}$$

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and, substituting (2.51) into (2.13), $(2.6)_1$, (2.7), we rewrite the system of corresponding field equations together with Clausius–Duhem entropy inequality in the following form:

$$\begin{aligned} \varrho_0 \partial_t \widetilde{\varepsilon}(\theta, w_\beta) + \partial_\alpha \widetilde{q}^\alpha(\theta, w_\beta) &= \varrho_0 \widetilde{r}(\theta, w_\beta), \\ \tau \partial_t w_\gamma - \partial_\alpha [f_1(\theta) \delta^\alpha{}_\gamma] &= c_\gamma(\theta, w_\beta), \end{aligned} \tag{2.52}$$

$$\varrho_0 \sigma := \varrho_0 \partial_t \widetilde{\eta} + \partial_\alpha \left[\frac{1}{\theta} \widetilde{q}^\alpha(\theta, w_\beta) \right] \geq \varrho_0 \frac{1}{\theta} \widetilde{r}(\theta, \mathbf{w}, t, X^\alpha). \tag{2.53}$$

In (2.51), (2.52) constant $\tau > 0$ can be interpreted as a thermal relaxation time. According to the analysis performed in Secs. 2.2.3, 2.2.4, 2.2.5, the model corresponding to the field equations (2.52) and consistent with the inequality (2.53) is completely determined by prescribing $f_1(\theta)$, constant τ, free energy function $\widetilde{\Psi}_C(\theta, \mathbf{w})$ which, without loss of generality, may be assumed as

$$\widetilde{\Psi}_C(\theta, \mathbf{w}) = \widetilde{\Psi}_{C0}(\theta) + \widetilde{\Psi}_1(\theta, \mathbf{w}), \tag{2.54}$$

and the source terms $\widetilde{r}(\theta, w_\beta, t, X^\alpha)$, $c_\gamma(\theta, w_\beta)$. This model comprises all particular models discussed in Secs. 2.2.4, 2.2.5 as special cases corresponding to specific forms of constitutive quantities $f_1(\theta)$, $\widetilde{\Psi}_1(\theta, \mathbf{w})$ and $\mathbf{c}(\theta, \mathbf{w})$. For comparison, those specific forms are presented in the Table 1.

The system of field equations (2.52) and the entropy inequality (2.53) can be considered as a representation of the class of phenomenological models of heat transport at finite speeds containing posible generalizations of both the mentioned models of practical applicability. Constitutive functions $f_1(\theta)$, $\widetilde{\Psi}_1(\theta, \mathbf{w})$, $\mathbf{c}(\theta, \mathbf{w})$ can be arranged to the form representing explicitly the corrections to, or derivations from those models. For example, taking $f_1(\theta) = -\tau\theta + m(\theta)$, $\widetilde{\Psi}_1(\theta, \mathbf{w}) = \dfrac{\kappa}{2\varrho\theta\tau}|\mathbf{w}|^2 + \xi(\theta, \mathbf{w})$ and $\mathbf{c}(\theta, \mathbf{w}) = -\mathbf{w} + \mathbf{d}(\theta, \mathbf{w})$, we may introduce corrections to the Maxwell–Cattoneo–Vernotte equation by means of functions $m(\theta)$, $\xi(\theta, \mathbf{w})$ and $\mathbf{d}(\theta, \mathbf{w})$.

In the remaining part of this paper, the source term in $(2.52)_2$ will be assumed as

$$\mathbf{c}(\theta, \mathbf{w}) = \mathbf{c}_1 \mathbf{w}, \qquad c_1 = \text{const}. \tag{2.55}$$

This simplifying assumption does not restrict the generality of further analysis concerning the application of the MDR for derivation of the thermodynamic restrictions, and of the procedures of symmetrization (together with conditions of symmetric hyperbolicity) applied in Part II. To obtain the results valid for arbitrary $\mathbf{c}(\theta, \mathbf{w})$ it suffices simply to replace $c_1\mathbf{w}$ by $\mathbf{c}(\theta, \mathbf{w})$. Hence, the symmetric systems obtained in Part II for particular source term (2.55) also apply to general $\mathbf{c}(\theta, \mathbf{w})$ if this replacement is done.

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Table 1.

Specific model (additional constitutive assumption)	$f_1(\theta)$	$\widetilde{\Psi}_1(\theta,\mathbf{w})$	$\mathbf{c}(\theta,\mathbf{w})$
(2.18), (2.24) $\widetilde{q}(\theta,\mathbf{w}) = \mathbf{S}(\theta)\mathbf{w}$ [4]	no specific form	$-\dfrac{\tau}{2\varrho_0\theta f_1'(\theta)}\mathbf{w}\cdot\mathbf{S}(\theta)\mathbf{w}$ (isotropic case) $-\dfrac{\tau\alpha(\theta)}{2\varrho_0\theta f_1'(\theta)}\lvert\mathbf{w}\rvert^2$	$f_1'(\theta)\mathbf{K}^{-1}(\theta)\mathbf{S}(\theta)\mathbf{w}$ (isotropic case) $\dfrac{f_1'(\theta)\alpha(\theta)}{\kappa(\theta)}\mathbf{w}$
(2.18), (2.24), (2.28) $\widetilde{q}(\theta,\mathbf{w}) = \mathbf{S}(\theta)\mathbf{w}$ and $\widetilde{\varepsilon}(\theta,\mathbf{w}) = \widetilde{\varepsilon}_0(\theta)$ [4, 5, 6],	no specific form	$\dfrac{1}{2\varrho_0}\theta\mathbf{w}\cdot\mathbf{B}\mathbf{w}$ (isotropic case) $\dfrac{1}{2\varrho_0}c_2\theta\lvert\mathbf{w}\rvert^2$	$-\dfrac{\theta^2[f_1'(\theta)]^2}{\tau}\mathbf{K}^{-1}(\theta)\mathbf{B}\mathbf{w}$ (isotropic case) $-\dfrac{c_2\theta^2[f_1'(\theta)]^2}{\tau\kappa(\theta)}\mathbf{w}$
(2.42), (2.43) Generalized Maxwell – Cattoneo – Vernotte equation [16, 17, 23]	no specific form	$-\dfrac{\tau}{2\varrho_0\theta f_1'(\theta)}\mathbf{w}\cdot\mathbf{S}_0\mathbf{w}$ (isotropic case) $-\dfrac{\tau\alpha_0}{2\varrho_0\theta f_1'(\theta)}\lvert\mathbf{w}\rvert^2$	$f_1'(\theta)\mathbf{K}^{-1}(\theta)\mathbf{S}_0\mathbf{w}$ (isotropic case) $\dfrac{\alpha_0 f_1'(\theta)}{\kappa(\theta)}\mathbf{w}$
(2.42), (2.43), (2.28) Generalized Maxwell – Cattoneo – Vernotte equation with $\widetilde{\varepsilon}(\theta,\mathbf{w}) = \widetilde{\varepsilon}_0(\theta)$ [16, 17]	$\dfrac{c_2\tau}{\theta}$	$\dfrac{c_2}{2\varrho_0}\theta\mathbf{w}\cdot\mathbf{S}_0\mathbf{w}$ (isotropic case) $\dfrac{{c_2}^2\alpha_0\theta}{2\varrho_0}\lvert\mathbf{w}\rvert^2$	$-\dfrac{{c_2}^2}{\theta^2}\mathbf{K}^{-1}(\theta)\mathbf{S}_0$ (isotropic case) $-\dfrac{{c_2}^2\alpha_0}{\theta^2\kappa(\theta)}$
(2.48), (2.49) Maxwell – Cattoneo – Vernotte equation [22]	$-\tau\theta$	$\dfrac{\kappa}{2\varrho_0\theta\tau}\lvert\mathbf{w}\rvert^2$	$-\mathbf{w}$

2.3. Relation to the model of a rigid heat conductor with "semi-empirical temperature"

Recently in a series of papers: KOSIŃSKI [24], CIMMELLI and KOSIŃSKI [25, 26], KOSIŃSKI and SAXTON [28], CIMMELLI, KOSIŃSKI and SAXTON [27, 29], a model of heat conduction with finite wave speed based on the concept of "semi-empirical temperature" has been developed. In this approach, a scalar internal state variable β called "semi-empirical temperature" is introduced and the equation relating the evolution of β to the temperature is postulated

$$\tau\dot{\beta} = f(\theta,\beta), \tag{2.56}$$

where the presence of the constant dimension parameter τ (thermal relaxation time) is motivated by dimensional analysis [27]. According to [25, 27], β is uniquely defined by (2.56) if a suitable initial condition

$$\beta(t_0, X^\alpha) = \beta_0(X^\alpha) \tag{2.57}$$

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is given, and $f(\cdot,\cdot)$ is Lipschitz continuous. It is also assumed [25, 27] that

$$\frac{\partial f}{\partial\theta} > 0, \qquad \frac{\partial f}{\partial\beta} \le 0 \tag{2.58}$$

since $(2.58)_1$ ensures the stability of the solutions of (2.56), (2.57). For $\tau = 0$, the inequality $(2.58)_2$ makes β an increasing function of θ and ensures the order relation between different temperatures.

In [25, 27], a rigid conductor of heat with "semi-empirical temperature" was considered with the following constitutive assumptions:

$$\begin{aligned} \varepsilon &= \widehat{\varepsilon}(\theta, \operatorname{grad}\beta),\\ \mathbf{q} &= \widehat{\mathbf{q}}(\theta, \operatorname{grad}\beta),\\ r &= \widehat{r}(\theta, \operatorname{grad}\beta, t, X^{\alpha}),\\ \eta &= \widehat{\eta}(\theta, \operatorname{grad}\beta), \end{aligned} \tag{2.59}$$

and the analysis performed in [27] showed that, in this case, $f(\theta,\beta)$ must be of the form

$$f(\theta,\beta) = f_1(\theta) + f_2(\beta). \tag{2.60}$$

Substitution of constitutive functions (2.59) into (2.1) and substitution of (2.60) into (2.56) yield the following system of two field equations for θ, β:

$$\begin{aligned} \varrho_0\dot{\widehat{\varepsilon}}(\theta, \operatorname{grad}\beta) + \operatorname{div}\widehat{\mathbf{q}}(\theta, \operatorname{grad}\beta) &= \varrho_0\widehat{r}(\theta, \operatorname{grad}\beta, t, X^{\alpha}),\\ \tau\dot{\beta} &= f_1(\theta) + f_2(\beta), \end{aligned} \tag{2.61}$$

subject to differential unilateral constraints resulting from substitution of (2.59) into the Clausius–Duhem entropy inequality (2.2)

$$\varrho_0\sigma := \varrho_0\dot{\widehat{\eta}}(\theta, \operatorname{grad}\beta) + \operatorname{div}\left[\frac{1}{\theta}\widehat{\mathbf{q}}(\theta, \operatorname{grad}\beta)\right] \ge \varrho_0\frac{1}{\theta}\widetilde{r}(\theta, \operatorname{grad}\beta, t, X^{\alpha}). \tag{2.62}$$

Assumption $(2.58)_1$ together with (2.60) enables one to invert the function $f_1(\cdot)$ ($f_1^{-1}(\cdot)$ denotes the inverse) and express θ in terms of β, $\dot{\beta}$, namely $\theta = f^{-1}(\tau\dot{\beta} - f_2(\beta))$, and, consequently, substitute $\theta = \widehat{\theta}(\beta,\dot{\beta})$ into (2.61), (2.62) thus obtaining second-order nonlinear partial differential equation for β subject to the second-order unilateral differential constraints. The alternative approach is to express the system (2.61) as an equivalent first-order quasi-linear system of partial differential equations. In general case, the obtained first-order system will be subject, besides the differential inequality resulting from (2.62), also to both evolutive (involving time derivative) and involutive (involving only spatial derivatives) constraints. In [28], the corresponding first-order system was derived for particular form of $f(\cdot,\cdot)$ in (2.56) while in [27], such system was discussed in the context

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of comparison of the model of a rigid conductor of heat with "semi-empirical temperature" and the model proposed by MORRO and RUGGIERI [4, 5, 6] expressed in terms of θ, $\mathbf{q}$ (see Sec. 2.2.4). For this comparison, the additional constitutive assumptions $\widehat{\mathbf{q}}(\theta, \operatorname{grad}\beta) = \alpha(\theta)\operatorname{grad}\beta$ and $\widehat{\varepsilon}(\theta, \operatorname{grad}\beta) = \widetilde{\varepsilon}(\theta)$ (corresponding to (2.18) and (2.28), respectively) were introduced. In [27], the new variable

$$\mathbf{w} := \operatorname{grad}\beta, \tag{2.63}$$

which is suggested in view of constitutive assumptions (2.59), together with the "prolonged" evolution equation obtained by spatial differentiation of $(2.61)_2$

$$\partial_\alpha[\tau\partial_t\beta] = f_1'(\theta)\partial_\alpha\theta + f_2'(\beta)\partial_\alpha\beta = \tau\partial_t[\partial_\alpha\beta], \quad \alpha = 1, 2, 3 \tag{2.64}$$

were employed in derivation of the equivalent first-order system. Substituting (2.63) into (2.64) and into $(2.61)_1$ we obtain the system of 4 field equations with the left-hand side exactly the same as the left-hand side of the system (2.52) and with the right-hand side of the evolution equation for $\mathbf{w}$ (resulting from (2.63), (2.64)) of the form $f_2'(\beta)\mathbf{w}$. Hence, the condition $f_2(\beta) = c_1\beta$, c_1 – constant, ((2.58) implies $c_1 \leq 0$) enables the transformation of the system (2.61) and the inequality (2.62) into the equivalent quasi-linear first-order conservative system for $\theta, \mathbf{w}$

$$\begin{aligned} \varrho_0\dot{\widehat{\varepsilon}}(\theta, \mathbf{w}) + \operatorname{div}\widehat{\mathbf{q}}(\theta, \mathbf{w}) &= \varrho_0\widehat{r}(\theta, \mathbf{w}, t, X^\alpha), \\ \tau\dot{w} - \operatorname{div}(f_1(\theta)\mathbf{I}) &= c_1\mathbf{w}, \end{aligned} \tag{2.65}$$

subject to the inequality

$$\varrho_0\sigma := \dot{\widehat{\eta}}(\theta, \mathbf{w}) + \operatorname{div}\left[\frac{1}{\theta}\widehat{\mathbf{q}}(\theta, \mathbf{w})\right] \geq \varrho_0\frac{1}{\theta}\widetilde{r}(\theta, \mathbf{w}, t, X^\alpha) \tag{2.66}$$

and involutive constraints implied by (2.63)

$$\partial_\alpha w_\beta - \partial_\beta w_\alpha = 0, \qquad \alpha, \beta = 1, 2, 3, \quad \alpha \neq \beta. \tag{2.67}$$

The constraints (2.67) enable integration of $(2.65)_2$ to the following evolution equation for a scalar β related to $\mathbf{w}$ by (2.63)

$$\tau\partial_t\beta = f_1(\theta) + c_1\beta. \tag{2.68}$$

Since the systems (2.65), (2.66) and (2.52), (2.53) for $c_1 = \text{const}$ have exactly the same form, the system (2.52), (2.53) with $c_1 = \text{const}$, when subject to involutive constraints (2.67), can be regarded as corresponding to the model of a rigid conductor of heat with "semi-empirical temperature" such that the evolution of the "semi-empirical temperature" is governed by (2.68), and this correspondence

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holds without any other additional constitutive assumptions (like $\mathbf{q} = \alpha(\theta)$, $\mathbf{w} = \alpha(\theta)\mathrm{grad}\beta$, $\varepsilon = \widehat{\varepsilon}(\theta)$ employed in [4, 5, 6, 17].

Properties of quasi-linear conservative systems with involutive constraints and entropy inequality or additional conservation equation were investigated by DAFERMOS [37], GODUNOV [38], BOILLAT [39–43] and STRUMIA [44, 45]. It is well known (for example, [38, 44]) that the involutive constraints hold for solutions at each time provided that they are satisfied by initial conditions. Hence, among the solutions $\theta(t, X^\alpha)$, $\mathbf{w}(t, X^\alpha)$ of the system (2.52), (2.53), a special class corresponding to the model with "semi-empirical temperature" can be distinguished. This class corresponds to initial value problems satisfying constraints (2.67) and, as a consequence, can be alternatively expressed in terms of the solutions $\beta(t, X^\alpha)$ of the respective initial value problems (2.57) (supplemented by required initial conditions for derivatives of β) for the third order nonlinear partial differential equation resulting from substitution of (2.68) into $(2.61)_1$. The involutive constraints do not affect the propagation speeds of weak discontinuity waves but may influence the shocks [39, 43]. Therfore, both models predict the same speeds of thermal waves of weak dicontinouity but may lead to different thermal shock behaviour. The involutive constraints (2.67) are irrelevant for the symmetrization of field equations (2.65) if the symmetric system for original field variables $\theta, \mathbf{w}$ is sought, but if the system of field equations is symmetrized by means of transformation of dependent variables then the involutive constraints (2.67) should be taken into account according to the procedures derived in [38–43].

3. The "main dependency relation" (MDR) and the restrictions on constitutive functions

3.1. The MDR

The general analysis of overdetermined systems of conservation equations has been provided by FREDRICHS [7]. In order to ensure the consistency of an overdetermined system of conservation equations ($N+1$ equations for N unknowns)

$$\begin{gathered}\partial_t g^{0\Lambda}(u^K) + \partial_\alpha g^{\alpha\Lambda}(u^K) = b^\Lambda(u^K, t, X^\gamma),\\ \Lambda = 1, 2, \ldots, N+1, \qquad K = 1, 2, \ldots, N, \qquad \alpha = 1, 2, \ldots, m\end{gathered} \tag{3.1}$$

he has introduced the MDR which requires the existence of $N+1$ functions $y_\Lambda : \mathbb{R}^N \to \mathbb{R}$, $\Lambda = 1, 2, \ldots, N+1$, not all identically zero, such that (Property CI in [7])

$$\begin{aligned} y_\Lambda(u^K)\frac{\partial g^{0\Lambda}(u^K)}{\partial u^M}\partial_t u^M + y_\Lambda(u^K)\frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M}\partial_\alpha u^M &\equiv 0,\\ y_\Lambda(u^K) b^K(u^K, t, X^\alpha) &\equiv 0,\end{aligned} \tag{3.2}$$

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holds for all functions $u^K(t, X^\alpha)$, $K = 1, 2, \ldots, N$. Since the values of $u^K(t, X^\alpha)$, $\partial_t u^K(t, X^\alpha)$ and $\partial_\alpha u^K(t, X^\alpha)$ can be taken arbitrary at each point (t, X^α), the identity $(3.2)_1$ is equivalent to the following system of identities (Property CI' in [7])

$$y_\Lambda(u^K)\frac{\partial g^{0\Lambda}(u^K)}{\partial u^M} \equiv 0, \qquad y_\Lambda(u^K)\frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M} \equiv 0. \tag{3.3}$$

The set of $N+1$ functions $y_\Lambda(u^K)$ can be obtained as a solution of the overdetermined system of linear homogeneous equations $(3.1)_2$, (3.2) and therefore, if it exists, it is not unique.

It would be convenient to introduce the following matrix notation:

$$\begin{gathered} \mathcal{A}^0(\mathbf{u}) = [\mathcal{A}^{0\Lambda}{}_M(u^K)] := \left[\frac{\partial g^{0\Lambda}(u^K)}{\partial u^M}\right], \\ \mathcal{A}^\alpha(\mathbf{u}) = [\mathcal{A}^{\alpha\Lambda}{}_M(u^K)] := \left[\frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M}\right], \\ \alpha = 1, 2, \ldots, m, \qquad \Lambda = 1, 2, \ldots, N+1, \qquad I, K = 1, 2, \ldots, N, \end{gathered} \tag{3.4}$$

and treat $N+1$ functions $y_\Lambda(u^K)$ as $(N+1)$ component row vector

$$\mathbf{y}^T(\mathbf{u}) = [y_\Lambda(u^K)]. \tag{3.5}$$

In matrix notation, $(3.2)_2$, (3.3) takes the form

$$\mathbf{y}^T(\mathbf{u})\mathcal{A}^0(\mathbf{u}) \equiv \mathbf{0}, \qquad \mathbf{y}^T(\mathbf{u})\mathcal{A}^\alpha(\mathbf{u}) \equiv \mathbf{0}, \qquad \mathbf{y}^T(\mathbf{u})\mathbf{b}(\mathbf{u}, t, X^\alpha) \equiv \mathbf{0}. \tag{3.6}$$

3.2. Equivalence of the entropy inequality and the MDR

The system of field equations (2.52) is a particular case of the quasi-linear system of N conservation equations for N unknowns

$$\partial_t \mathbf{f}^0(\mathbf{u}) + \partial_\alpha \mathbf{f}^\alpha(\mathbf{u}) = \mathbf{d}(\mathbf{u}, t, \mathbf{X}), \tag{3.7}$$

while the Clausius–Duhem entropy inequality (2.53) is a particular case of imposed unilateral differential constraints

$$\partial_t h^0(\mathbf{u}) + \partial_\alpha h^\alpha(\mathbf{u}) \geq \mu(\mathbf{u}, t, \mathbf{X}), \tag{3.8}$$

where $\mathbf{X}$ stands for $[X^\alpha]$. The unknowns $\mathbf{u}$ take values in an open bounded neighbourhood $\mathcal{O}$ of the origin in $\mathbb{R}^N$, and $\mathbf{f}^0$, $\mathbf{f}^\alpha$, h^0, h^α are presumed to have continuous second derivatives with respect to their argument.

Performing the respective differentiation, we rewrite the system (3.7) in a matrix form

$$\begin{gathered} \mathbf{B}^0(\mathbf{u})\partial_t \mathbf{u} + \mathbf{B}^\alpha(\mathbf{u})\partial_\alpha \mathbf{u} = \mathbf{d}(\mathbf{u}, t, \mathbf{X}), \\ \mathbf{B}^0(\mathbf{u}) = \nabla_\mathbf{u} \mathbf{f}^0(\mathbf{u}), \qquad \mathbf{B}^\alpha(\mathbf{u}) = \nabla_\mathbf{u} \mathbf{f}^\alpha(\mathbf{u}), \end{gathered} \tag{3.9}$$

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and the inequality (3.8) as

$$\text{(3.10)}\qquad \begin{gathered} \mathbf{k}^{0T}(\mathbf{u})\partial_t\mathbf{u} + \mathbf{k}^{\alpha T}(\mathbf{u})\partial_\alpha\mathbf{u} \geq \mu(\mathbf{u},t,\mathbf{X}), \\ \mathbf{k}^{0T}(\mathbf{u}) = \nabla_{\mathbf{u}} h^0(\mathbf{u}), \qquad \mathbf{k}^{\alpha T}(\mathbf{u}) = \nabla_{\mathbf{u}} h^\alpha(\mathbf{u}), \end{gathered}$$

where $\nabla_{\mathbf{u}}$ denotes differentiation with respect to $\mathbf{u}$.

OBSERVATION. If the system (3.7) satisfies the condition

$$\text{(3.11)}\qquad \det \mathbf{B}^0(\mathbf{u}) \neq 0 \qquad \text{for all } \mathbf{u}$$

then every Lipschitz continuous solution $\mathbf{u}(t,\mathbf{X})$ of (3.7) satisfies the inequality (3.8) if and only if the system of $N+1$ conservation equations composed of (3.7) and of the following conservation equation:

$$\text{(3.12)}\qquad \begin{gathered} \partial_t h^0(\mathbf{u}) + \partial_\alpha h^\alpha(\mathbf{u}) = \chi(\mathbf{u},t,\mathbf{X}), \\ \chi(\mathbf{u},t,\mathbf{X}) = \nabla_{\mathbf{u}} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}\cdot\mathbf{d}(\mathbf{u},t,\mathbf{X}), \end{gathered}$$

with

$$\text{(3.13)}\qquad \chi(\mathbf{u},\cdot,\cdot) - \mu(\mathbf{u},\cdot,\cdot) \geq 0,$$

satisfies the MDR.

It should be noted that the condition (3.11) is necessary for hyperbolicity of the system (3.7).

The proof of the Observation is given in the Appendix.

3.3. Thermodynamic restrictions via the MDR

In wiew of the Observation, thermodynamic restrictions imposed by Clausius–Duhem entropy inequality can be obtained from the requirement that the system of 5 first-order partial differential equations for 4 unknown fields $\theta(t,X^\alpha)$, $w_\gamma(t,X^\alpha)$

$$\text{(3.14)}\qquad \begin{aligned} \varrho_0\partial_t\widetilde{\varepsilon} + \partial_\alpha\widetilde{q}^\alpha &= \varrho_0\widetilde{r}, \\ \tau\partial_t w_\gamma - \partial_\alpha[f_1(\theta)\delta^\alpha_\gamma] &= c_1 w_\gamma, \\ \varrho_0\partial_t\widetilde{\eta} + \partial_\alpha\left[\frac{1}{\theta}\widetilde{q}^\alpha\right] &= \varrho_0\widetilde{\sigma}, \end{aligned}$$

where $\widetilde{\varepsilon}$, $\widetilde{q}^\alpha$, $\widetilde{r}$, and $\widetilde{\eta}$ are postulated as the constitutive functions (2.3) and $\widetilde{\sigma} = \widetilde{\sigma}(\theta, w_\gamma, t, X^\alpha)$, should satisfy the MDR, provided that constitutive function $\widetilde{\varepsilon}(\theta, w_\gamma)$ satisfies the condition

$$\text{(3.15)}\qquad \det\begin{bmatrix} \varrho_0\dfrac{\partial\widetilde{\varepsilon}(\theta,w_\gamma)}{\partial\theta} & \varrho_0\dfrac{\partial\widetilde{\varepsilon}(\theta,w_\gamma)}{\partial w_\alpha} \\ 0 & \tau\delta^\alpha{}_\gamma \end{bmatrix} \neq 0 \quad \text{for all } \theta, w_\gamma,$$

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which corresponds to (3.11), and which simply means that internal energy ε depends on temperature θ monotonically for each w_γ.

The system (3.14) can be written as an overdetermined system of conservation equations ($N+1$ equations for N unknowns)

$$
\begin{gathered}
\partial_t g^{0\Lambda}(u^K) + \partial_\alpha g^{\alpha\Lambda}(u^K) = b^\Lambda(u^K, t, X^\gamma), \\
\Lambda = 1,2,\ldots,N+1, \qquad K = 1,2,\ldots,N, \qquad \alpha = 1,2,\ldots,m
\end{gathered}
\tag{3.16}
$$

where, in our case, $N=4$, $m=3$ and

$$
\begin{aligned}
[u^K] &= [\theta, w_\gamma], \\
[g^{0\Lambda}(u^K)] &= [\varrho_0 \widetilde{\varepsilon}(\theta, w_\beta), \tau w_\gamma, \varrho_0 \widetilde{\eta}(\theta, w_\beta)], \\
[g^{\alpha\Lambda}(u^K)] &= \left[\widetilde{q}^\alpha(\theta, w_\beta), -f_1(\theta)\delta^\alpha{}_\gamma, \frac{1}{\theta}\widetilde{q}^\alpha(\theta, w_\beta)\right], \\
[b^\Lambda(u^K, t, X^\alpha)] &= [\varrho_0 \widetilde{r}(\theta, w_\beta, t, X^\alpha), c_1 w_\gamma, \varrho_0 \widetilde{\sigma}(\theta, w_\beta, t, X^\alpha)], \\
\gamma,\ \beta &= 1,2,3.
\end{aligned}
\tag{3.17}
$$

Performing the respective differentiation we rewrite the system (3.14) in a matrix form

$$
\begin{gathered}
\mathcal{A}^{0\Lambda}{}_M(u^K)\partial_t u^M + \mathcal{A}^{\alpha\Lambda}{}_M(u^K)\partial_\alpha u^M = b^\Lambda(u^K, t, X^\gamma), \\
\mathcal{A}^{0\Lambda}{}_M(u^K) = \frac{\partial g^{0\Lambda}(u^K)}{\partial u^M}, \qquad \mathcal{A}^{\alpha\Lambda}{}_M(u^K) = \frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M},
\end{gathered}
\tag{3.18}
$$

with the 5×4 matrices $[\mathcal{A}^{0\Lambda}{}_M]$, $[\mathcal{A}^{\alpha\Lambda}{}_M]$ of the form

$$
\begin{gathered}
\left[\mathcal{A}^{0\Lambda}{}_M\right] = \left[\frac{\partial g^{0\Lambda}}{\partial u^M}\right] = \begin{bmatrix}
\varrho_0 \frac{\partial \widetilde{\varepsilon}}{\partial \theta} & \varrho_0 \frac{\partial \widetilde{\varepsilon}}{\partial w_1} & \varrho_0 \frac{\partial \widetilde{\varepsilon}}{\partial w_2} & \varrho_0 \frac{\partial \widetilde{\varepsilon}}{\partial w_3} \\
0 & \tau & 0 & 0 \\
0 & 0 & \tau & 0 \\
0 & 0 & 0 & \tau \\
\varrho_0 \frac{\partial \widetilde{\eta}}{\partial \theta} & \varrho_0 \frac{\partial \widetilde{\eta}}{\partial w_1} & \varrho_0 \frac{\partial \widetilde{\eta}}{\partial w_2} & \varrho_0 \frac{\partial \widetilde{\eta}}{\partial w_3}
\end{bmatrix}, \\
\left[\mathcal{A}^{\alpha\Lambda}{}_M\right] = \left[\frac{\partial g^{\alpha\Lambda}}{\partial u^M}\right] = \begin{bmatrix}
\frac{\partial \widetilde{q}^\alpha}{\partial \theta} & \frac{\partial \widetilde{q}^\alpha}{\partial w_1} & \frac{\partial \widetilde{q}^\alpha}{\partial w_2} & \frac{\partial \widetilde{q}^\alpha}{\partial w_3} \\
-\delta^\alpha{}_1 f_1'(\theta) & 0 & 0 & 0 \\
-\delta^\alpha{}_2 f_1'(\theta) & 0 & 0 & 0 \\
-\delta^\alpha{}_3 f_1'(\theta) & 0 & 0 & 0 \\
-\frac{1}{\theta^2}\widetilde{q}^\alpha + \frac{1}{\theta}\frac{\partial \widetilde{q}^\alpha}{\partial \theta} & \frac{1}{\theta}\frac{\partial \widetilde{q}^\alpha}{\partial w_1} & \frac{1}{\theta}\frac{\partial \widetilde{q}^\alpha}{\partial w_2} & \frac{1}{\theta}\frac{\partial \widetilde{q}^\alpha}{\partial w_3}
\end{bmatrix}.
\end{gathered}
\tag{3.19}
$$

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In order to investigate the MDR for the system (3.16), (3.17), we introduce the following notation for y_Λ, $[y_\Lambda] = [\lambda, z^\gamma, \mu]$ and require the existence of 5 functions $\lambda(\theta, w_\gamma)$, $z^\beta(\theta, w_\gamma)$ and $\mu(\theta, w_\gamma)$, not all identically zero, such that the identities

$$\begin{gathered}\lambda[\varrho_0\partial_t\widetilde{\varepsilon} + \partial_\alpha\widetilde{q}^\alpha] + z^\gamma\{\tau\partial_t w_\gamma - \partial_\alpha[f_1(\theta)\delta^\alpha_\gamma]\} + \mu\left[\varrho_0\partial_t\widetilde{\eta} + \partial_\alpha\left(\frac{1}{\theta}\widetilde{q}^\alpha\right)\right] \equiv 0,\\ \varrho_0\lambda\widetilde{r} + c_1 z^\gamma w_\gamma + \varrho_0\mu\widetilde{\sigma} \equiv 0,\end{gathered} \tag{3.20}$$

hold for all $[\theta, w_\gamma]$. Solving the MDR $(3.2)_2$, (3.3) for (3.20), we obtain the following relations

$$\begin{aligned}\lambda\frac{\partial\widetilde{\varepsilon}}{\partial\theta} + \mu\frac{\partial\widetilde{\eta}}{\partial\theta} &= 0,\\ \lambda\frac{\partial\widetilde{\varepsilon}}{\partial w_\gamma} + \mu\frac{\partial\widetilde{\eta}}{\partial w_\gamma} + z^\gamma\frac{\tau}{\varrho_0} &= 0,\\ \left(\lambda + \frac{\mu}{\theta}\right)\frac{\partial\widetilde{q}^\alpha}{\partial\theta} - \frac{\mu}{\theta^2}\widetilde{q}^\alpha - z^\alpha f_1'(\theta) &= 0,\\ \left(\lambda + \frac{\mu}{\theta}\right)\frac{\partial\widetilde{q}^\alpha}{\partial w_\gamma} &= 0.\end{aligned} \tag{3.21}$$

According to $(3.21)_{1,2}$, the differential of entropy $d\widetilde{\eta}$ can be expressed in terms of the differentials $d\widetilde{\varepsilon}$ and dw_γ

$$\begin{aligned}\mu d\widetilde{\eta} = \mu\frac{\partial\widetilde{\eta}}{\partial\theta}d\theta + \mu\frac{\partial\widetilde{\eta}}{\partial w_\gamma}dw_\gamma &= -\lambda\frac{\partial\widetilde{\varepsilon}}{\partial\theta}d\theta - \lambda\frac{\partial\widetilde{\varepsilon}}{\partial w_\gamma}dw_\gamma - z_\gamma\frac{\tau}{\varrho_0}dw_\gamma\\ &= -\lambda d\widetilde{\varepsilon} - z_\gamma\frac{\tau}{\varrho_0}dw_\gamma,\end{aligned} \tag{3.22}$$

and rearranged to the form of generalized Gibbs relation

$$-\frac{\mu}{\lambda}d\widetilde{\eta} = d\widetilde{\varepsilon} + \frac{\tau}{\varrho_0}\frac{1}{\lambda}z_\gamma dw_\gamma, \tag{3.23}$$

which shows that the factor $-\mu/\lambda$ corresponds to the temperature θ. This fact also follows from $(3.21)_4$.

3.4. Family of solutions of the MDR

In view of (3.21), we obtain

$$\begin{aligned}\theta &= -\frac{\mu}{\lambda},\\ z^\gamma &= -\lambda\frac{\varrho_0}{\tau}\left(\frac{\partial\widetilde{\varepsilon}}{\partial w_\gamma} - \theta\frac{\partial\widetilde{\eta}}{\partial w_\gamma}\right) = -\lambda\frac{\varrho_0}{\tau}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma} = \mu\frac{\varrho_0}{\tau}\frac{1}{\theta}\left(\frac{\partial\widetilde{\varepsilon}}{\partial w_\gamma} - \theta\frac{\partial\widetilde{\eta}}{\partial w_\gamma}\right)\\ &= \mu\frac{\varrho_0}{\tau}\frac{1}{\theta}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma} = \lambda\frac{1}{\theta f_1'(\theta)}\widetilde{q}^\gamma = -\mu\frac{1}{\theta^2 f_1'(\theta)}\widetilde{q}^\gamma,\end{aligned} \tag{3.24}$$

where the free energy $\widetilde{\Psi}_C$ is introduced by (2.8).

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It follows from (3.24) that the family of solutions of the MDR can be written as

$$(3.25)\qquad [y_\Lambda] = -\lambda\left[-1,\ \frac{\varrho_0}{\tau}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma},\ \theta\right] = -\mu\left[\frac{1}{\theta},\ -\frac{1}{\theta}\frac{\varrho_0}{\tau}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma},\ -1\right],$$

where λ and μ are arbitrary functions of $[\theta, w_\gamma]$ and $\lambda = -1$, $[\widehat{y}_\Lambda] = \left[-1, \frac{\varrho_0}{\tau}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma}, \theta\right]$ corresponds to the case when the equation of balance of energy is treated as the additional conservation equation, while $\mu = -1$, $[\breve{y}_\Lambda] = \left[\frac{1}{\theta}, -\frac{1}{\theta}\frac{\varrho_0}{\tau}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma}, -1\right]$ corresponds to the case when the equation of balance of entropy is treated as the additional conservation equation.

3.5. Restrictions on constitutive functions

3.5.1. General restrictions. It follows from (3.21), (3.24) that constitutive functions $\widetilde{\varepsilon}$, $\widetilde{\eta}$ and $\widetilde{q}^\alpha$ must satisfy the following relations

$$(3.26)\qquad \begin{aligned} \widetilde{q}^\gamma &= -\frac{\varrho_0\theta f_1'(\theta)}{\tau}\left(\frac{\partial\widetilde{\varepsilon}}{\partial w_\gamma} - \theta\frac{\partial\widetilde{\eta}}{\partial w_\gamma}\right) = -\frac{\varrho_0\theta f_1'(\theta)}{\tau}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma},\\ \frac{\partial\widetilde{\eta}}{\partial\theta} &= \frac{1}{\theta}\frac{\partial\widetilde{\varepsilon}}{\partial\theta}, \end{aligned}$$

and

$$(3.27)\qquad \widetilde{\eta} = -\frac{\partial\widetilde{\Psi}_C}{\partial\theta}, \qquad \widetilde{\varepsilon} = \widetilde{\Psi}_C - \theta\frac{\partial\widetilde{\Psi}_C}{\partial\theta}.$$

Identity $(3.20)_2$ of the MDR together with (3.24), (3.26) yields

$$(3.28)\qquad \varrho_0\frac{\widetilde{r}}{\theta} + c_1\frac{1}{\theta^2 f_1'(\theta)}\widetilde{q}^\gamma w_\gamma - \varrho_0\widetilde{\sigma} \equiv 0,$$

and, in view of (3.13), it gives the entropy production inequality in the following form:

$$(3.29)\qquad \frac{c_1}{\theta^2 f_1'(\theta)}\widetilde{q}^\gamma w_\gamma = -\frac{\varrho_0}{\tau}\frac{c_1}{\theta}\left(\frac{\partial\widetilde{\varepsilon}}{\partial w_\gamma} - \theta\frac{\partial\widetilde{\eta}}{\partial w_\gamma}\right)w_\gamma = -\frac{\varrho_0}{\tau}\frac{c_1}{\theta}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma}w_\gamma \geq 0.$$

Of course, the restrictions (3.26) – (3.29) on constitutive functions $\widetilde{q}(\theta, \mathbf{w})$, $\widetilde{\eta}(\theta, \mathbf{w})$ and $\widetilde{\varepsilon}(\theta, \mathbf{w})$ derived here with the aid of the MDR coincide with the corresponding restrictions (2.9), (2.15), (2.16) obtained in [22, 4] as direct consequences of the Clausius – Duhen entropy inequality if we substitute $f_1(\theta) = -\tau f(\theta)$, $\mathbf{b}(\theta, \mathbf{w}) = \tau^{-1}\mathbf{c}(\theta, \mathbf{w}) = \tau^{-1}\mathbf{w}$.

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3.5.2. Additional constitutive assumptions

A. $\widetilde{\varepsilon}(\theta, w_\gamma) = \widetilde{\varepsilon}_0(\theta)$

In Secs. 2.2.3, 2.2.4, 2.2.5, the the additional constitutive assumptions concerning the form of $\widetilde{\mathbf{q}}(\theta, \mathbf{w})$ and $\widetilde{\varepsilon}(\theta, \mathbf{w})$ were analysed in detail. In those considerations the condition (2.28) ($\widetilde{\varepsilon}(\theta, \mathbf{w}) = \widetilde{\varepsilon}_0(\theta)$) was taken as supplementary to the condition (2.18) ($\widetilde{\mathbf{q}} = \mathbf{S}(\theta)\mathbf{w}$). In this section, the consequences of the assumption (2.28) are studied without prior asumption (2.18).

As a consequence of the assumption (2.28), the entropy $\widetilde{\eta}(\theta, w_\gamma)$ splits into two parts

$$\widetilde{\eta}(\theta, w^\gamma) = \overset{\approx}{\eta}_0(\theta) + \overset{\approx}{\eta}_1(w_\gamma), \tag{3.30}$$

since, according to (3.26), we have in this case

$$\frac{\partial \widetilde{\eta}(\theta, w^\gamma)}{\partial \theta} = \frac{1}{\theta} \widetilde{\varepsilon}_0'(\theta), \tag{3.31}$$

and therefore

$$\frac{\partial^2 \widetilde{\eta}(\theta, w^\gamma)}{\partial \theta \, \partial w_\alpha} = 0. \tag{3.32}$$

Hence, the free energy $\widetilde{\Psi}_C$ given by (2.8) also splits,

$$\begin{aligned} \widetilde{\Psi}_C(\theta, w_\gamma) &= \widetilde{\Psi}_{C0}(\theta) - \theta \, \overset{\approx}{\eta}_1(w_\gamma), \\ \widetilde{\Psi}_{C0}(\theta) &= \overset{\approx}{\varepsilon}(\theta) - \theta \, \overset{\approx}{\eta}_0(\theta). \end{aligned} \tag{3.33}$$

Moreover, the assumption (2.28) together with its consequences (3.30), (3.33) yields

$$\widetilde{q}^\gamma = \frac{\varrho_0 \theta^2 f_1'(\theta)}{\tau} \frac{\partial \overset{\approx}{\eta}_1(w_\alpha)}{\partial w_\gamma}. \tag{3.34}$$

Hence, the heat flux vector $\widetilde{\mathbf{q}} = [\widetilde{q}^\alpha]$ is collinear with $\nabla_{\mathbf{w}} \overset{\approx}{\eta}_1 = \left[\dfrac{\partial \overset{\approx}{\eta}_1}{\partial w_\alpha}\right]$.

B. $\widetilde{\varepsilon}(\theta, w_\gamma) = \widetilde{\varepsilon}_0(\theta)$ *for isotropic body*

If it is additionally assumed that the body is isotropic then

$$\overset{\approx}{\eta}_1(w_\alpha) = \overset{\approx}{\eta}_1(w^\alpha w_\alpha), \tag{3.35}$$

$$\widetilde{r}(\theta, w_\gamma) = \overset{\approx}{r}(\theta, w_\gamma w^\gamma, t, X^\alpha), \tag{3.36}$$

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where $\overset{\approx}{\eta}_1$ is a function of one variable and, as a consequence, (3.30), (3.33) and (3.34) take the form

$$\begin{aligned} \widetilde{\eta}(\theta, w_\alpha) &= \overset{\approx}{\eta}_0(\theta) + \overset{\approx}{\eta}_1(w^\alpha w_\alpha), \\ \widetilde{\Psi}_C(\theta, w_\alpha) &= \widetilde{\Psi}_{C0}(\theta) - \theta\, \overset{\approx}{\eta}_1(w^\alpha w_\alpha), \\ \widetilde{q}^\gamma &= 2\frac{\varrho_0 \theta^2 f_1'(\theta)}{\tau}\, \overset{\approx}{\eta}{}_1'(w_\alpha w^\alpha) w^\gamma. \end{aligned} \tag{3.37}$$

Therefore, the heat flux vector is collinear with internal state variable $\mathbf{w}$ (but does not depend linearly on $\mathbf{w}$) in this case and, in view of (3.37), (3.36) and (3.29), the following entropy production inequality is obtained

$$2\frac{\varrho_0}{\tau} c_1\, \overset{\approx}{\eta}{}_1'(w_\alpha w^\alpha) w_\gamma w^\gamma \geq 0. \tag{3.38}$$

The model of a rigid conductor of heat corresponding to the constitutive assumption (3.30), (3.35) can be regarded as a particular generalization of the model developed in [4] in a sense that, like in [4], internal energy depends only on the temperature and the body is isotropic, but a more general dependence $(3.37)_3$ of $\widetilde{\mathbf{q}}$ on $\mathbf{w}$ is assumed instead of (2.24).

C. $\widetilde{\varepsilon}(\theta, w_\gamma) = \widetilde{\varepsilon}_0(\theta)$, *isotropic body,* $\overset{\approx}{\eta}_1$ *quadratic in* $\mathbf{w}$

Assuming a particular form of function $\overset{\approx}{\eta}_1$ namely $\overset{\approx}{\eta}_1(\xi) = -\frac{1}{2}c_2\xi$, $c_2 > 0$, in (3.35), we obtain from (3.37), (3.38),

$$\begin{aligned} \widetilde{\eta}(\theta, w_\alpha) &= \overset{\approx}{\eta}_0(\theta) - \frac{1}{2}c_2 w_\gamma w^\gamma, \\ \widetilde{q}^\gamma(\theta, w_\alpha) &= -\frac{\varrho_0\theta^2 f_1'(\theta) c_2}{\tau} w^\gamma, \\ -\frac{c_1 c_2}{\tau} w^\alpha w_\alpha &\geq 0. \end{aligned} \tag{3.39}$$

In this case $\widetilde{\mathbf{q}}$ is linear in $\mathbf{w}$ and it follows from (2.42) that the model developed in [4] is obtained if c_1 is replaced by $\dfrac{c_2\theta^2[f_1'(\theta)]^2}{\tau\kappa(\theta)}$.

Appendix

A.1. Results for the system (3.7) in normal form

For the system (3.7) in normal (Cauchy) form

$$\mathbf{f}^0(\mathbf{u}) = \mathbf{u}, \tag{A.1}$$

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Kosiński [30] has formulated the following Lemma 0 (the proof is given in [31]):

Every Lipshitz continuous solution $\mathbf{u}(t, \mathbf{X})$ of the system (3.7), (A.1) satisfies (3.8) if and only if

$$\nabla_{\mathbf{u}} h^0(\mathbf{u}) \nabla_{\mathbf{u}} \mathbf{f}^\alpha(\mathbf{u}) = \nabla_{\mathbf{u}} h^\alpha(\mathbf{u}) \tag{A.2}$$

and

$$\nabla_{\mathbf{u}} h^\alpha(\mathbf{u}) \cdot \mathbf{d}(\mathbf{u}, \cdot, \cdot) - \mu(\mathbf{u}, \cdot, \cdot) \geq 0. \tag{A.3}$$

Then, as a consequence, he proved [30, 31] (by contracting both sides of (3.7) with $\nabla_{\mathbf{u}} h^0(\mathbf{u})$) the following:

Corollary. Every Lipschitz continuous solution $\mathbf{u}(t, \mathbf{X})$ of the system (3.7), (A.1) consistent with (3.8) satisfies the additional conservation equation (3.11), (3.12) almost everywhere.

By "consistency" we mean that (A.2), (A.3) hold. The system (2.52) considered in this paper is not in normal (Cauchy) form and therefore Lemma 0 and Corollary do not directly apply to our case.

A.2. Generalization of Lemma 0 and Corollary to the systems (3.7) not in normal form

Assumption (3.11) implies that the mapping $\mathbf{f}^0 : \mathcal{O} \to \mathbb{R}^N$ is invertible and therefore $\mathbf{v} := \mathbf{f}^0$ can be taken as new dependent variable related to $\mathbf{u}$ by the inverse mapping $\mathbf{f}^{0^{-1}} : \mathbb{R}^N \to \mathcal{O}$, $\mathbf{u}(\mathbf{v}) = \mathbf{f}^{0^{-1}}(\mathbf{v})$. In new dependent variables, system (3.7) and the inequality (3.8) assume the following form:

$$\partial_t \mathbf{v} + \partial_\alpha \widehat{\mathbf{f}}^\alpha(\mathbf{v}) = \widehat{\mathbf{d}}(\mathbf{v}, t, \mathbf{x}), \tag{A.4}$$

$$\partial_t \widehat{h}^0(\mathbf{v}) + \partial_\alpha \widehat{h}^\alpha(\mathbf{v}) \geq \widehat{\mu}(\mathbf{v}, t, \mathbf{X}), \tag{A.5}$$

where

$$\begin{aligned}
\widehat{\mathbf{f}}^\alpha(\mathbf{v}) &= \mathbf{f}^\alpha(\mathbf{u}(\mathbf{v})) = \mathbf{f}^\alpha(\mathbf{f}^{0^{-1}}(\mathbf{v})), \\
\widehat{\mathbf{d}}(\mathbf{v}, t, \mathbf{X}) &= \mathbf{d}(\mathbf{u}(\mathbf{v}), t, \mathbf{x}) = \mathbf{d}(\mathbf{f}^{0^{-1}}(\mathbf{v}), t, \mathbf{X}), \\
\widehat{h}^0(\mathbf{v}) &= h^0(\mathbf{u}(\mathbf{v})) = h^0(\mathbf{f}^{0^{-1}}(\mathbf{v})), \\
\widehat{h}^\alpha(\mathbf{v}) &= h^\alpha(\mathbf{u}(\mathbf{v})) = h^\alpha(\mathbf{f}^{0^{-1}}(\mathbf{v})), \\
\widehat{\mu}(\mathbf{v}, t, \mathbf{X}) &= \mu(\mathbf{u}(\mathbf{v}), t, \mathbf{x}) = \mu(\mathbf{f}^{0^{-1}}(\mathbf{v}), t, \mathbf{X}).
\end{aligned} \tag{A.6}$$

Since (A.4) is in normal (Cauchy) form, Lemma 0 and Corollary do apply to (A.4), (A.5) and, as a consequence, the necessary and sufficient conditions for the Lipschitz continuous solutions $\mathbf{v}(t, \mathbf{X})$ of (A.4) to satisfy (A.5) are

$$\begin{aligned}
\nabla_{\mathbf{v}} \widehat{h}^0(\mathbf{v}) \nabla_{\mathbf{v}} \widehat{\mathbf{f}}^0(\mathbf{v}) &= \nabla_{\mathbf{v}} \widehat{h}^\alpha(\mathbf{v}), \\
\nabla_{\mathbf{v}} \widehat{h}^0(\mathbf{v}) \cdot \widehat{\mathbf{d}}(\mathbf{v}, \cdot, \cdot) - \widehat{\mu}(\mathbf{v}, \cdot, \cdot) &\geq 0,
\end{aligned} \tag{A.7}$$

and the additional conservation equation

$$\partial_t \widehat{h}^0(\mathbf{v}) + \partial_\alpha \widehat{h}^\alpha(\mathbf{v}) = \nabla_{\mathbf{v}} \widehat{\eta}(\mathbf{v}) \cdot \widehat{\mathbf{d}}(\mathbf{v}, t, \mathbf{X}), \tag{A.8}$$

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where $\nabla_{\mathbf{v}}$ denotes differentiation with respect to $\mathbf{v}$, is satisfied by Lipschitz continuous solutions of (A.4). Taking into account (A.6) and employing the chain rule, we obtain from (A.7)

$$\text{(A.9)} \qquad \begin{gathered} [\nabla_{\mathbf{u}} h^0(\mathbf{u}(\mathbf{v}))\nabla_{\mathbf{v}}\mathbf{u}(\mathbf{v})][\nabla_{\mathbf{u}}\mathbf{f}^{\alpha}(\mathbf{u}(\mathbf{v}))\nabla_{\mathbf{u}}\mathbf{u}(\mathbf{v})] = \nabla_{\mathbf{v}} h^{\alpha}(\mathbf{u}(\mathbf{v}))\nabla_{\mathbf{v}}\mathbf{u}(\mathbf{v}), \\ \nabla_{\mathbf{u}} h^0(\mathbf{u}(\mathbf{v}))\nabla_{\mathbf{v}}\mathbf{u}(\mathbf{v})\cdot\mathbf{d}(\mathbf{u}(\mathbf{v}),\cdot,\cdot) - \mu(\mathbf{u}(\mathbf{v}),\cdot,\cdot) \geq 0. \end{gathered}$$

Returning in (A.9) to original variable $\mathbf{u}$ and taking into account that

$$\text{(A.10)} \qquad \nabla_{\mathbf{v}}\mathbf{u}(\mathbf{v}) = \nabla_{\mathbf{v}}\mathbf{f}^{0^{-1}}(\mathbf{v}) = [\nabla_{\mathbf{u}} f^0(\mathbf{u}(\mathbf{v}))]^{-1} = [B^0(\mathbf{u}(\mathbf{v}))]^{-1},$$

we obtain

$$\text{(A.11)} \qquad \begin{gathered} \nabla_{\mathbf{u}} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}\nabla_{\mathbf{u}}\mathbf{f}^{\alpha}(\mathbf{u}) = \nabla_{\mathbf{u}} h^{\alpha}(\mathbf{u}), \\ \nabla_{\mathbf{u}} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}\cdot\mathbf{d}(\mathbf{u},\cdot,\cdot) - \mu(\mathbf{u},\cdot,\cdot) \geq 0. \end{gathered}$$

Hence, the relations (A.11) are generalizations of the relations (A.2), (A.3) of Lemma 0 to the case of the system (3.7) which is not in normal (Cauchy) form and satisfies the condition (3.11) and, consequently, every Lipschitz continuous solution of the system (3.7), (3.11) satisfies (3.8) iff (A.11) holds. Similarly, taking into account (A.6), $(\text{A.11})_1$, (A.10), employing the chain rule and then returning into original dependent variable $\mathbf{u}$, we obtain from (A.8) the following conservation equation

$$\text{(A.12)} \qquad \partial_t h^0(\mathbf{u}) + \partial_{\alpha} h^{\alpha}(\mathbf{u}) = \nabla_{\mathbf{u}} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}\cdot\mathbf{d}(\mathbf{u},t,\mathbf{X}),$$

which corresponds to contraction of both sides of (3.7) with $\nabla_{\mathbf{u}} h^0(\mathbf{u})[B^0(\mathbf{u})]^{-1}$. Hence, every Lipschitz continuous solution $\mathbf{u}(t,\mathbf{X})$ of the system (3.7) satysfying (3.11) and consitent with inequality (3.8) (that is, for which (A.11) holds) satisfies the additional conservation equation (3.12) with

$$\text{(A.13)} \qquad \chi(\mathbf{u},t,\mathbf{X}) = \nabla_{\mathbf{u}} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}\cdot\mathbf{d}(\mathbf{u},t,\mathbf{X})$$

almost everywhere, and

$$\text{(A.14)} \qquad \chi(\mathbf{u},\cdot,\cdot) - \mu(\mathbf{u},\cdot,\cdot) \geq 0,$$

according to $(\text{A.11})_2$.

A.3. Consistency with the MDR

The system (3.7) together with the additional conservation equation (A.12), (A.13), (A.14) can be considered as the system of $N+1$ conservation equations for N unknowns (3.1) which, without loss of generality, can be written in the

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form

$$
\text{(A.15)}\quad
\begin{gathered}
\mathbf{g}^0(\mathbf{u}) = [g^{0\Lambda}(u^K)] = \begin{bmatrix} \mathbf{f}^0(\mathbf{u}) \\ h^0(\mathbf{u}) \end{bmatrix} = \begin{bmatrix} f^{0I}(u^K) \\ h^0(u^K) \end{bmatrix},\\
\mathbf{g}^\alpha(\mathbf{u}) = [g^{\alpha\Lambda}(u^K)] = \begin{bmatrix} \mathbf{f}^\alpha(\mathbf{u}) \\ h^\alpha(\mathbf{u}) \end{bmatrix} = \begin{bmatrix} f^{\alpha I}(u^K) \\ h^\alpha(u^K) \end{bmatrix},\\
\mathbf{b}(\mathbf{u},\cdot,\cdot) = [b^\Lambda(u^K,\cdot,\cdot)] = \begin{bmatrix} \mathbf{d}(\mathbf{u},\cdot,\cdot) \\ \chi(\mathbf{u},\cdot,\cdot) \end{bmatrix} = \begin{bmatrix} d^I(u^K,\cdot,\cdot) \\ \chi(u^K,\cdot,\cdot) \end{bmatrix},\\
I,K = 1,2,\ldots,N, \qquad \Lambda = 1,2,\ldots,N+1.
\end{gathered}
$$

It follows from (A.11), (A.12), (A.13), (3.9), (3.10) that the system (3.1), (A.15) corresponding to (3.7), (A.12), (A.13) satisfies the conditions (3.6) of the MDR for

$$
\text{(A.16)}\qquad \mathbf{y}^T(\mathbf{u}) = [\nabla_\mathbf{u} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}, -1] = [\mathbf{k}^{0^T}(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}, -1]
$$

since, in the case of (A.15), the $(N+1)\times N$ matrices (3.1.4) $\mathcal{A}^0(\mathbf{u})$, $\mathcal{A}^\alpha(\mathbf{u})$ assume the form

$$
\text{(A.17)}\quad
\begin{gathered}
\mathcal{A}^0(\mathbf{u}) = \begin{bmatrix} \nabla_\mathbf{u}\mathbf{f}^0(\mathbf{u}) \\ \nabla_\mathbf{u} h^0(\mathbf{u}) \end{bmatrix} = \begin{bmatrix} \mathbf{B}^0(\mathbf{u}) \\ \mathbf{k}^{0^T}(\mathbf{u}) \end{bmatrix},\\
\mathcal{A}^\alpha(\mathbf{u}) = \begin{bmatrix} \nabla_\mathbf{u}\mathbf{f}^\alpha(\mathbf{u}) \\ \nabla_\mathbf{u} \mathbf{h}^\alpha(\mathbf{u}) \end{bmatrix} = \begin{bmatrix} \mathbf{B}^\alpha(\mathbf{u}) \\ \mathbf{k}^{\alpha^T}(\mathbf{u}) \end{bmatrix},
\end{gathered}
$$

and (A.13) holds. Therefore, the system (3.7), (A.12), (A.13) satisfies the MDR provided that the conditions (A.11) hold. In this way we have proved that if the system (3.7) satisfies the condition (3.11) and every Lipschitz continuous solution of (3.7) satisfies the inequality (3.8), then the system (3.7), (3.12) (3.13) of $N+1$ conservation equations satisfies the MDR.

For the converse, we assume that the system (3.1), (A.15) satisfies the condition (3.11) and the MDR, and that the inequality (3.13) is satisfied. The MDR implies that there exist such $\mathbf{y}(\mathbf{u})$ with the components not all identically zero which satisfy (3.6), $(3.2)_2$ for $\mathbf{b}(\mathbf{u})$ given by $(\text{A.15})_3$ and for $\mathcal{A}^0(\mathbf{u})$, $\mathcal{A}^\alpha(\mathbf{u})$ given by (A.17). For convenience, we denote $y_{N+1}(\mathbf{u}^K) = \lambda(\mathbf{u})$ and $y_I(\mathbf{u}) = p_I(\mathbf{u})$, $I = 1,2,\ldots,N$, $\mathbf{p}^T(\mathbf{u}) = [p_I(\mathbf{u})]$. Then, it follows from (3.6), (A.15), (A.17) that

$$
\text{(A.18)}\quad
\begin{gathered}
\mathbf{p}^T(\mathbf{u})\mathbf{B}^0(\mathbf{u}) = -\lambda(\mathbf{u})\mathbf{k}(\mathbf{u}), \qquad \mathbf{p}^T(\mathbf{u})\mathbf{B}^\alpha(\mathbf{u}) = -\lambda(\mathbf{u})\mathbf{k}(\mathbf{u}),\\
\mathbf{p}^T(\mathbf{u})\mathbf{d}(\mathbf{u},\cdot,\cdot) = -\chi(\mathbf{u},\cdot,\cdot).
\end{gathered}
$$

The condition (3.11) implies that $\lambda(\mathbf{u})$ is not identically zero and therefore both sides of (A.18) can be multiplied by $-[\lambda(\mathbf{u})]^{-1}$. Denoting $l_I(u^K) = -[\lambda(\mathbf{u})]^{-1}p_I(\mathbf{u})$ and taking into account (3.9), (3.10) we obtain from (A.18)

$$
\text{(A.19)}\qquad \mathbf{l}^T(\mathbf{u}) = \mathbf{k}^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1} = \nabla_\mathbf{u} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}
$$

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and the following relations corresponding to (A.11):

$$\text{(A.20)} \qquad \begin{aligned} \nabla_{\mathbf{u}} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}\nabla_{\mathbf{u}}\mathbf{f}^\alpha(\mathbf{u}) &= \nabla_{\mathbf{u}} h^\alpha(\mathbf{u}), \\ \nabla_{\mathbf{u}} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}\mathbf{d}(\mathbf{u},\cdot,\cdot) &= \chi(\mathbf{u},\cdot,\cdot). \end{aligned}$$

It then follows from the generalization of Lemma 0 that every Lipschitz continuous solution of (3.7), (3.11) satisfies the inequality (3.8), in view of (3.13), what completes the proof.

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Symmetric forms of the equations of heat transport in a rigid conductor of heat with internal state variables
II. Alternative symmetric systems

W. LARECKI (WARSZAWA)

In Part I of this series, it has been shown that the field equations corresponding to the model of a rigid conductor of heat with (vector) internal state variable subject to the entropy inequality can be represented as the respective system of $N+1$ conservation equations for N unknowns, on which the "main dependency relation" (MDR) is imposed. In this paper (Part II), it is demonstrated how two families of symmetric systems corresponding to the consistent system of N conservation equations (family of symmetric systems for original unknowns and the family of $N+1$ symmetric conservative systems for transformed unknowns) can be directly derived with the aid of the MDR. The condition of equivalence of symmetric systems to the original system of conservation equations is analysed and alternatively formulated. For the considered model of a rigid conductor of heat, the conditions on free energy that assure symmetric hyperbolicity of symmetric systems are established, and it is shown that they are stronger than the conditions required for equivalence of symmetric systems to the original system of conservation equations. Two alternative symmetric conservative systems are derived for the considered model of a rigid conductor of heat and the conditions of symmetric hyperbolicity for those systems are established with the aid of the relation between convexity (concavity) of the respective generating potentials, and with the aid of the relation between symmetric hyperbolicity of the symmetric systems for original unknowns and symmetric conservative system for the transformed unknowns.

1. Introduction

In Part I of this paper [1], we have shown that the field equations corresponding to the model of a rigid conductor of heat with vector internal state variables, with the Clausius–Duhem entropy inequality taken into account, can be represented by an overdetermined system of conservation equations ($N+1$ equations for N unknowns)

$$(1.1)\qquad \begin{gathered}\partial_t g^{0\Lambda}(u^K) + \partial_\alpha g^{\alpha\Lambda}(u^K) = b^\Lambda(u^K, t, X^\gamma),\\ \Lambda = 1,2,\ldots,N+1, \qquad K = 1,2,\ldots,N, \qquad \alpha,\gamma = 1,2,\ldots,m,\end{gathered}$$

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where $N = 4$, $m = 3$ and

$$
\begin{aligned}
[u^K] &= [\theta, w_\gamma],\\
[g^{0\Lambda}(u^K)] &= [\varrho_0\widetilde{\varepsilon}(\theta, w_\beta), \tau w_\gamma, \varrho_0\widetilde{\eta}(\theta, w_\beta)],\\
[g^{\alpha\Lambda}(u^K)] &= \left[\widetilde{q}^\alpha(\theta, w_\beta), -f_1(\theta)\delta^\alpha{}_\gamma, \frac{1}{\theta}\widetilde{q}^\alpha(\theta, w_\beta)\right],\\
[b^\Lambda(u^K, t, X^\alpha)] &= [\varrho_0\widetilde{r}(\theta, w_\beta, t, X^\alpha), c_1 w_\gamma, \varrho_0\widetilde{\sigma}(\theta, w_\beta, t, X^\alpha)],\\
\gamma,\ \beta &= 1, 2, 3,
\end{aligned}
\tag{1.2}
$$

on which the "main dependency relation" (MDR) is imposed. In Eqs. (1.1), (1.2) $\theta(t, X^\alpha)$ represents the temperature field, $\mathbf{w}(t, X^\alpha) = [w_\gamma(t, X^\alpha)]$ is the field of internal state variables. In this model f_1, $\widetilde{\varepsilon}$, $\widetilde{q}^\alpha$, $\widetilde{r}$, and $\widetilde{\sigma}$ are postulated as the constitutive functions

$$
\begin{aligned}
&\widetilde{\varepsilon} = \widetilde{\varepsilon}(\theta, w_\gamma), \quad \widetilde{q}^\alpha = \widetilde{q}^\alpha(\theta, w_\gamma), \quad \widetilde{r} = \widetilde{r}(\theta, w_\gamma, t, X^\alpha),\\
&\widetilde{\eta} = \widetilde{\eta}(\theta, w_\gamma), \quad \widetilde{\sigma} = \widetilde{\sigma}(\theta, w_\gamma, t, X^\alpha), \quad f_1 = f_1(\theta).
\end{aligned}
\tag{1.3}
$$

The first equation of the system (1.1), (1.2) corresponds to balance of energy, the next three equations are the components of the evolution equation for internal state variable $\mathbf{w}$, and the last equation can be interpreted as the equation of balance of entropy.

In this paper, the possibilities of expressing the system (1.1), (1.2) in the form of symmetric systems of the first order partial differential equations are investigated. The emphasis is put on the employed method of symmetrization which directly leads to alternative symmetric systems. This method is based on direct application of the MDR which, as we have proved in Sec. 3.2 of Part I, can be used for derivation of thermodynamic restrictions. From the point of view of thermodynamics, the MDR formulated by FRIEDRICHS [2], can be regarded as a generalization of the "entropy principle" of extended thermodynamics of MÜLLER and LIU [3, 4, 5], in the sense that it assigns an analogue of Lagrange–Liu multiplier also to the balance of entropy. The advantage of employing the MDR instead of the "entropy principle" is that it enables one to derive equivalent (for classical solutions) alternative symmetric systems directly. In this approach, a family of symmetric systems (parameterized by real differentiable functions) with respect to original dependent variables as well as a family of $N+1$ symmetric conservative systems with respect to transformed dependent variables can be obtained, offering a possibility of selecting or constructing symmetric systems optimal from the point of view of the chosen numerical method and/or which are most suitable for the considered initial-boundary value problem. The symmetrization may be utilized in the design and analysis of numerical solutions. As it is mentioned by HARTEN [6], it offers the possibility of linearizing

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locally the equations in a way which preserves the hyperbolicity and conservation properties if the symmetric system is in the conservative form. Since the Cauchy problem is locally well-posed for the quasi-linear symmetric hyperbolic systems [7, 8], symmetrization enables the application of this result provided that the condition of symmetric hyperbolicity is satisfied.

In Secs. 2.1, 2.2 and 2.3 we recall restrictions on constitutive functions (1.3) and the family of solutions of the "main dependency relation" derived in Part I. Then, in Sec. 3.1, the general procedure of obtaining symmetric systems for the original unknowns is presented and a family of symmetric systems with respect to $[\theta, w_\gamma]$, parameterized by differentiable functions is given in Sec. 3.2.

The condition of equivalence of the symmetric systems to original system of conservation equations is discussed in Sec. 3.3. Two Observations, which provide alternative formulation of this condition are proved with the aid of three Lemmas given in the Appendix A. Then, in Sec. 3.4, the restrictions on the free energy $\widetilde{\Psi}_C(\theta, w_\gamma)$ that assure equivalence of the symmetric systems for $[\theta, w_\gamma]$ to the system (1.1), (1.2) are derived. Further conditions on the free energy $\widetilde{\Psi}_C(\theta, w_\gamma)$ that assure symmetric hyperbolicity of the symmetric systems for $[\theta, w_\gamma]$ are derived in Secs. 3.5.1 – 3.5.4. It is shown that the restriction on $\widetilde{\Psi}_C(\theta, w_\gamma)$ imposed by the condition of symmetric hyperbolicity is much stronger than the condition ensuring the equivalence of the symmetric systems and the original system of conservation equations or, in other words, that symmetric hyperbolicity implies equivalence to the original system of conservation equations (1.1).

In Secs. 4.1.1, 4.1.2, we present the general procedure of simultaneous derivation of $N+1$ alternative symmetric conservative systems corresponding to the system (1.1) that satisfies the MDR. Taking into account Observations stated in Sec. 3.3, we show that the condition of equivalence of symmetric systems for original unknowns to (1.1), together with the assumption that (1.1) contains N independent equations (is determined), suffices for transformation of (1.1) into symmetric conservative form in $N+1$ ways. In Sec. 4.1.4, we derive the general relations between alternative symmetric conservative systems that enable transformation of the given symmetric conservative system into the remaining N symmetric conservative systems, as well as to establish the relation between convexity (concavity) of the respective potentials (hence, symmetric hyprbolicity of those systems).

The procedure developed in Secs. 4.1.1, 4.1.2 is employed for derivation of two alternative symmetric conservative systems governing heat transport in a rigid heat conductor with (vector) internal state variable. The first one (Sec. 4.2.2) corresponds to the case of the equation of balance of energy treated as the additional conservation equation implied by the equation of balance of entropy and the equation of evolution for the internal state variable, while the second (Sec. 4.2.3) corresponds to the case of the equation of balance of entropy treated

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as the additional conservation equation implied by the equation of balance of energy and by the equation of evolution for the internal state variable. The possibility of interchanging ("switching") the role of the ("original") conserved quantity (like energy) and the role of the additional ("derived") conserved quantity (like entropy) in analysing systems of conservation equations admitting the additional conservation equation ("equations with convex extensions"), was for the first time discussed by FRIEDRICHS and LAX [9].

Finally, the conditions of symmetric hyperbolicity for both the symmetric conservative systems are established with the aid of the relation between the respective potentials generating those two systems (Sec. 4.3.1), and with the aid of the relation between symmetric hyperbolicity of the symmetric system for original dependent variables and the corresponding symmetric conservative system (Sec. 4.3.2). The proof that one of the potentials is covex (concave) if and only if the second one is concave(convex) is given in the Appendix B.

It has been shown in Part I that the model of a rigid conductor of heat with (vector) internal state variable considered here comprises, as special cases, various phenomenological models proposed in the literature. In particular, the equations (1.1), (1.2), (1.3), when supplemented by the respective involutive constraints, can be interpreted as corresponding to the special case of the model of a rigid conductor of heat with scalar internal state variable called "semi-empirical temperature". Symmetrization of the first-order system of equations corresponding to the more general version of the "semi-empirical temperature" model (five conservation equations for five unknowns) has been considered by DOMAŃSKI, JABŁOŃSKI and KOSIŃSKI [14] with the aid of the converse to the condition of Friedrichs and Lax, and the results obtained there are discussed by the author in a separate note [15].

2. Basic equations, restrictions on constitutive functions and solutions of the MDR

2.1. Basic equations in a matrix form

Performing the respective differentiation, we rewrite the system (1.1) in a matrix form

$$
\begin{gathered}
\mathcal{A}^{0\Lambda}{}_M(u^K)\partial_t u^M + \mathcal{A}^{\alpha\Lambda}{}_M(u^K)\partial_\alpha u^M = b^\Lambda(u^K, t, X^\alpha), \\
\mathcal{A}^{0\Lambda}{}_M(u^K) = \frac{\partial g^{0\Lambda}(u^K)}{\partial u^M}, \qquad \mathcal{A}^{\alpha\Lambda}{}_M(u^K) = \frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M},
\end{gathered}
\tag{2.1}
$$

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with the 5×4 matrices $[\mathcal{A}^{0\Lambda}{}_M]$, $[\mathcal{A}^{\alpha\Lambda}{}_M]$ of the form

$$\left[\mathcal{A}^{0\Lambda}{}_M\right] = \left[\frac{\partial g^{0\Lambda}}{\partial u^M}\right] = \begin{bmatrix} \varrho_0 \dfrac{\partial \widetilde{\varepsilon}}{\partial \theta} & \varrho_0 \dfrac{\partial \widetilde{\varepsilon}}{\partial w_1} & \varrho_0 \dfrac{\partial \widetilde{\varepsilon}}{\partial w_2} & \varrho_0 \dfrac{\partial \widetilde{\varepsilon}}{\partial w_3} \\ 0 & \tau & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \tau \\ \varrho_0 \dfrac{\partial \widetilde{\eta}}{\partial \theta} & \varrho_0 \dfrac{\partial \widetilde{\eta}}{\partial w_1} & \varrho_0 \dfrac{\partial \widetilde{\eta}}{\partial w_2} & \varrho_0 \dfrac{\partial \widetilde{\eta}}{\partial w_3} \end{bmatrix},$$

(2.2)

$$\left[\mathcal{A}^{\alpha\Lambda}{}_M\right] = \left[\frac{\partial g^{\alpha\Lambda}}{\partial u^M}\right] = \begin{bmatrix} \dfrac{\partial \widetilde{q}^\alpha}{\partial \theta} & \dfrac{\partial \widetilde{q}^\alpha}{\partial w_1} & \dfrac{\partial \widetilde{q}^\alpha}{\partial w_2} & \dfrac{\partial \widetilde{q}^\alpha}{\partial w_3} \\ -\delta^\alpha{}_1 f_1'(\theta) & 0 & 0 & 0 \\ -\delta^\alpha{}_2 f_1'(\theta) & 0 & 0 & 0 \\ -\delta^\alpha{}_3 f_1'(\theta) & 0 & 0 & 0 \\ -\dfrac{1}{\theta^2}\widetilde{q}^\alpha + \dfrac{1}{\theta}\dfrac{\partial \widetilde{q}^\alpha}{\partial \theta} & \dfrac{1}{\theta}\dfrac{\partial \widetilde{q}^\alpha}{\partial w_1} & \dfrac{1}{\theta}\dfrac{\partial \widetilde{q}^\alpha}{\partial w_2} & \dfrac{1}{\theta}\dfrac{\partial \widetilde{q}^\alpha}{\partial w_3} \end{bmatrix}.$$

For completness, we recall that, according to FRIEDRICHS [2], the MDR requires the existence of $N+1$ functions $y_\Lambda : \mathbb{R}^N \to \mathbb{R}$, $\Lambda = 1, 2, \ldots, N+1$, not all identically zero, such that (Property CI in [2])

$$\begin{aligned} y_\Lambda(u^K)\frac{\partial g^{0\Lambda}(u^K)}{\partial u^M}\partial_t u^M + y_\Lambda(u^K)\frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M}\partial_\alpha u^M &\equiv 0, \\ y_\Lambda(u^K) b^\Lambda(u^K) &\equiv 0, \end{aligned} \tag{2.3}$$

holds for all functions $u^K(t, X^\alpha)$, $K = 1, 2, \ldots, N$.

The identity $(2.3)_1$ is equivalent to the following system of identities (Property CI' in [2]):

$$y_\Lambda(u^K)\frac{\partial g^{0\Lambda}(u^K)}{\partial u^M} \equiv 0, \qquad y_\Lambda(u^K)\frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M} \equiv 0. \tag{2.4}$$

The set of $N+1$ functions $y_\Lambda(u^K)$ is obtained as a solution of the overdetermined system of linear homogeneous equations $(2.3)_2$, (2.4) and therefore, if it exists, it is not unique.

2.2. Restrictions on constitutive functions

In Secs. 3.3, 3.4 of Part I, it has been found that the system (1.1) (1.2) satisfies the MDR if and only if constitutive functions $\widetilde{\varepsilon}$, $\widetilde{\eta}$ and $\widetilde{q}^{\,\alpha}$ satisfy the following relations:

$$\begin{aligned} \widetilde{q}^{\,\gamma} = -\frac{\varrho_0 \theta f_1'(\theta)}{\tau}\left(\frac{\partial \widetilde{\varepsilon}}{\partial w_\gamma} - \theta \frac{\partial \widetilde{\eta}}{\partial w_\gamma}\right) &= -\frac{\varrho_0 \theta f_1'(\theta)}{\tau}\frac{\partial \widetilde{\Psi}_C}{\partial w_\gamma}, \\ \frac{\partial \widetilde{\eta}}{\partial \theta} &= \frac{1}{\theta}\frac{\partial \widetilde{\varepsilon}}{\partial \theta}, \end{aligned} \tag{2.5}$$

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and

$$\widetilde{\eta} = -\frac{\partial \widetilde{\Psi}_C}{\partial \theta}, \qquad \widetilde{\varepsilon} = \widetilde{\Psi}_C - \theta \frac{\partial \widetilde{\Psi}_C}{\partial \theta}, \tag{2.6}$$

where the free energy $\widetilde{\Psi}_C$ is introduced

$$\widetilde{\Psi}_C(\theta, w_\gamma) = \widetilde{\varepsilon}(\theta, w_\gamma) - \theta \widetilde{\eta}(\theta, w_\gamma). \tag{2.7}$$

The entropy inequality implies

$$\frac{c_1}{\theta^2 f_1'(\theta)} \widetilde{q}^{\,\gamma} w_\gamma = -\frac{\varrho_0}{\tau} \frac{c_1}{\theta} \left(\frac{\partial \widetilde{\varepsilon}}{\partial w_\gamma} - \theta \frac{\partial \widetilde{\eta}}{\partial w_\gamma} \right) w_\gamma = -\frac{\varrho_0}{\tau} \frac{c_1}{\theta} \frac{\partial \widetilde{\Psi}_C}{\partial w_\gamma} w_\gamma \geq 0. \tag{2.8}$$

The restrictions on constitutive functions (2.5), (2.6), (2.7) enable us to transform the systems (1.1), (1.2) into quasi-linear symmetric systems of 4 equations for 4 unknowns. We recall that, in Part I (Sec. 3.2) of this paper, we have assumed that $\dfrac{\partial \widetilde{\varepsilon}(\theta, w_\gamma)}{\partial \theta} \neq 0$ (or, equivalently $\dfrac{\partial^2 \widetilde{\Psi}_C(\theta, w_\gamma)}{\partial \theta^2} \neq 0$) for all θ, w_γ in order to make it possible to derive thermodynamic restrictions via the MDR.

2.3. Family of solutions of the MDR

In Sec. 3.4 of Part I, a family of solutions $\mathbf{y}^T = [y_\Lambda]$ of the MDR (2.3), (2.4) has been calculated. Introducing the notation for y_Λ, $[y_\Lambda] = [\lambda, z^\gamma, \mu]$ where by λ, z^γ, μ we mean functions $\lambda(\theta, w_\gamma)$, $z^\beta(\theta, w_\gamma)$ and $\mu(\theta, w_\gamma)$, we have obtained

$$\begin{aligned} \theta &= -\frac{\mu}{\lambda} \\ z^\gamma &= -\lambda \frac{\varrho_0}{\tau} \left(\frac{\partial \widetilde{\varepsilon}}{\partial w_\gamma} - \theta \frac{\partial \widetilde{\eta}}{\partial w_\gamma} \right) = -\lambda \frac{\varrho_0}{\tau} \frac{\partial \widetilde{\Psi}_C}{\partial w_\gamma} \\ &= \mu \frac{\varrho_0}{\tau} \frac{1}{\theta} \left(\frac{\partial \widetilde{\varepsilon}}{\partial w_\gamma} - \theta \frac{\partial \widetilde{\eta}}{\partial w_\gamma} \right) = \mu \frac{\varrho_0}{\tau} \frac{1}{\theta} \frac{\partial \widetilde{\Psi}_C}{\partial w_\gamma} \\ &= \lambda \frac{1}{\theta f_1'(\theta)} \widetilde{q}^{\,\gamma} = -\mu \frac{1}{\theta^2 f_1'(\theta)} \widetilde{q}^{\,\gamma}, \end{aligned} \tag{2.9}$$

Thus, the family of solutions of the MDR can be written as

$$[y_\Lambda] = -\lambda \left[-1,\ \frac{\varrho_0}{\tau} \frac{\partial \widetilde{\Psi}_c}{\partial w_\gamma},\ \theta \right] = -\mu \left[\frac{1}{\theta},\ -\frac{1}{\theta} \frac{\varrho_0}{\tau} \frac{\partial \widetilde{\Psi}_c}{\partial w_\gamma},\ -1 \right], \tag{2.10}$$

where λ and μ are arbitrary functions of $[\theta, w_\gamma]$, and $\lambda = -1$ and $[\widehat{y}_\Lambda] = \left[-1, \dfrac{\varrho_0}{\tau} \dfrac{\partial \widetilde{\Psi}_c}{\partial w_\gamma}, \theta \right]$ correspond to the case when the equation of balance of energy is treated as the additional conservation equation, while $\mu = -1$ and $[\breve{y}_\Lambda] = \left[\dfrac{1}{\theta}, -\dfrac{1}{\theta} \dfrac{\varrho_0}{\tau} \dfrac{\partial \widetilde{\Psi}_c}{\partial w_\gamma}, -1 \right]$ correspond to the case when the equation of balance of entropy is treated as the additional conservation equation.

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3. Symmetric systems for unknowns $[\theta, w_\gamma]$

3.1. General procedure

The procedure of symmetrization of consistent systems of $N+1$ conservation equations for N unknown fields, that is systems of the type (1.1) given by Fredrichs [2], is employed to obtain the symmetric systems of the equations for $[\theta, w_\gamma]$. Differentiation of the identities (2.4) yields

$$
\begin{aligned}
\frac{\partial y_\Lambda}{\partial u^S}\frac{\partial g^{0\Lambda}}{\partial u^M} &= -y_\Lambda \frac{\partial^2 g^{0\Lambda}}{\partial u^S \partial u^M},\\
\frac{\partial y_\Lambda}{\partial u^S}\frac{\partial g^{\alpha\Lambda}}{\partial u^M} &= -y_\Lambda \frac{\partial^2 g^{\alpha\Lambda}}{\partial u^S \partial u^M}.
\end{aligned}
\tag{3.1}
$$

It follows from (3.1) that the $N \times (N+1)$ matrix $[\mathcal{K}_{S\Lambda}(u^K)]$ with entries

$$
\mathcal{K}_{S\Lambda}(u^K) = \frac{\partial y_\Lambda(u^K)}{\partial u^S}
\tag{3.2}
$$

is the left symmetrizer of the matrices $[\mathcal{A}^{0\Lambda}{}_M]$, $[\mathcal{A}^{\alpha\Lambda}{}_M]$. Left multiplication of the system (2.1) by the matrix $[\mathcal{K}_{S\Lambda}]$ gives therefore a symmetric system of N first order partial differential equations for N unknowns

$$
A^0{}_{SM}(u^K)\partial_t u^M + A^\alpha{}_{SM}(u^K)\partial_t u^M = p_S(u^K, t, X^\alpha),
\tag{3.3}
$$

where

$$
\begin{aligned}
A^0{}_{SM}(u^K) &= \mathcal{K}_{S\Lambda}(u^K)\mathcal{A}^{0\Lambda}{}_M(u^K) = \frac{\partial y_\Lambda(u^K)}{\partial u^S}\frac{\partial g^{0\Lambda}(u^K)}{\partial u^M},\\
A^\alpha{}_{SM}(u^K) &= \mathcal{K}_{S\Lambda}(u^K)\mathcal{A}^{\alpha\Lambda}{}_M(u^K) = \frac{\partial y_\Lambda(u^K)}{\partial u^S}\frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M},\\
p_S(u^K, t, X^\alpha) &= \mathcal{K}_{S\Lambda}(u^K) b^\Lambda(u^K, t, \alpha) = \frac{\partial y_\Lambda(u^K)}{\partial u^S} b^\Lambda(u^K, t, X^\alpha).
\end{aligned}
\tag{3.4}
$$

The set of $N+1$ functions $y_\Lambda(u^K)$ which are solution of the MDR (2.3), (2.4), can be treated as a vector in $\mathbb{R}^{N+1}$. Since it is a solution of a homogeneous system of equations (2.4) and $(2.3)_2$, it is not unique. It can be easily verified that, in the case of the system (1.1) containing N independent equations, the solution set of the MDR takes the form of a family of collinear vectors. As a system (1.1) containing N independent equations, we understand here the system (1.1) for which at least one of the $(N+1) \times N$ matrices $\mathcal{A}^{0\Lambda}{}_M(u^K)$, $\mathcal{A}^{\alpha\Lambda}{}_M(u^K)$ is of rank N for all u^K. It follows from (2.2) that, in the case of the considered system (1.1), (1.2), rank $\mathcal{A}^{0\Lambda}{}_M(u^K) = N$ for all u^K if $\dfrac{\partial \widetilde{\varepsilon}(\theta, w_\gamma)}{\partial \theta} \neq 0$ for all θ, w_γ, what means that internal energy $\widetilde{\varepsilon}$ is a monotone function of the temperature θ for all w_γ. This

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condition coincides with the condition (3.14) of Part I, which has been assumed as necessary for application of the MDR for derivation of the restrictions on constitutive functions implied by the Clausius–Duhem entropy inequality. Therefore the considered system (1.1), (1.2) is assumed to contain N independent equations and if $\widehat{\mathbf{y}} = [\widehat{y}_\Lambda]$ is a particular solution of $(2.3)_2$ and (2.4), then any other solution $\check{\mathbf{y}} = [\check{y}_\Lambda]$ of $(2.3)_2$, (2.4) can be expressed as $\check{\mathbf{y}} = \xi\widehat{\mathbf{y}}, \check{y}_\Lambda(u^K) = \xi(u^K)\widehat{y}_\Lambda(u^K)$, $\Lambda = 1, 2, ..., N+1$ and $\xi(u^K)$ is a differentiable function of u^K, not identically zero. It immediately follows from (2.1), (2.3) and (2.4) that symmetric systems corresponding to $[\check{y}_\Lambda]$ and $[\widehat{y}_\Lambda]$, that is $\widehat{A}^0_{SM}(u^K)\partial_t u^M + \widehat{A}^\alpha_{SM}(u^K)\partial_\alpha u^M = \widehat{p}_S(u^K)$ and $\check{A}^0_{SM}(u^K)\partial_t u^M + \check{A}^\alpha_{SM}(u^K)\partial_\alpha u^M = \check{p}_S(u^K)$, respectively, differ by a scalar factor $\xi(u^K)$ since $\check{A}^0_{SM}(u^K) = \xi(u^K)\widehat{A}^0_{SM}(u^K)$, $\check{A}^\alpha_{SM}(u^K) = \xi(u^K)\widehat{A}^\alpha_{SM}(u^K)$ and $\check{p}_S(u^K) = \xi(u^K)\widehat{p}_S(u^K)$. Hence, a symmetric system (3.3) can be associated to each solution of the MDR and, therefore, we have one-parameter family of symmetric systems (3.3) parameterized by suitably differentiable functions $\xi(u^K)$.

3.2. Family of symmetric systems for $[\theta, w_\gamma]$

In order to symmetrize the system (2.1), (2.2), (2.5), (2.6), (2.8), we calculate the matrix

$$
(3.5)\qquad \left[\widehat{\mathcal{K}}_{S\Lambda}\right] = \left[\frac{\partial \widehat{y}_\Lambda}{\partial u^S}\right] = \begin{bmatrix}
0 & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_1} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_2} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_3} & 1 \\
0 & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1^2} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_2} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_3} & 0 \\
0 & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_2\partial w_1} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_2^2} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_2\partial w_3} & 0 \\
0 & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_3\partial w_1} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_3\partial w_2} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_3^2} & 0
\end{bmatrix}
$$

for particular solution of the MDR $[\widehat{y}_\Lambda] = \left[-1,\ \dfrac{\varrho_0}{\tau}\dfrac{\partial\widetilde{\Psi}_c}{\partial w_\gamma},\ \theta\right]$ and, taking into account (2.5), (2.6), express the matrices $\mathcal{A}^{0\Lambda}_M$ and $\mathcal{A}^{\alpha\Lambda}_M$ in terms of θ, $f_1(\theta)$ and derivatives of $\widetilde{\Psi}_C$

$$
[\mathcal{A}^{0\Lambda}_M] = \begin{bmatrix}
-\varrho_0\theta\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta^2} & \varrho_0\left(\dfrac{\partial\widetilde{\Psi}_C}{\partial w_1} - \theta\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_1}\right) & \varrho_0\left(\dfrac{\partial\widetilde{\Psi}_C}{\partial w_2} - \theta\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_2}\right) & \varrho_0\left(\dfrac{\partial\widetilde{\Psi}_C}{\partial w_3} - \theta\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_3}\right) \\
0 & \tau & 0 & 0 \\
0 & 0 & \tau & 0 \\
0 & 0 & 0 & \tau \\
-\varrho_0\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta^2} & -\varrho_0\dfrac{\partial\widetilde{\Psi}_C}{\partial\theta\partial w_1} & -\varrho_0\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_2} & -\varrho_0\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_3}
\end{bmatrix},
$$

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$$[\mathcal{A}^{\alpha\Lambda}_{M}] = \begin{bmatrix} a^{\alpha}_{11} & -\frac{\varrho_0}{\tau}\theta f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial w_1} & -\frac{\varrho_0}{\tau}\theta f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial w_2} & -\frac{\varrho_0}{\tau}\theta f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial w_3} \\ -f_1'(\theta)\delta^{\alpha}_1 & 0 & 0 & 0 \\ -f_1'(\theta)\delta^{\alpha}_2 & 0 & 0 & 0 \\ -f_1'(\theta)\delta^{\alpha}_3 & 0 & 0 & 0 \\ a^{\alpha}_{51} & -\frac{\varrho_0}{\tau} f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial w_1} & -\frac{\varrho_0}{\tau} f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial w_2} & -\frac{\varrho_0}{\tau} f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial w_3} \end{bmatrix},$$

(3.6)
$$\begin{aligned} a^{\alpha}_{11} &= -\frac{\varrho_0}{\tau} f_1'(\theta)\frac{\partial\widetilde{\Psi}_C}{\partial w_\alpha} - \frac{\varrho_0}{\tau}\theta f_1''(\theta)\frac{\partial\widetilde{\Psi}_C}{\partial w_\alpha} - \frac{\varrho_0}{\tau}\theta f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial\theta}, \\ a^{\alpha}_{51} &= -\frac{\varrho_0}{\tau} f_1''(\theta)\frac{\partial\widetilde{\Psi}_C}{\partial w_\alpha} - \frac{\varrho_0}{\tau} f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial\theta}. \end{aligned}$$

Then, we obtain the matrices $\widehat{A}^0_{SM}$, $\widehat{A}^{\alpha}_{SM}$ and the production term $\widehat{p}_S$ of the symmetric system of four equations for four unknowns $[u^K] = [\theta, w_\gamma]$

$$\widehat{A}^0_{SM}(u^K)\partial_t u^M + \widehat{A}^{\alpha}_{SM}(u^K)\partial_\alpha u^M = \widehat{p}_S(u^K, t, X^\alpha),$$

$$[\widehat{A}^0_{SM}] = [\widehat{\mathcal{K}}_{S\Lambda}][\mathcal{A}^{0\Lambda}_{M}] = \varrho_0 \begin{bmatrix} -\frac{\partial^2\widetilde{\Psi}_C}{\partial\theta^2} & 0 & 0 & 0 \\ 0 & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_1^2} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_1 \partial w_2} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_1 \partial w_3} \\ 0 & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_2 \partial w_1} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_2^2} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_2 \partial w_3} \\ 0 & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_3 \partial w_1} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_3 \partial w_2} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_3^2} \end{bmatrix},$$

$$[\widehat{A}^{\alpha}_{SM}] = [\widehat{\mathcal{K}}_{S\Lambda}][\mathcal{A}^{\alpha\Lambda}_{M}] = -\frac{\varrho_0}{\tau} f_1'(\theta) \begin{bmatrix} \widehat{a}^{\alpha}_{11} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial w_1} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial w_2} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial w_3} \\ \frac{\partial^2\widetilde{\Psi}_C}{\partial w_1 \partial w_\alpha} & 0 & 0 & 0 \\ \frac{\partial^2\widetilde{\Psi}_C}{\partial w_2 \partial w_\alpha} & 0 & 0 & 0 \\ \frac{\partial^2\widetilde{\Psi}_C}{\partial w_3 \partial w_\alpha} & 0 & 0 & 0 \end{bmatrix},$$

(3.7)
$$\widehat{a}^{\alpha}_{11} = 2\frac{\partial^2\widetilde{\Psi}_C}{\partial\theta \partial w_\alpha} + \frac{f_1''(\theta)}{f_1'(\theta)}\frac{\partial\widetilde{\Psi}_C}{\partial w_\alpha},$$

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(3.7) [cont.]
$$[\widehat{p}_S] = [\widehat{\mathcal{K}}_{S\Lambda}][b^\Lambda] = c_1\frac{\varrho_0}{\tau}\left[\frac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_\alpha}w_\alpha - \frac{1}{\theta}\frac{\partial\widetilde{\Psi}_C}{\partial w_\alpha}w_\alpha + \frac{\tau}{c_1}\widetilde{r},\right.$$
$$\left.\frac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_\alpha}w_\alpha,\ \frac{\partial^2\widetilde{\Psi}_C}{\partial w_2\partial w_\alpha}w_\alpha,\ \frac{\partial^2\widetilde{\Psi}_C}{\partial w_3\partial w_\alpha}w_\alpha\right].$$

It should be noted that if we take the 4×4 minor $[\widehat{K}_{SM}]$ of the 4×5 matrix $[\widehat{\mathcal{K}}_{S\Lambda}]$, obtained by deleting the first column composed of zeros and take the 4×4 minors $[A^{0K}_M]$ and $[A^{\alpha K}_M]$ of the respective 5×4 matrices $[\mathcal{A}^{0\Lambda}_M]$ and $[\mathcal{A}^{\alpha\Lambda}_M]$ obtained by deleting the first row in each matrix, then the symmetric 4×4 matrices $[\widehat{A}^0_{SM}]$ and $[\widehat{A}^\alpha_{SM}]$ in the system (4.8) can be expressed as $[\widehat{A}^\alpha_{SM}] = [\widehat{K}_{SP}][A^{0P}_M]$ and $[\widehat{A}^\alpha_{SM}] = [\widehat{K}_{SP}][A^{\alpha P}_M]$. That is, $[\widehat{K}_{SP}]$ is the left symmetrizer of the square matrices $[A^{0P}_M]$, $[A^{\alpha P}_M]$.

Since the considered system (1.1), (1.2), (2.5), (2.6), (2.8) contains four independent equations, all symmetric systems with respect to the unknowns $[u^K] = [\theta, w_\gamma]$ can be obtained from (3.7) as the family of symmetric systems parametrized by differentiable functions $\xi(u^K)$,

$$\xi(u^K)\widehat{A}^0_{SM}(u^K)\partial_t u^M + \xi(u^K)\widehat{A}^\alpha_{SM}(u^K)\partial_\alpha u^M = \xi(u^K)\widehat{p}_S(u^K, t, X^\alpha), \tag{3.8}$$

where $\widehat{A}^0_{SM}$, $\widehat{A}^\alpha_{SM}$ and $\widehat{p}_S$ are given by (3.7). It follows from (2.10), (3.4), (3.7) that, for $\xi = 1$ we obtain the symmetric system corresponding to the case of balance of energy treated as the additional conservation equation and, for $\xi = -1/\theta$, we obtain the symmetric system corresponding to the case of balance of entropy treated as the additional conservation equation.

3.3. Equivalence of symmetric systems to original system of conservation equations

Sufficient condition. It has been proved by FRIEDRICHS [2] that nonsingularity of the $(N+1)\times(N+1)$ matrix composed of $y_\Lambda(u^K)$, $\dfrac{\partial y_\Lambda(u^K)}{\partial u^M}$

$$\begin{bmatrix} y_\Lambda(u^K) \\ \mathcal{K}_{M\Lambda}(u^K)\end{bmatrix} = \begin{bmatrix} y_\Lambda(u^K) \\ \dfrac{\partial y_\Lambda(u^K)}{\partial u^M}\end{bmatrix}, \tag{3.9}$$
$$\Lambda = 1,2,\ldots,N+1, \qquad K,M = 1,2,\ldots,N$$

for all u^K, for given solution $\mathbf{y}^T(u^K) = [y_\Lambda(u^K)]$ of the MDR, ensures the equivalence, for the classical (differentiable) solutions, of the symmetric system (3.3), (3.4) corresponding to $\mathbf{y}^T(u^K)$, to the original system of conservation equations (1.1). Equivalence is understood here in the sense that every classical (differentiable) solution of the symmetric system (3.8) satisfies the original system (1.1), and conversely. The matrices (3.9) have two interesting properties which can be formulated as the following Observations:

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OBSERVATION 1. For the system (1.1) containing N independent equations, nonsingularity (singularity) of the matrix (3.9) for particular solution of the MDR implies nonsingularity (singularity) of all matrices of the type (3.9) corresponding to the family of solutions of the MDR.

OBSERVATION 2. Alternative sufficient condition of equivalence. The matrix (3.9) is nonsingular if and only if among $N+1$ functions

$$[y_1(u^K), y_2(u^K), \ldots, y_{\Sigma-1}(u^K), y_\Sigma(u^K), y_{\Sigma+1}(u^K), \ldots, y_N(u^K), y_{N+1}(u^K)]$$

there are N independent functions (say,

$$[y_1(u^K), y_2(u^K), \ldots, y_{\Sigma-1}(u^K), y_{\Sigma+1}(u^K), \ldots, y_N(u^K), y_{N+1}(u^K)]$$

are independent, what means that the transformation

$$[u^1, u^2, \ldots, u^N] \to [y_1, y_2, \ldots, y_{\Sigma-1}, y_{\Sigma+1}, \ldots, y_N, y_{N+1}]$$

is invertible), and the remaining one (say, y_Σ) is not a homogeneous function of degree one of those independent functions (that is, the function $y_\Sigma(u^K(y_1, y_2, \ldots, y_{\Sigma-1}, y_{\Sigma+1}, \ldots, y_N, y_{N+1})) = \bar{y}_\Sigma(y_1, y_2, \ldots, y_{\Sigma-1}, y_{\Sigma+1}, \ldots, y_N, y_{N+1})$ is not a homogeneous function of degree one of the arguments $y_1, y_2, \ldots, y_{\Sigma-1}, y_{\Sigma+1}, \ldots, y_N, y_{N+1}$).

Observation 1 is a direct consequence of Lemma 1, and Observation 2 immediately follows from Lemma 2 and Lemma 3 proved in the Appendix A.

If rank $[\mathcal{K}_{S\Lambda}(u^K)] = N$ for all $u^K \in \mathcal{D}$ ($\mathcal{D}$ – convex domain in $\mathbb{R}^N$) and consequently, $y_1, \ldots, y_{\Sigma-1}, y_{\Sigma+1}, \ldots, y_{N+1}$ can be taken as new dependent variables, then the condition that $\bar{y}_\Sigma(y_1, \ldots, y_{\Sigma-1}, y_{\Sigma+1}, \ldots, y_{N+1})$ is not a homogeneous function of degree one with respect to all arguments expresses the fact that, roughly speaking, all equations of the system (1.1) are nontrivially involved in the MDR. It means that in the system (1.1) none of the equations is merely a linear combination with certain numerical factors of other equations or, in other words, any system of N equations selected from $N+1$ equations of (1.1) is the system of N independent equations. Equivalently, the solution set of the MDR takes the form of a family of collinear vectors in $\mathbb{R}^{N+1}$ in which there are no components y_Λ differing only by a numerical factor, and/or none of the components y_Λ identically vanishes ($\bar{y}_\Lambda(z_K) \equiv 0$ is a homogeneous function of degree one of $z_1, z_2, \ldots z_N$).

3.4. Equivalence of the family of symmetric systems for $[\theta, w_\gamma]$

It follows from Observation 1 that in order to establish the sufficient condition of equivalence for the system (1.1), (1.2), (2.5), (2.6), (2.8), it suffices to consider the matrix of the type (3.9) for one particular solution of the MDR selected from

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the family (2.10). Hence, we select the solution $[\widehat{y}] = [-1, \widehat{z}_\gamma, \widehat{\mu}]$, $\widehat{z}_\gamma = \dfrac{\varrho_0}{\tau}\dfrac{\partial\widetilde{\psi}_C}{\partial w_\gamma}$, $\widehat{\mu} = \theta$, and take the following 5×5 matrix

$$(3.10)\qquad \begin{bmatrix} \widehat{y}_\Lambda \\ \widehat{\mathcal{K}}_{S\Lambda} \end{bmatrix} = \begin{bmatrix} -1 & \widehat{z}_1 & \widehat{z}_2 & \widehat{z}_3 & \widehat{\mu} \\ 0 & \dfrac{\partial\widehat{z}_1}{\partial\theta} & \dfrac{\partial\widehat{z}_2}{\partial\theta} & \dfrac{\partial\widehat{z}_3}{\partial\theta} & \dfrac{\partial\widehat{\mu}}{\partial\theta} \\ 0 & \dfrac{\partial\widehat{z}_1}{\partial w_1} & \dfrac{\partial\widehat{z}_2}{\partial w_1} & \dfrac{\partial\widehat{z}_3}{\partial w_1} & \dfrac{\partial\widehat{\mu}}{\partial w_1} \\ 0 & \dfrac{\partial\widehat{z}_1}{\partial w_2} & \dfrac{\partial\widehat{z}_2}{\partial w_2} & \dfrac{\partial\widehat{z}_3}{\partial w_2} & \dfrac{\partial\widehat{\mu}}{\partial w_2} \\ 0 & \dfrac{\partial\widehat{z}_1}{\partial w_3} & \dfrac{\partial\widehat{z}_2}{\partial w_3} & \dfrac{\partial\widehat{z}_3}{\partial w_3} & \dfrac{\partial\widehat{\mu}}{\partial w_3} \end{bmatrix}$$

$$= \frac{\varrho_0}{\tau} \begin{bmatrix} -\dfrac{\tau}{\varrho_0} & \dfrac{\partial\widetilde{\Psi}_C}{\partial w_1} & \dfrac{\partial\widetilde{\Psi}_C}{\partial w_2} & \dfrac{\partial\widetilde{\Psi}_C}{\partial w_3} & \dfrac{\tau}{\varrho_0}\theta \\ 0 & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial\theta} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_2\partial\theta} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_3\partial\theta} & \dfrac{\tau}{\varrho_0} \\ 0 & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial^2 w_1} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_2} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_3} & 0 \\ 0 & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_2} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial^2 w_2} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_2\partial w_3} & 0 \\ 0 & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_3} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_2\partial w_3} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial^2 w_3} & 0 \end{bmatrix}.$$

The matrix (3.10) is nonsingular and, as a consequence, the equivalence of the symmetric systems is ensured if the constitutive function $\widetilde{\Psi}_C(\theta, w_\gamma)$ satisfies the condition

$$(3.11)\qquad \det\left[\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha\partial w_\beta}\right] = \det\left[\frac{\partial^2\widetilde{\varepsilon}}{\partial w_\alpha\partial w_\beta} - \theta\frac{\partial^2\widetilde{\eta}}{\partial w_\alpha\partial w_\beta}\right] \neq 0 \quad \text{for all} \quad [\theta, w_\gamma] \in D.$$

3.5. Symmetric hyperbolicity of symmetric systems for $[\theta, w_\gamma]$

3.5.1. Definition. According to the definition of FRIEDRICHS [2], a symmetric system (3.3) is symmetric hyperbolic (in time direction t) in an open convex domain $\mathcal{D} \subset \mathbb{R}^N$ if the matrix $[A^0_{SM}]$ is positive definite for all $u^K \in \mathcal{D}$.

3.5.2. Symmetric hyperbolicity of the system (3.7). It follows from $(3.7)_2$ that $[\widehat{A}^0_{SM}]$ is positive (negative) definite for $[\theta, w_1, w_2, w_3]$ from a convex domain $\mathcal{D} \subset \mathbb{R}^4$ if an only if $\widetilde{\Psi}_C(\theta, w_1, w_2, w_3)$ has the following properties:

$$\frac{\partial^2\widetilde{\Psi}_C}{\partial\theta^2} < 0 \qquad \left(\frac{\partial^2\widetilde{\Psi}_C}{\partial\theta^2} > 0\right)$$

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($\widetilde{\Psi}_C$ is a concave (convex) function of θ while w_1, w_2, w_3 play the role of fixed parameters) and $\widetilde{\Psi}_C$ is a convex (concave) function of the three arguments w_1, w_2, w_3 while θ plays the role of a fixed parameter, for all $[\theta, w_1, w_2, w_3] \in \mathcal{D}$. In the case of $[\widehat{A}^0_{SM}]$ positive definite, the system $(3.7)_1$ is symmetric hyperbolic while, in the case of $[\widehat{A}^0_{SM}]$ negative definite, it can be transformed to the equivalent symmetric hyperbolic system by multiplying each equation by the factor (-1).

3.5.3. Symmetric hyperbolicity of the family (3.8). In choosing $\xi(u^K)$, the condition of symmetric hyperbolicity (positive definite $[\xi \widehat{A}^0_{SM}]$) should be taken into account. Since the conditions of positive (negative) definiteness of the matrix $[\widehat{A}^0_{SM}]$ are established above, the conditions on $\xi(u^K)$ ensuring positive definiteness of $[\xi \widehat{A}^0_{SM}]$ follow from the fact that, if $[\widehat{A}^0_{SM}(u^K)]$ is positive (negative) definite for $[u^K] \in \mathcal{D}$ then $[\xi(u^K)\widehat{A}^0_{SM}(u^K)]$ is positive (negative) for $\xi(u^K) > 0$, $[u^K] \in \mathcal{D}$ and negative (positive) definite for $\xi(u^K) < 0$, $[u^K] \in \mathcal{D}$.

3.5.4. Relation to the condition of equivalence to the original system of conservation equations (1.1). It should be noted that the condition rank $[\mathcal{K}_{S\Lambda}(u^K)] = N$ for all $u^K \in \mathcal{D}$ (which means that among $N+1$ functions $y(u^K)$ there are N independent functions of u^K, $u^K \in \mathcal{D}$) is necessary for symmetric hyperbolicity of the symmetric system (3.3), (3.4) (necessary for positive definiteness of $[A^0_{SM}] = [\mathcal{K}_{S\Lambda}][\mathcal{A}^0_{SM}]$) and necessary for nonsingularity of the matrix (3.9). It is the necessary and sufficient condition for transformation of dependent variables $u^K = u^K(y_1, \ldots, y_{\Sigma-1}, y_{\Sigma+1}, \ldots, y_{N+1})$ (where $y_1(u^K), \ldots, y_{\Sigma-1}(u^K)$, $y_{\Sigma+1}(u^K), \ldots$, $y_{N+1}(u^K)$ are independent functions of u^K) and such transformations will be employed in the following to obtain symmetric conservative systems. The restriction on $\widetilde{\Psi}_C(\theta, w_\gamma)$ imposed by the condition of symmetric hyperbolicity of the system (3.7) is much stronger (positive definiteness of $(\partial^2 \widetilde{\Psi}_C)/(\partial w_\alpha \partial w_\beta)$ for all $[\theta, w_\gamma] \in \mathcal{D}$) than the condition (3.11) ensuring the equivalence of the symmetric systems and the original system of conservation equations or, in other words, symmetric hyperbolicity of (3.7) implies equivalence to original system of conservation equations (1.1).

4. Alternative symmetric conservative systems

4.1. Basic relations

4.1.1. Class of symmetric conservative systems. In [10], GODUNOV introduced the class of quasi-linear symmetric conservative systems of the following form

$$\partial_t \left(\frac{\partial \bar{\varphi}^0(l_M)}{\partial l_K} \right) + \partial_\alpha \left(\frac{\partial \bar{\varphi}^\alpha(l_M)}{\partial l_K} \right) = b^K(l_M), \tag{4.1}$$
$$\alpha = 1, 2, \ldots, m, \qquad I, K = 1, 2, \ldots, N.$$

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The system (4.1) of N equations for N unknowns $l_K(t, X^\alpha)$ is specified completely by $m+1$ functions $\bar{\varphi}^0(l_M)$, $\bar{\varphi}^\alpha(l_M)$ and N functions $b^K(l_M)$. It implies the additional conservation equation which is obtained by multiplying (4.1) by the row vector $[l_1, l_2, \ldots, l_N]$

$$\partial_t\left(l_K\frac{\partial\bar{\varphi}^0(l_M)}{\partial l_K} - \bar{\varphi}^0\right) + \partial_\alpha\left(l_K\frac{\partial\bar{\varphi}^\alpha(l_M)}{\partial l_K} - \bar{\varphi}^\alpha\right) = l_K b^K(l_M). \tag{4.2}$$

4.1.2. Direct derivation of $N+1$ alternative symmetric conservative systems. For derivation of alternative symmetric conservative systems, we assume that the system of $N+1$ conservation equations (1.1) contains N independent equations, satisfies the MDR (2.3), (2.4) (hence, solution set of the MDR takes the form of a family of collinear row vectors in $\mathbb{R}^{N+1}$) and the matrices (3.9) corresponding to the solutions of the MDR are nonsingular for all $u^K \in \mathcal{D}$. In order to derive alternative symmetric conservative systems, we define the potentials

$$\varphi^0 = y_\Lambda(u^K)g^{0\Lambda}(u^K), \qquad \varphi^\alpha = y_\Lambda(u^K)g^{\alpha\Lambda}(u^K). \tag{4.3}$$

According to Observation 1 and Observation 2, nonsingularity of the matrix (3.9) (for any solution of the MDR) for all $u^K \in \mathcal{D}$ implies that the family of solutions of the MDR contains $N+1$ vectors $\overset{(\Sigma)}{\mathbf{y}}$ such that Σ-th component is -1, namely, $\overset{(\Sigma)}{y} = \left[\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, -1, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right]$, $\Sigma = 1, 2, \ldots, N+1$, and each of $N+1$ transformations $[u^1, \ldots, u^N] \to \left[\overset{(\Sigma)}{y}_1 \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right]$, is invertible. Hence, the components $[\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}]$ can be taken as new dependent variables and the respective potentials (4.3) can be expressed as functions of these new variables,

$$\begin{aligned}
\overset{(\Sigma)}{\varphi^0} = \sum_{\substack{\Lambda=1\\ \Lambda\neq\Sigma}}^{N+1} \overset{(\Sigma)}{y}_\Lambda\, g^{0\Lambda}\left(u^K\left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right)\right)&\\
- g^{0\Sigma}\left(u^K\left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right)\right)&,\\
\overset{(\Sigma)}{\varphi^\alpha} = \sum_{\substack{\Lambda=1\\ \Lambda\neq\Sigma}}^{N+1} \overset{(\Sigma)}{y}_\Lambda\, g^{\alpha\Lambda}\left(u^K\left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right)\right)&\\
- g^{\alpha\Sigma}\left(u^K\left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right)\right)&.
\end{aligned} \tag{4.4}$$

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Differentiating (4.4) with respect to $\overset{(\Sigma)}{y}_{\Delta}$, $\Delta = 1, 2, \ldots, \Sigma - 1, \Sigma + 1, \ldots, N + 1$, and taking into account the MDR (2.4), we obtain

$$
\begin{aligned}
\frac{\partial \overset{(\Sigma)}{\varphi^0}}{\partial \overset{(\Sigma)}{y_\Delta}} &= g^{0\Delta}\left(u^K\left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right)\right), \\
\frac{\partial \overset{(\Sigma)}{\varphi^\alpha}}{\partial \overset{(\Sigma)}{y_\Delta}} &= g^{\alpha\Delta}\left(u^K\left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right)\right),
\end{aligned}
\tag{4.5}
$$

and

$$
\begin{aligned}
\sum_{\substack{\Delta=1\\ \Delta\neq\Sigma}}^{N+1} \overset{(\Sigma)}{y}_\Delta \frac{\partial \overset{(\Sigma)}{\varphi^0}}{\partial \overset{(\Sigma)}{y_\Delta}} - \overset{(\Sigma)}{\varphi^0} &= g^{0\Sigma}\left(u^K\left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right)\right), \\
\sum_{\substack{\Delta=1\\ \Delta\neq\Sigma}}^{N+1} \overset{(\Sigma)}{y_\Delta} \frac{\partial \overset{(\Sigma)}{\varphi^\alpha}}{\partial \overset{(\Sigma)}{y_\Delta}} - \overset{(\Sigma)}{\varphi^\alpha} &= g^{\alpha\Sigma}\left(u^K\left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right)\right).
\end{aligned}
\tag{4.6}
$$

Thus, the introduced transformation of dependent variables enables us to express the equations $\partial_t g^{0\Delta}(u^K) + \partial_\alpha g^{\alpha\Delta}(u^K) = b^\Delta(u^K, t, X^\gamma)$, $\Delta = 1, 2, \ldots, \Sigma - 1, \Sigma + 1, \ldots, N + 1$ of the system (1.1) in symmetric conservative form (4.1)

$$
\partial_t\left(\frac{\partial \overset{(\Sigma)}{\varphi^0}}{\partial \overset{(\Sigma)}{y_\Delta}}\right) + \partial_\alpha\left(\frac{\partial \overset{(\Sigma)}{\varphi^\alpha}}{\partial \overset{(\Sigma)}{y_\Delta}}\right) = b^\Delta\left(u^K\left(\overset{(\Sigma)}{y_\Omega}\right), t, X^\gamma\right),
\tag{4.7}
$$
$$
\Delta, \Omega = 1, 2, \ldots, \Sigma - 1, \Sigma + 1, \ldots, N + 1,
$$

while the equation $\partial_t g^{0\Sigma}(u^K) + \partial_\alpha g^{\alpha\Sigma}(u^K) = b^\Sigma(u^K, t, X^\gamma)$ takes the form (4.2) and is interpreted as the additional conservation equation

$$
\partial_t\left(\sum_{\substack{\Delta=1\\ \Delta\neq\Sigma}}^{N+1} \overset{(\Sigma)}{y}_\Delta \frac{\partial \overset{(\Sigma)}{\varphi^0}}{\partial \overset{(\Sigma)}{y_\Delta}} - \overset{(\Sigma)}{\varphi^0}\right) + \partial_\alpha\left(\sum_{\substack{\Delta=1\\ \Delta\neq\Sigma}}^{N+1} \overset{(\Sigma)}{y_\Delta} \frac{\partial \overset{(\Sigma)}{\varphi^\alpha}}{\partial \overset{(\Sigma)}{y_\Delta}} - \overset{(\Sigma)}{\varphi^\alpha}\right)
= \sum_{\substack{\Delta=1\\ \Delta\neq\Sigma}}^{N+1} \overset{(\Sigma)}{y_\Delta}\, b^\Delta\left(u^K\left(\overset{(\Sigma)}{y_\Omega}\right), t, X^\gamma\right).
\tag{4.8}
$$

In this way, $N + 1$ alternative symmetric conservative systems corresponding to the system (1.1) can be obtained.

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4.1.3. Symmetric hyperbolicity of symmetric conservative systems. According to the definition of symmetric hyperboliccity of symmetric systems (see Sec. 3.5.1), the symmetric conservative system (4.7) is symmetric hyperbolic if the Hessian matrix

$$\left[\frac{\partial \overset{(\Sigma)}{\varphi}{}^{0}}{\partial \overset{(\Sigma)}{y}_{\Delta}\, \partial \overset{(\Sigma)}{y}_{\Omega}}\right]$$

is positive definite, what is equivalent to convexity of the potential $\overset{(\Sigma)}{\varphi}{}^{0}$. In the case of concave $\overset{(\Sigma)}{\varphi}{}^{0}$ and

$$\left[\frac{\partial^2 \overset{(\Sigma)}{\varphi}{}^{0}}{\partial \overset{(\Sigma)}{y}_{\Delta}\, \partial \overset{(\Sigma)}{y}_{\Omega}}\right]$$

negative definite, the system (4.7) can be brought into equivalent symmetric conservative system simply by multiplying each equation by the factor (-1).

4.1.4. Relations between alternative symmetric conservative systems. In order to demonstrate the relations betwen different symmetric conservative systems corresponding to the same system of $N+1$ conservation equations (satisfying the assumptions mentioned in Sec. 4.1.2), we take two arbitrarily chosen solutions of the MDR

$$\overset{(\Sigma)}{\mathbf{y}} = \left[\overset{(\Sigma)}{y}_{1}, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, -1, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right]$$

and

$$\overset{(\Delta)}{\mathbf{y}} = \left[\overset{(\Delta)}{y}_{1}, \ldots, \overset{(\Delta)}{y}_{\Delta-1}, -1, \overset{(\Delta)}{y}_{\Delta+1}, \ldots, \overset{(\Delta)}{y}_{N+1}\right]$$

(without loss of generality, we may assume $1 \le \Sigma < \Delta \le N+1$). The symmetric conservative system can be assigned to each of them according to (4.4) – (4.8). The components of $\overset{(\Sigma)}{\mathbf{y}}$ and $\overset{(\Delta)}{\mathbf{y}}$ are mutualy related

$$\begin{gathered}\overset{(\Delta)}{y}_{\Lambda} = -\frac{\overset{(\Sigma)}{y}_{\Lambda}}{\overset{(\Sigma)}{y}_{\Delta}}, \qquad \overset{(\Sigma)}{y}_{\Lambda} = -\frac{\overset{(\Delta)}{y}_{\Lambda}}{\overset{(\Delta)}{y}_{\Sigma}}, \qquad \Lambda \neq \Delta, \qquad \Lambda \neq \Sigma, \\ \overset{(\Delta)}{y}_{\Sigma}\ \overset{(\Sigma)}{y}_{\Delta} = 1, \qquad \overset{(\Delta)}{y}_{\Delta} = \overset{(\Sigma)}{y}_{\Sigma} = -1.\end{gathered} \tag{4.9}$$

According to (4.3), (4.4), the respective potentials $\left[\overset{(\Sigma)}{\varphi}{}^{0}, \overset{(\Sigma)}{\varphi}{}^{\alpha}\right]$ and $\left[\overset{(\Delta)}{\varphi}{}^{0}, \overset{(\Delta)}{\varphi}{}^{\alpha}\right]$, corresponding to $\overset{(\Sigma)}{\mathbf{y}}$ and $\overset{(\Delta)}{\mathbf{y}}$, can be obtained and, in view of (4.9), they satisfy

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the following relations:

(4.10)
$$\begin{aligned}
&\overset{(\Delta)}{\varphi}{}^{0}\left(\overset{(\Delta)}{y}_{1},\ldots,\overset{(\Delta)}{y}_{\Sigma+1},\overset{(\Delta)}{y}_{\Sigma},\overset{(\Delta)}{y}_{\Sigma+1},\ldots,\overset{(\Delta)}{y}_{\Delta-1},\overset{(\Delta)}{y}_{\Delta+1},\ldots,\overset{(\Delta)}{y}_{N+1}\right)\\
&\quad=-\overset{(\Delta)}{y}_{\Sigma}\overset{(\Sigma)}{\varphi}{}^{0}\left(-\frac{\overset{(\Delta)}{y}_{1}}{\overset{(\Delta)}{y}_{\Sigma}},\ldots,-\frac{\overset{(\Delta)}{y}_{\Sigma-1}}{\overset{(\Delta)}{y}_{\Sigma}},-\frac{\overset{(\Delta)}{y}_{\Sigma+1}}{\overset{(\Delta)}{y}_{\Sigma}},\ldots,\right.\\
&\qquad\left.-\frac{\overset{(\Delta)}{y}_{\Delta-1}}{\overset{(\Delta)}{y}_{\Sigma}},\frac{1}{\overset{(\Delta)}{y}_{\Sigma}},-\frac{\overset{(\Delta)}{y}_{\Delta+1}}{\overset{(\Delta)}{y}_{\Sigma}},\ldots,-\frac{\overset{(\Delta)}{y}_{N+1}}{\overset{(\Delta)}{y}_{\Sigma}}\right),\\
&\overset{(\Delta)}{\varphi}{}^{\alpha}\left(\overset{(\Delta)}{y}_{1},\ldots,\overset{(\Delta)}{y}_{\Sigma+1},\overset{(\Delta)}{y}_{\Sigma},\overset{(\Delta)}{y}_{\Sigma+1},\ldots,\overset{(\Delta)}{y}_{\Delta-1},\overset{(\Delta)}{y}_{\Delta+1},\ldots,\overset{(\Delta)}{y}_{N+1}\right)\\
&\quad=-\overset{(\Delta)}{y}_{\Sigma}\overset{(\Sigma)}{\varphi}{}^{\alpha}\left(-\frac{\overset{(\Delta)}{y}_{1}}{\overset{(\Delta)}{y}_{\Sigma}},\ldots,-\frac{\overset{(\Delta)}{y}_{\Sigma-1}}{\overset{(\Delta)}{y}_{\Sigma}},-\frac{\overset{(\Delta)}{y}_{\Sigma+1}}{\overset{(\Delta)}{y}_{\Sigma}},\ldots,\right.\\
&\qquad\left.-\frac{\overset{(\Delta)}{y}_{\Delta-1}}{\overset{(\Delta)}{y}_{\Sigma}},\frac{1}{\overset{(\Delta)}{y}_{\Sigma}},-\frac{\overset{(\Delta)}{y}_{\Delta+1}}{\overset{(\Delta)}{y}_{\Sigma}},\ldots,-\frac{\overset{(\Delta)}{y}_{N+1}}{\overset{(\Delta)}{y}_{\Sigma}}\right),
\end{aligned}$$

and

(4.11)
$$\begin{aligned}
&\overset{(\Sigma)}{\varphi}{}^{0}\left(\overset{(\Sigma)}{y}_{1},\ldots,\overset{(\Sigma)}{y}_{\Sigma-1},\overset{(\Sigma)}{y}_{\Sigma+1},\ldots,\overset{(\Sigma)}{y}_{\Delta-1},\overset{(\Sigma)}{y}_{\Delta},\overset{(\Sigma)}{y}_{\Delta+1},\ldots,\overset{(\Sigma)}{y}_{N+1}\right)\\
&\quad=-\overset{(\Sigma)}{y}_{\Delta}\overset{(\Delta)}{\varphi}{}^{0}\left(-\frac{\overset{(\Sigma)}{y}_{1}}{\overset{(\Sigma)}{y}_{\Delta}},\ldots,-\frac{\overset{(\Sigma)}{y}_{\Sigma-1}}{\overset{(\Sigma)}{y}_{\Delta}},\frac{1}{\overset{(\Sigma)}{y}_{\Delta}},-\frac{\overset{(\Sigma)}{y}_{\Sigma+1}}{\overset{(\Sigma)}{y}_{\Delta}},\ldots,\right.\\
&\qquad\left.-\frac{\overset{(\Sigma)}{y}_{\Delta-1}}{\overset{(\Sigma)}{y}_{\Delta}},-\frac{\overset{(\Sigma)}{y}_{\Delta+1}}{\overset{(\Sigma)}{y}_{\Delta}},\ldots,-\frac{\overset{(\Sigma)}{y}_{N+1}}{\overset{(\Sigma)}{y}_{\Delta}}\right),\\
&\overset{(\Sigma)}{\varphi}{}^{\alpha}\left(\overset{(\Sigma)}{y}_{1},\ldots,\overset{(\Sigma)}{y}_{\Sigma+1},\overset{(\Sigma)}{y}_{\Sigma+1},\ldots,\overset{(\Sigma)}{y}_{\Delta-1},\overset{(\Sigma)}{y}_{\Delta},\overset{(\Sigma)}{y}_{\Delta+1},\ldots,\overset{(\Sigma)}{y}_{N+1}\right)\\
&\quad=-\overset{(\Sigma)}{y}_{\Sigma}\overset{(\Delta)}{\varphi}{}^{\alpha}\left(-\frac{\overset{(\Sigma)}{y}_{1}}{\overset{(\Sigma)}{y}_{\Delta}},\ldots,-\frac{\overset{(\Sigma)}{y}_{\Sigma-1}}{\overset{(\Sigma)}{y}_{\Delta}},\frac{1}{\overset{(\Sigma)}{y}_{\Delta}},-\frac{\overset{(\Sigma)}{y}_{\Sigma+1}}{\overset{(\Sigma)}{y}_{\Delta}},\ldots,\right.\\
&\qquad\left.-\frac{\overset{(\Sigma)}{y}_{\Delta-1}}{\overset{(\Sigma)}{y}_{\Delta}},-\frac{\overset{(\Sigma)}{y}_{\Delta+1}}{\overset{(\Sigma)}{y}_{\Delta}},\ldots,-\frac{\overset{(\Sigma)}{y}_{N+1}}{\overset{(\Sigma)}{y}_{\Delta}}\right).
\end{aligned}$$

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If the system (4.7), (4.8) corresponding to $\overset{(\Sigma)}{\mathbf{y}}$ is given, then, with the aid of the relations (4.9), (4.10), (4.11), the "new" system (4.7), (4.8) corresponding to $\overset{(\Delta)}{\mathbf{y}}$ can be obtained. Of course, in such transformations the "role" of the equations in the system (4.7), (4.8) changes, namely, the equation (4.8) of the system (4.7), (4.8) corresponding to $\overset{(\Sigma)}{\mathbf{y}}$ will be, according to (4.9), (4.10), (4.11), transformed to a "member" of the system (4.7) for $\overset{(\Delta)}{\mathbf{y}}$, while this member of the system (4.7) for $\overset{(\Sigma)}{\mathbf{y}}$ which corresponds to the conservation equation $\partial_t g^{0\Delta}(u^K) + \partial_\alpha g^{\alpha\Delta}(u^K) = b^\Delta(u^K, t, X^\gamma)$ in (1.1) will be transformed to the equation (4.8) of the system (4.7), (4.8) for $\overset{(\Delta)}{\mathbf{y}}$. This transformation can be easily demonstrated by substituting (4.9) into (4.7), (4.8), performing the respective differentiations and taking into account (4.9), (4.10), (4.11), (4.4).

With the aid of $(4.10)_1$, $(4.11)_1$, the relation between convexity (concavity) of $\overset{(\Delta)}{\varphi}{}^0$ and concavity (convexity) of $\overset{(\Sigma)}{\varphi}{}^0$ can be established provided that either $\overset{(\Delta)}{y}{}_\Sigma(u^K) > 0$ or $\overset{(\Delta)}{y}{}_\Sigma(u^K) < 0$ for all u^K. Since the proof of this relation in a general case requires lengthy calculations, we restrict ourselvs to demonstrating this relation for the case of two symmetric conservative systems given in Secs. 4.2.2 and 4.2.3. Therefore, for those systems we derive the relations (4.9), (4.10), (4.11) in Sec. 4.3.1 and examine convexity (concavity) of the respective potentials in Appendix B. The reasoning given in Appendix B can be directly generalized for $(4.10)_1$, $(4.11)_1$. It should be noted that the problem of transformation of the given symmetric conservative system (4.1) in one spatial dimension into another symmetric conservative system has been considered by Godunov and Sultangazian [11, 12]. For such transformation, the relations differing by numerical factor (-1) from $(4.9)_{1,2}$ were postulated in [11, 12]. Hence, the procedure of derivation of "new" symmetric conservative system from the "old" one proposed by Godunov and Sultangazin [11, 12] results in changing the sign of that conservation equation in "new" symmetric conservative system which is the additional conservation equation implied by the "old" symmetric conservative system and, as a consequence, the potential $\varphi^0(l_K)$ (see, (4.1)) for "new" symmetric conservative system and the potential $\varphi^0(l_K)$ for "old" symmetric conservative system are both convex (concave). In our approach to the derivation of alternative symmetric conservative systems, the signs of all $N+1$ conservation equations are preserved.

4.2. Two alternative symmetric conservative systems corresponding to (1.1), (1.2)

4.2.1. Family of potentials corresponding to (1.1), (1.2). According to the general procedure developed in Sec. 4.1.2, we derive in this paper two alternative symmetric conservative systems, one corresponding to the case of balance of en-

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ergy treated as the additional conservation equation and described by $\mathbf{y} = \widehat{\mathbf{y}} = [-1, \widehat{z}^{\gamma}, \widehat{\mu}]$, and the second one corresponding to the case of balance of entropy treated as the additional conservation equation and described by $\mathbf{y} = \check{\mathbf{y}} = [\check{\lambda}, \check{z}^{\gamma}, -1]$.

In order to transform the system of field equations (1.1), (1.2) with constitutive functions satisfying the conditions (2.5), (2.8) into symmetric conservative systems with respect to the field variables $[\check{\lambda}, \check{z}^{\gamma}]$ or $[\check{z}^{\gamma}, \check{\mu}]$, we therefore introduce the following potentials (4.3):

$$(4.12)\qquad \begin{aligned} \varphi_C^0 &= \lambda \varrho_0 \widetilde{\varepsilon} + z^{\gamma} \tau w_{\gamma} + \mu \varrho_0 \widetilde{\eta}, \\ \varphi_C^{\alpha} &= \lambda \widetilde{q}^{\alpha} - z^{\alpha} f_1(\theta) + \mu \frac{1}{\theta} \widetilde{q}^{\alpha}. \end{aligned}$$

In view of (2.9), (2.5), (2.6), potentials (4.12) can be expressed as

$$(4.13)\qquad \begin{aligned} \varphi_C^0 &= \lambda \widetilde{\varphi}_C^0 = -\frac{\mu}{\theta} \widetilde{\varphi}_C^0, \\ \varphi_C^{\alpha} &= -z^{\alpha} f_1(\theta) = \lambda \widetilde{\varphi}_C^{\alpha} = -\frac{\mu}{\theta} \widetilde{\varphi}_C^{\alpha}, \end{aligned}$$

where

$$(4.14)\qquad \begin{aligned} \widetilde{\varphi}_C^0 &= \varrho_0 \widetilde{\varepsilon} + \frac{f_1(\theta)}{\theta f_1'(\theta)} \widetilde{q}^{\gamma} w_{\gamma} - \varrho_0 \theta \widetilde{\eta} = \varrho_0 \left[\widetilde{\Psi}_C - \frac{\partial \widetilde{\Psi}_C}{\partial w_{\gamma}} w_{\gamma} \right], \\ \widetilde{\varphi}_C^{\alpha} &= -\frac{f_1(\theta)}{\theta f_1'(\theta)} \widetilde{q}^{\alpha} = \frac{\varrho_0}{\tau} f_1(\theta) \frac{\partial \widetilde{\Psi}_C}{\partial w_{\alpha}}. \end{aligned}$$

4.2.2. Balance of energy treated as the additional conservation equation. When the first equation of the system (1.1), (1.2) corresponding to the balance of energy is treated as the additional conservation equation implied by the system of four remaining equations, we put $\lambda = -1$ in (2.10), (2.9), (4.13), and obtain

$$(4.15)\qquad \begin{aligned} \mu &= \widehat{\mu} = \theta, \\ z^{\gamma} &= \widehat{z}^{\gamma} = \frac{\varrho_0}{\tau} \left(\frac{\partial \widetilde{\varepsilon}}{\partial w_{\gamma}} - \theta \frac{\partial \widetilde{\eta}}{\partial w_{\gamma}} \right) = \frac{\varrho_0}{\tau} \frac{\partial \widetilde{\Psi}_C}{\partial w_{\gamma}} = \frac{1}{\theta f_1'(\theta)} \widetilde{q}^{\gamma}, \\ \widehat{\varphi}_C^0 &= -\widetilde{\varphi}_C^0 = -\varrho_0 \widetilde{\varepsilon} - \frac{\tau}{\theta f_1'(\theta)} \widetilde{q}^{\alpha} w_{\gamma} + \varrho_0 \theta \widetilde{\eta} = -\varrho_0 \left[\widetilde{\Psi}_C - \frac{\partial \widetilde{\Psi}_C}{\partial w_{\gamma}} w_{\gamma} \right], \\ \widehat{\varphi}_C^{\alpha} &= -\widetilde{\varphi}_C^{\alpha} = \frac{f_1(\theta)}{\theta f_1'(\theta)} \widetilde{q}^{\alpha} = -\frac{\varrho_0}{\tau} f_1(\theta) \frac{\partial \widetilde{\Psi}_C}{\partial w_{\alpha}}. \end{aligned}$$

In $(4.15)_{1,2}$ $\widehat{\mu}$ and $\widehat{z}^{\gamma}$ are functions of θ and w_{γ}. The condition (3.11) that the matrix (3.10) is nonsingular coincides in this case with the condition that Jacobian of the mapping $[\theta, w_{\alpha}] \to [\widehat{z}_{\gamma}, \widehat{\mu}]$ is nonsingular. Hence, there exists the inverse

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$[\widehat{z}^\gamma, \widehat{\mu}] \to [\theta, w_\alpha]$. Assuming that constitutive functions $\widehat{\varepsilon}(\theta, w_\gamma)$ and $\widehat{\eta}(\theta, w_\gamma)$ are chosen such that (3.11) is satisfied, we define functions $\theta = \widehat{\theta}(\widehat{\mu}) = \widehat{\mu}$ and $w_\gamma = \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)$, and use them to express $\widehat{\varphi}^0_C$, $\widehat{\varphi}^\alpha_C$ as functions of $\widehat{\mu}$ and $\widehat{z}^\alpha$,

$$
\begin{aligned}
\widehat{\varphi}^0_C(\widehat{\mu}, \widehat{z}^\alpha) &= -\varrho_0 \widetilde{\varepsilon}(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) + \varrho_0 \widehat{\mu} \widetilde{\eta}(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) \\
&\qquad - \frac{\tau}{\widehat{\mu} f_1'(\widehat{\mu})} \widetilde{q}^\beta(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) \widehat{w}_\beta(\widehat{\mu}, \widehat{z}^\alpha) \\
&= -\varrho_0 \widetilde{\varepsilon}(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) + \varrho_0 \widehat{\mu} \widetilde{\eta}(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) + \tau \widehat{z}^\gamma \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha), \\
\widehat{\varphi}^\alpha_C(\widehat{\mu}, \widehat{z}^\alpha) &= \frac{f_1(\widehat{\mu})}{\widehat{\mu} f_1'(\widehat{\mu})} \widetilde{q}^\alpha(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) = f_1(\widehat{\mu}) \widehat{z}^\alpha,
\end{aligned}
\tag{4.16}
$$

since it follows from $(4.15)_{1,2}$ that

$$
\widetilde{q}^\gamma(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) = -\widehat{\mu} f_1'(\widehat{\mu}) \widehat{z}^\gamma.
\tag{4.17}
$$

Differentiating (4.16) with respect to $\widehat{z}^\alpha$ and $\widehat{\mu}$, and taking into account (2.5), (2.6), $(4.15)_{1,2}$, we obtain

$$
\begin{aligned}
\frac{\partial \widehat{\varphi}^0_C}{\partial \widehat{z}^\beta} &= \tau \widehat{w}_\beta(\widehat{\mu}, \widehat{z}^\alpha) = \tau w_\beta, \\
\frac{\partial \widehat{\varphi}^\alpha_C}{\partial \widehat{z}^\beta} &= -f_1(\widehat{\mu}) \delta^\alpha{}_\beta = -f_1(\theta) \delta^\alpha{}_\beta, \\
\frac{\partial \widehat{\varphi}^0_C}{\partial \widehat{\mu}} &= \varrho_0 \widetilde{\eta}(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) = \varrho_0 \widetilde{\eta}, \\
\frac{\partial \widehat{\varphi}^\alpha_C}{\partial \widehat{\mu}} &= \frac{1}{\widehat{\mu}} \widetilde{q}^\alpha(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) = \frac{1}{\theta} \widetilde{q}^\alpha.
\end{aligned}
\tag{4.18}
$$

Hence, the system of four equations of (1.1), $(1.2)_{2,3,4,5}$ can be written as an equivalent symmetric conservative system for four unknown fields $\widehat{z}^\alpha(t, X^\alpha)$, $\widehat{\mu}(t, X^\alpha)$,

$$
\begin{aligned}
&\partial_t \left(\frac{\partial \widehat{\varphi}^0_C}{\partial \widehat{z}^\beta} \right) + \partial_\alpha \left(\frac{\partial \widehat{\varphi}^\alpha_C}{\partial \widehat{z}^\beta} \right) = c_1 \widehat{w}_\beta(\widehat{\mu}, \widehat{z}^\alpha) = \frac{c_1}{\tau} \frac{\partial \widehat{\varphi}^0_C}{\partial \widehat{z}^\beta}, \\
&\partial_t \left(\frac{\partial \widehat{\varphi}^0_C}{\partial \widehat{\mu}} \right) + \partial_\alpha \left(\frac{\partial \widehat{\varphi}^\alpha_C}{\partial \widehat{\mu}} \right) = \varrho_0 \widetilde{\sigma}(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha), t, X^\beta),
\end{aligned}
\tag{4.19}
$$

while the equation

$$
\begin{aligned}
\partial_t \left(\widehat{\mu} \frac{\partial \widehat{\varphi}^0_C}{\partial \widehat{\mu}} + \widehat{z}^\beta \frac{\partial \widehat{\varphi}^0_C}{\partial \widehat{z}^\beta} - \widehat{\varphi}^0_C \right) + \partial_\alpha \left(\widehat{\mu} \frac{\partial \widehat{\varphi}^\alpha_C}{\partial \widehat{\mu}} + \widehat{z}^\beta \frac{\partial \widehat{\varphi}^\alpha_C}{\partial \widehat{z}^\beta} - \widehat{\varphi}^\alpha_C \right) \\
= c_1 \widehat{z}^\beta \widehat{w}_\beta + \varrho_0 \widehat{\mu} \widetilde{\sigma} = \varrho_0 \widetilde{r},
\end{aligned}
\tag{4.20}
$$

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which is a consequence of the system (4.19), corresponds to the first equation of the system (1.1), (1.2), since

$$\begin{aligned}\widehat{\mu}\frac{\partial\widehat{\varphi}^0_C}{\partial\widehat{\mu}} + \widehat{z}^\beta\frac{\partial\widehat{\varphi}^0_C}{\partial\widehat{z}^\beta} - \widehat{\varphi}^0_c &= \varrho_0\widetilde{\varepsilon}(\widehat{\theta}(\widehat{\mu}),\widehat{w}_\gamma(\widehat{\mu},\widehat{z}^\alpha)) = \varrho_0\widehat{\varepsilon},\\ \widehat{\mu}\frac{\partial\widehat{\varphi}^\alpha_C}{\partial\widehat{\mu}} + \widehat{z}^\beta\frac{\partial\widehat{\varphi}^\alpha_C}{\partial\widehat{z}^\beta} - \widehat{\varphi}^\alpha_C &= \widetilde{q}^\alpha(\widehat{\theta}(\widehat{\mu}),\widehat{w}_\gamma(\widehat{\mu},\widehat{z}^\beta)) = \widetilde{q}^\alpha.\end{aligned} \tag{4.21}$$

4.2.3. Balance of entropy treated as the additional conservation equation. In order to obtain an alternative symmetric conservative system corresponding to (1.1), (1.2), in which the balance of entropy is treated as the additional conservation equation, we put $\mu = -1$ in (2.10), (2.9), (4.13), and obtain

$$\begin{aligned}\lambda &= \check{\lambda} = \frac{1}{\theta},\\ z^\gamma &= \check{z}^\gamma = -\frac{\varrho_0}{\tau}\frac{1}{\theta}\left(\frac{\partial\widetilde{\varepsilon}}{\partial w_\gamma} - \theta\frac{\partial\widetilde{\eta}}{\partial w_\gamma}\right) = -\frac{\varrho_0}{\tau}\frac{1}{\theta}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma} = \frac{1}{\theta^2 f_1'(\theta)}\widetilde{q}^\gamma,\\ \check{\varphi}^0_C &= \frac{1}{\theta}\widetilde{\varphi}^0_C = \varrho_0\frac{1}{\theta}\widetilde{\varepsilon} + \frac{\tau\widetilde{q}^\gamma w_\gamma}{\theta^2 f_1'(\theta)} - \varrho_0\widetilde{\eta} = \varrho_0\frac{1}{\theta}\left[\widetilde{\Psi}_C - \frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma}w_\gamma\right],\\ \check{\varphi}^\alpha_C &= \frac{1}{\theta}\widetilde{\varphi}^\alpha_C = -\frac{f_1(\theta)}{\theta^2 f_1'(\theta)}\widetilde{q}^\alpha = \frac{\varrho_0}{\tau}\frac{f_1(\theta)}{\theta}\frac{\partial\widetilde{\Psi}_C}{\partial w_\alpha},\end{aligned} \tag{4.22}$$

where, $\check{\lambda}$ and $\check{z}^\gamma$ are functions of θ and w_γ. As it follows from Observation 1, for the system of conservation equations considered in this paper, nonsingularity of the matrix of the type (3.9) for one particular solution of the MDR implies nonsingularity of that matrix for all other solutions of the MDR. Hence, the condition (3.11) is necessary and sufficient for nonsingularity of the matrix (3.9) for $\check{\mathbf{y}} = [\check{\lambda}, \check{z}^\gamma, -1]$ and, as in the case of $\widehat{\mathbf{y}} = [-1, \widehat{z}^\gamma, \widehat{\mu}]$, it obviously coincides with the condition that Jacobian of the mapping $[\theta, w_\alpha] \to [\check{\lambda}, \check{z}^\gamma]$ is nonsingular. As in the previous case, assuming that constitutive functions $\widetilde{\varepsilon}(\theta, w_\gamma)$, $\widetilde{\eta}(\theta, w_\gamma)$ are chosen such that this condition holds, we define functions $\theta = \check{\theta}(\check{\lambda}) = \frac{1}{\check{\lambda}}$ and $w_\gamma = \check{w}_\gamma(\check{\lambda}, \check{z}^\alpha)$. Then, we express potentials $\check{\varphi}^0_C$, $\check{\varphi}^\alpha_C$ as functions of $\check{\lambda}$ and $\check{z}^\alpha$,

$$\begin{aligned}\check{\varphi}^0_C(\check{\lambda},\check{z}^\alpha) &= \varrho_0\check{\lambda}\widetilde{\varepsilon}(\check{\theta}(\check{\lambda}),\check{w}_\gamma(\check{\lambda},\check{z}^\alpha)) - \varrho_0\widetilde{\eta}(\check{\theta}(\check{\lambda}),\check{w}_\gamma(\check{\lambda},\check{z}^\alpha))\\ &\quad + \tau\check{z}^\gamma\check{w}_\gamma(\check{\lambda},\check{z}^\alpha),\\ \check{\varphi}^\alpha_C(\check{\lambda},\check{z}^\alpha) &= -f_1\left(\frac{1}{\check{\lambda}}\right)\check{z}^\alpha,\end{aligned} \tag{4.23}$$

taking into account that

$$\widetilde{q}^\gamma = f_1'\left(\frac{1}{\check{\lambda}}\right)\frac{1}{\check{\lambda}^2}\check{z}^\gamma, \tag{4.24}$$

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according to $(4.22)_{1,2}$. Differentiating (4.23) with respect to $\check{\lambda}$ and $\check{z}^\alpha$ and employing (2.5), (2.6), (4.22), we obtain

$$
(4.25)\qquad
\begin{aligned}
\frac{\partial \check{\varphi}^0_C}{\partial \check{\lambda}} &= \varrho_0 \widetilde{\varepsilon}(\check{\theta}(\check{\lambda}), \check{w}_\gamma(\check{\lambda}, \check{z}^\alpha)) = \varrho_0 \widetilde{\varepsilon},\\
\frac{\partial \check{\varphi}^\alpha_C}{\partial \check{\lambda}} &= f_1'\left(\frac{1}{\check{\lambda}}\right)\frac{\check{z}^\alpha}{\check{\lambda}^2} = \widetilde{q}^\alpha,\\
\frac{\partial \check{\varphi}^0_C}{\partial \check{z}^\gamma} &= \tau \check{w}_\gamma(\check{\lambda}, \check{z}^\alpha) = \tau w_\gamma,\\
\frac{\partial \check{\varphi}^\alpha_C}{\partial \check{z}^\beta} &= -f_1\left(\frac{1}{\check{\lambda}}\right)\delta^\alpha_\gamma = -f_1(\theta)\delta^\alpha_\gamma\,.
\end{aligned}
$$

The system of the first four equations of (1.1), (1.2) can therefore be written as a symmetric conservative system for four unknown fields $\check{\lambda}(t, X^\alpha)$, $\check{z}^\alpha(t, X^\alpha)$

$$
(4.26)\qquad
\begin{aligned}
\partial_t\left(\frac{\partial \check{\varphi}^0_C}{\partial \check{\lambda}}\right) + \partial_\alpha\left(\frac{\partial \check{\varphi}^\alpha_C}{\partial \check{\lambda}}\right) &= \varrho_0 \widetilde{r}(\check{\theta}(\check{\lambda}), \check{w}_\gamma(\check{\lambda}, \check{z}^\alpha), t, X^\beta),\\
\partial_t\left(\frac{\partial \check{\varphi}^0_C}{\partial \check{z}^\gamma}\right) + \partial_\alpha\left(\frac{\partial \check{\varphi}^\alpha_C}{\partial \check{z}^\gamma}\right) &= c_1 \check{w}_\gamma(\check{\lambda}, \check{z}^\alpha) = \frac{c_1}{\tau}\frac{\partial \check{\varphi}^0_C}{\partial \check{z}^\gamma}\,.
\end{aligned}
$$

The conservation equation implied by (4.26)

$$
(4.27)\qquad
\begin{aligned}
\partial_t\left(\check{\lambda}\frac{\partial \check{\varphi}^0_C}{\partial \check{\lambda}} + \check{z}^\gamma \frac{\partial \check{\varphi}^0_C}{\partial \check{z}^\gamma} - \check{\varphi}^0_C\right) + \partial_\alpha\left(\frac{\partial \check{\varphi}^\alpha_C}{\partial \check{\lambda}} + \check{z}^\gamma \frac{\partial \check{\varphi}^\alpha_C}{\partial \check{z}^\gamma} - \check{\varphi}^\alpha_C\right)&\\
= \varrho_0 \check{\lambda}\widetilde{r} + c_1 \check{z}^\gamma \check{w}^\gamma = \varrho_0 \widetilde{\sigma}&,
\end{aligned}
$$

corresponds to the equation of balance of entropy (the last equation in the system (1.1), (1.2)) since

$$
(4.28)\qquad
\begin{aligned}
\check{\lambda}\frac{\partial \check{\varphi}^0_C}{\partial \check{\lambda}} + \check{z}^\gamma \frac{\partial \check{\varphi}^0_C}{\partial \check{z}^\gamma} - \check{\varphi}^0_C &= \varrho_0 \widetilde{\eta}(\check{\theta}(\check{\lambda}), \check{w}_\gamma(\check{\lambda}, \check{z}^\alpha)) = \varrho_0 \widetilde{\eta},\\
\check{\lambda}\frac{\partial \check{\varphi}^\alpha_C}{\partial \check{\lambda}} + \check{z}^\gamma \frac{\partial \check{\varphi}^\alpha_C}{\partial \check{z}^\gamma} - \check{\varphi}^\alpha_C &= f_1'\left(\frac{1}{\check{\lambda}}\right)\frac{1}{\check{\lambda}}\check{z}^\alpha = \frac{1}{\theta}\widetilde{q}^\alpha.
\end{aligned}
$$

It should be noted that the potential $\check{\varphi}^0_C(\theta, w_\gamma)$ is not the Legendre transform of the entropy $\widetilde{\eta}$ since "multipliers" $\check{\lambda}(\theta, w_\gamma)$, $\check{z}^\alpha(\theta, w_\gamma)$ are not derivatives of the entropy with respect to primitive field variables θ, w_γ, respectively. This is because of the fact that internal energy $\widetilde{\varepsilon}$, which is an extensive variable, is not a primitive field variable of the theory. It is well known (see, for example, [13]) that, in thermodynamic theories formulated in terms of extensive quantities as primitive fields and balances of them, the Lagrange multipliers of the variational problem of maximization of entropy or, equivalently, Liu multipliers in the entropy principle of extended thermodynamics, which are intensive quantities, are

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the respective derivatives of the entropy with respect to extensive quantities. As a consequence, the potential, which gives extensive quantities as derivatives with respect to intensive quantities, is the Legendre transform of the entropy.

4.3. Symmetric hyperbolicity of symmmetric conservative systems (4.19), (4.26)

In Sec. 3.5, we have discussed the conditions upon which symmetric systems (3.8) for unknowns $[u^K(t, X^\alpha)]$ are symmetric hyperbolic. Now, the question of symmetric hyperbolicity of symmetric conservative systems (4.19) and (4.26) arises. There are various possibilities of establishing the conditions which ensure that systems (4.26), (4.19) are symmetric hyperbolic (or can be transformed to symmetric hyperbolic systems by multiplication by numerical factor (-1)), namely: a) to investigate directly either convexity (concavity) of $\widehat{\varphi}^0_C(\widehat{z}, \widehat{\mu})$ and $\check{\varphi}^0_C(\check{\lambda}, \check{z}^\gamma)$ or positive (negative) definiteness of the respective Hessians, b) to investigate convexity (concavity) of one of the potentials (either $\widehat{\varphi}^0_C(\widehat{z}, \widehat{\mu})$ or $\check{\varphi}^0_C(\check{\lambda}, \check{z}^\gamma)$) and employ the relations between $\widehat{\varphi}^0_C(\widehat{z}, \widehat{\mu})$ and $\check{\varphi}^0_C(\check{\lambda}, \check{z}^\gamma)$,, and c) to make use of the relation between symmetric hyperbolicity of the symmetric systems (3.8) and symmetric hyperbolicity of the symmetric conservative systems (4.7). Since, in this paper, the emphasis is put on various aspects of the employed method of symmetrization, we shall demonstrate here indirect examination of symmetric hyperbolicity of the systems (4.19) and (4.26) with the aid of the relation between potentials $\widehat{\varphi}^0_C(\widehat{z}, \widehat{\mu})$ and $\check{\varphi}^0_C(\check{\lambda}, \check{z}^\gamma)$, and with the aid of the relation between symmetric hyperbolicity of symmetric systems (3.8) and symmetric conservative systems (4.7).

4.3.1. Relation between potentials generating alternative symmetric conservative systems. It follows from (4.15), (4.16), (4.22), (4.23) that the potentials $\check{\varphi}^0_C(\widehat{\mu}, \widehat{z}^\alpha)$ and $\check{\varphi}^0_C(\check{\lambda}, \check{z}^\gamma)$ as well as the new field variables $\widehat{\mu}(t, X^\alpha)$, $\widehat{z}^\gamma(t, X^\alpha)$ and $\check{\lambda}(t, X^\alpha)$, $\check{z}^\gamma(t, X^\alpha)$ of the respective symmetric conservative systems (4.19), (4.26) are mutually related in accordance with the relations derived in Sec. 4.1.4.

$$\check{\lambda} = \frac{1}{\widehat{\mu}}, \qquad \check{z}^\gamma = -\frac{\widehat{z}^\gamma}{\widehat{\mu}},$$
$$\check{\varphi}^0_C(\check{\lambda}, \check{z}^\gamma) = -\check{\lambda}\widehat{\varphi}^0_C\left(\frac{1}{\check{\lambda}}, -\frac{\check{z}^\gamma}{\check{\lambda}}\right), \qquad \check{\varphi}^\alpha_C(\check{\lambda}, \check{z}^\gamma) = -\check{\lambda}\widehat{\varphi}^\alpha_C\left(\frac{1}{\check{\lambda}}, -\frac{\check{z}^\gamma}{\check{\lambda}}\right) \tag{4.29}$$

and

$$\widehat{\mu} = \frac{1}{\check{\lambda}}, \qquad \widehat{z}^\gamma = \frac{\check{z}^\gamma}{\check{\lambda}},$$
$$\widehat{\varphi}^0_C(\widehat{\mu}, \widehat{z}^\gamma) = -\widehat{\mu}\check{\varphi}^0_C\left(\frac{1}{\widehat{\mu}}, -\frac{\widehat{z}^\gamma}{\widehat{\mu}}\right), \qquad \widehat{\varphi}^\alpha_C(\widehat{\mu}, \widehat{z}^\gamma) = -\widehat{\mu}\check{\varphi}^\alpha_C\left(\frac{1}{\widehat{\mu}}, -\frac{\widehat{z}^\gamma}{\widehat{\mu}}\right), \tag{4.30}$$

where $\widehat{\mu} = \theta$ is positive. The relations (4.29), (4.30) lead to the following

PROPERTY. The potential $\widehat{\varphi}^0(\widehat{\mu}, \widehat{z}^\gamma)$ is convex (concave) if and only if the potential $\check{\varphi}^0(\check{\lambda}, \check{z}^\gamma)$ is concave (convex).

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The proof of the Property is given in the Appendix B. Since we have established the relation between convexity (concavity) of $\widehat{\varphi}^0_C(\widehat{z},\widehat{\mu})$ and concavity (convexity) of $\check{\varphi}^0_C(\check{\lambda},\check{z}^\gamma)$, it now suffices to examine only the conditions of symmetric hyperbolicity for the system (4.19) (positive definiteness of Hessian matrix of $\widehat{\varphi}^0_C(\widehat{z},\widehat{\mu})$).

4.3.2. Symmetric hyperbolicity of the symmetric system for $[\theta, w_\gamma]$ and of the symmetric conservative system, both corresponding to the same solution of the MDR. We recall that the symmetric system (3.7) for unknowns $[u_K(t,X^\alpha)] = [\theta(t,X^\alpha), w_\gamma(t,X^\alpha)]$ and the symmetric conservative system (4.19) for unknowns $[\widehat{z}^\gamma(t,X^\alpha), \widehat{\mu}(t,X^\alpha)]$ correspond to the same solution of the MDR, namely,

$$\mathbf{y} = \widehat{\mathbf{y}} = [-1, \widehat{z}^\gamma(\theta, w_\gamma), \widehat{\mu}(\theta, w_\gamma)] = \left[-1, \frac{\varrho_0}{\tau}\frac{\partial \widetilde{\Psi}_C}{\partial w_\gamma}, \theta\right].$$

It follows from (4.18) that the components of Hessian of $\widehat{\varphi}^0_C(\widehat{z}^\gamma,\widehat{\mu}))$ satisfy the relations

$$\begin{aligned}
\frac{\partial^2\widehat{\varphi}^0_C}{\partial\widehat{z}^\alpha\partial\widehat{z}^\beta} &= \tau\frac{\partial\widehat{w}_\alpha(\widehat{\mu},\widehat{z}^\gamma)}{\partial\widehat{z}^\beta} = \tau\frac{\partial\widehat{w}_\beta(\widehat{\mu},\widehat{z}^\gamma)}{\partial\widehat{z}^\beta},\\
\frac{\partial^2\widehat{\varphi}^0_C}{\partial\widehat{\mu}\partial\widehat{z}^\alpha} &= \tau\frac{\partial\widehat{w}_\alpha(\widehat{\mu},\widehat{z}^\gamma)}{\partial\widehat{\mu}} = \varrho_0\frac{\partial\widetilde{\eta}(\widehat{\theta}(\widehat{\mu}),\widehat{w}_\beta(\widehat{\mu},\widehat{z}^\lambda))}{\partial\widehat{w}_\gamma}\frac{\partial\widehat{w}_\gamma(\widehat{\mu},\widehat{z}^\lambda)}{\partial\widehat{z}^\alpha},\\
\frac{\partial^2\widehat{\varphi}^0_C}{\partial\widehat{\mu}^2} &= \varrho_0\frac{\partial\widetilde{\eta}(\widehat{\theta}(\widehat{\mu}),\widehat{w}_\beta(\widehat{\mu},\widehat{z}^\lambda))}{\partial\widehat{\theta}}\frac{\partial\widehat{\theta}}{\partial\widehat{\mu}} + \varrho_0\frac{\partial\widetilde{\eta}(\widehat{\theta}(\widehat{\mu}),\widehat{w}_\beta(\widehat{\mu},\widehat{z}^\lambda))}{\partial\widehat{w}_\gamma}\frac{\partial\widehat{w}_\gamma(\widehat{\mu},\widehat{z}^\lambda)}{\partial\widehat{\mu}},
\end{aligned} \tag{4.31}$$

which enable us to represent this Hessian as the following product:

$$[\widehat{B}^0_{PQ}] = \begin{bmatrix} \dfrac{\partial^2\widehat{\varphi}^0_C}{\partial\widehat{z}^\alpha\partial\widehat{z}^\beta} & \dfrac{\partial^2\widehat{\varphi}^0_C}{\partial\widehat{z}^\alpha\partial\widehat{\mu}} \\ \dfrac{\partial^2\widehat{\varphi}^0_C}{\partial\widehat{z}^\beta\partial\widehat{\mu}} & \dfrac{\partial^2\widehat{\varphi}^0_C}{\partial\widehat{\mu}^2} \end{bmatrix} \tag{4.32}$$

$$= \begin{bmatrix} 0 & \tau & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \tau \\ \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial\widehat{\theta}} & \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial\widehat{w}_1} & \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial\widehat{w}_2} & \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial\widehat{w}_3} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \dfrac{\partial\widehat{\theta}}{\partial\widehat{\mu}} \\ \dfrac{\partial\widehat{w}_1}{\partial\widehat{z}^1} & \dfrac{\partial\widehat{w}_1}{\partial\widehat{z}^2} & \dfrac{\partial\widehat{w}_1}{\partial\widehat{z}^3} & \dfrac{\partial\widehat{w}_1}{\partial\widehat{\mu}} \\ \dfrac{\partial\widehat{w}_2}{\partial\widehat{z}^1} & \dfrac{\partial\widehat{w}_2}{\partial\widehat{z}^2} & \dfrac{\partial\widehat{w}_2}{\partial\widehat{z}^3} & \dfrac{\partial\widehat{w}_2}{\partial\widehat{\mu}} \\ \dfrac{\partial\widehat{w}_3}{\partial\widehat{z}^1} & \dfrac{\partial\widehat{w}_3}{\partial\widehat{z}^2} & \dfrac{\partial\widehat{w}_3}{\partial\widehat{z}^3} & \dfrac{\partial\widehat{w}_3}{\partial\widehat{\mu}} \end{bmatrix}.$$

The first term in the product (4.38) is the matrix $[A^{0M}{}_S]$ which, as we have mentioned in Sec. 3.2, is a 4×4 minor of the 5×4 matrix $[\mathcal{A}^{0\Lambda}{}_S]$, given by (2.2),

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obtained by deleting the first row in $[\mathcal{A}^{0\Lambda}{}_S]$. The second term is the Jacobi matrix of the transformation $[\widehat{z}^\gamma, \widehat{\mu}] \to [\theta, w_\gamma]$ defined as $\theta = \widehat{\theta}(\widehat{\mu}) = \widehat{\mu}$, $w_\gamma = \widehat{w}_\gamma(\widehat{z}^\gamma, \widehat{\mu})$ in (4.16). In Sec. 3.2, the 4×4 matrix $[\widehat{K}_{SM}]$ is introduced as a 4×4 minor of the 4×5 matrix $[\widehat{\mathcal{K}}_{S\Lambda}]$ given by (3.5), which is obtained by deleting the first column in $[\widehat{\mathcal{K}}_{S\Lambda}]$. This matrix can be obtained as a 4×4 minor of the 5×5 matrix (3.10) by deleting the first row and the first column. It follows from (3.10) that $[\widehat{K}_{S\Lambda}]$ corresponds to the transposed Jacobi matrix of the transformation $[\theta, w_\gamma] \to [\widehat{z}^\gamma, \widehat{\mu}]$. It will be convenient to employ the following notation for the matrices $\widehat{\mathbf{B}}^0 := [\widehat{B}^0_{PQ}]$, $\widehat{\mathbf{A}}^0 := [\widehat{A}^0_{PQ}]$, $\mathbf{A}^0 := [A^{0M}_R]$, $\widehat{\mathbf{K}} := [\widehat{K}_{SQ}]$ and rewrite (4.32) in the form

$$\widehat{\mathbf{B}}^0 = \mathbf{A}^0(\widehat{\mathbf{K}}^T)^{-1} = \widehat{\mathbf{B}}^{0T}, \tag{4.33}$$

since the transformation $[\widehat{z}^\gamma, \widehat{\mu}] \to [\theta, w_\gamma]$ is the inverse of the transformation $[\theta, w_\gamma] \to [\widehat{z}^\gamma, \widehat{\mu}]$ and $\widehat{\mathbf{B}}^0$ is symmetric. It is noted in Sec. 3.2 that $\widehat{\mathbf{K}}$ is the left symmetrizer of $\mathbf{A}^0$ and the symmetric matrix $\widehat{\mathbf{A}}^0$ of the symmetric system (3.7) can be obtained as

$$\widehat{\mathbf{A}}^0 = \widehat{\mathbf{K}}\mathbf{A}^0 = \widehat{\mathbf{A}}^{0T}. \tag{4.34}$$

The relations (4.33), (4.34) imply that the symmetric matrices $\widehat{\mathbf{A}}^0$ and $\widehat{\mathbf{B}}^0$ are mutually related by the following congruent transformation

$$\begin{aligned} \widehat{\mathbf{B}}^0 &= \widehat{\mathbf{K}}^{-1}\widehat{\mathbf{A}}^0(\widehat{\mathbf{K}}^{-1})^T, \\ \widehat{\mathbf{A}}^0 &= \widehat{\mathbf{K}}\widehat{\mathbf{B}}^0\widehat{\mathbf{K}}^T, \end{aligned} \tag{4.35}$$

where $\widehat{\mathbf{K}}$ is nonsingular. Positive (negative) definiteness of $\widehat{\mathbf{A}}^0$ means that $\mathbf{x}^T\widehat{\mathbf{A}}^0\mathbf{x} > 0$ (< 0) for all $\mathbf{x} \neq 0$ and therefore $\mathbf{x}^T\widehat{\mathbf{K}}\widehat{\mathbf{B}}^0\widehat{\mathbf{K}}^T\mathbf{x} > 0$ (< 0) for all $\mathbf{x} \neq \mathbf{0}$. Hence, $\mathbf{y}^T\widehat{\mathbf{B}}^0\mathbf{y} > 0$ (< 0) for all $\mathbf{y} \neq \mathbf{0}$ since $\mathbf{y} = \widehat{\mathbf{K}}^T\mathbf{x}$ represents all nonzero vectors for all $\mathbf{x} \neq \mathbf{0}$. This shows that positive (negative) definiteness of $\widehat{\mathbf{A}}^0$ implies positive (negative) definiteness of $\widehat{\mathbf{B}}^0$. The converse can be proved in the same way. Therefore, the restrictions on constitutive functions $\widetilde{\Psi}_C(\theta, w_\gamma)$ (or $\widetilde{\varepsilon}(\theta, w_\gamma)$ and $\widetilde{\eta}(\theta, w_\gamma)$) established in Sec. 3.5.2, which ensure symmetric hyperbolicity of the system (3.7) (positive definiteness of $[\widehat{A}^0_{SM}]$), also ensure symmetric hyperbolicity of the symmetric conservative system (4.19) (convexity of $\widehat{\varphi}^0_C(\widehat{z}^\gamma, \widehat{\mu})$ or, equivalently, positive definiteness of the Hessian matrix of $\widehat{\varphi}^0_C(\widehat{z}^\gamma, \widehat{\mu})$).

The reasoning similar to that presented here for particular systems (3.7) and (4.19) can be directly applied to each of the $N+1$ symmetric conservative systems derived in Sec. 4.1.2 and the corresponding symmetric systems for original unknowns u^K (derived in Sec. 3.1), thus leading to the following observation.

OBSERVATION 3. Symmetric conservative system (4.7) for unknowns

$$\overset{(\Sigma)}{y}_1(t, X^\alpha), \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}(t, X^\alpha), \overset{(\Sigma)}{y}_{\Sigma+1}(t, X^\alpha), \ldots, \overset{(\Sigma)}{y}_{N+1}(t, X^\alpha)$$

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is symmetric hyperbolic if and only if the symmetric system (3.3), (3.4) for unknowns $u^1(t, X^\alpha), \dots, u^N(t, X^\alpha)$ corresponding to the solution

$$\mathbf{y} = \overset{(\Sigma)}{\mathbf{y}} = \left[\overset{(\Sigma)}{y}_1(u^K), \dots, \overset{(\Sigma)}{y}_{\Sigma-1}(u^K), -1, \overset{(\Sigma)}{y}_{\Sigma+1}(u^K), \dots, \overset{(\Sigma)}{y}_{N+1}(u^K)\right]$$

of the MDR is symmetric hyperbolic.

Now, it follows from the relation between convexity (concavity) of $\widehat{\varphi}_C^0(\widehat{z}^\gamma, \widehat{\mu})$ and concavity (convexity) of $\check{\varphi}_C^0(\check{\lambda}, \check{z}^\gamma)$, that the systems (3.7) and (4.19) are symmetric hyperbolic if and only if the system (4.26) can be brought to symmetric hyperbolic form by multiplication by numerical factor (-1) and, conversely, the systems (3.7) and (4.19) can be brought to symmetric hyperbolic form by multiplication by numerical factor (-1) if and only if the system (4.26) is symmetric hyperbolic.

As it is emphasized by GODUNOV and SULTANGAZIAN [11, 12], alternative symmetric conservative systems corresponding to the same consistent system of $N+1$ conservation equations for N unknowns are not equivalent when weak (discontinuous) solutions are considered. This nonequivalence manifests itself in that the different equations are to be replaced by inequalities for different symmetric conservative systems. Namely, for the symmetric conservative system (4.19) corresponding to the equation of balance of energy treated as the additional conservation equation (energy being a “derived” conserved quantity), Eqs. (4.19) are now satisfied in the weak sense while the additional conservation equation (4.20) should be replaced by the inequality (≥ 0), also in the weak sense. When the symmetric conservative system (4.26) corresponding to the equation of balance of entropy in the role of the additional conservation equation (entropy being a “derived” conserved quantity) is considered for weak solutions, the equations (4.26) are now satisfied in the weak sense but the equation (4.27) should be replaced by the inequality (≥ 0) in a weak sense. A physical interpretation of those inequalities is mentioned in [9]. In the case of the energy taken as a “derived” conserved quantity, we have an increase of energy across the surface of discontinuity. This may be interpreted as energy production on the shock. Hence, in the processes, in which the energy is conserved, it must be taken as one of the “original” conserved quantities. If the entropy is taken as a “derived” conserved quantity, the inequality for discontinuous solutions means that entropy increases on the shock.

Appendix A

LEMMA 1. If the solution set of the MDR (2.3) (2.4) for the system (1.1) has a form of a family of collinear $N+1$ component row vectors $\mathbf{y}^T(u^K) = [y_\Lambda(u^K)]$ parametrized by differentiable functions $\alpha(u^K)$ (equivalently, if the system (1.1)

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contains N independent equations), then every matrix (3.9) corresponding to the solution of the MDR has the same kernel.

P r o o f. Let $\mathbf{y}^{*T}(\mathbf{u})$ and $\mathbf{y}^{**T}(\mathbf{u})$ be the two arbitrary different solutions of the "main dependency relation" (2.3), (2.4) for the system (1.1). It is assumed that there exists a differentiable function $\alpha(u^K)$, $\alpha(u^K) \not\equiv 0$, such that $\mathbf{y}^{**T}(u^K) = \alpha(u^K)\mathbf{y}^{*T}(u^K)$. Let $\mathcal{M}^*(u^K)$ and $\mathcal{M}^{**}(u^K)$ be the $(N+1)\times(N+1)$ matrices (3.9) corresponding to $\mathbf{y}^{*T}(u^K)$ and $\mathbf{y}^{**T}(u^K)$, respectively. Those matrices can be written in the following form:

$$\text{(A.1)} \qquad \mathcal{M}^*(u^K) := \begin{bmatrix} \mathbf{y}^{*T}(u^K) \\ (\nabla_{\mathbf{u}}\mathbf{y}^*(u^K))^T \end{bmatrix},$$

$$\text{(A.2)} \qquad \mathcal{M}^{**}(u^K) := \begin{bmatrix} \mathbf{y}^{**T}(u^K) \\ (\nabla_{\mathbf{u}}\mathbf{y}^{**}(u^K))^T \end{bmatrix} = \begin{bmatrix} \alpha(u^K)\mathbf{y}^{*T}(u^K) \\ \alpha(u^K)(\nabla_{\mathbf{u}}\mathbf{y}^*(u^K))^T + [\nabla_{\mathbf{u}}\alpha(u^K)]\otimes\mathbf{y}^{*T}(u^K) \end{bmatrix},$$

where $\nabla_{\mathbf{u}}$ denotes differentiation with respect to u^K. Let $\mathbf{z}^*(u^K) \in \operatorname{Ker}\mathcal{M}^*(u^K)$. It follws from (A.1) that

$$\text{(A.3)} \qquad \mathbf{y}^{*T}(u^K)\mathbf{z}^*(u^K) \equiv 0, \qquad [\nabla_{\mathbf{u}}\mathbf{y}^*(u^K)]^T\mathbf{z}^*(u^K) \equiv \mathbf{0}$$

and (A.3), (A.2) imply $\mathcal{M}^{**}(u^K)\mathbf{z}^*(u^K) \equiv \mathbf{0}$. Therefore, $\mathbf{z}^*(u^K) \in \operatorname{Ker}\mathcal{M}^{**}(u^K)$. In the same way it can be proved that if $\mathbf{z}^{**}(u^K) \in \operatorname{Ker}\mathcal{M}^{**}(u^K)$ then $\mathbf{z}^{**}(u^K) \in \operatorname{Ker}\mathcal{M}^*(u^K)$. Hence $\operatorname{Ker}\mathcal{M}^*(u^K) = \operatorname{Ker}\mathcal{M}^{**}(u^K)$.

LEMMA 2. The necessary condition for the $(N+1)\times(N+1)$ matrix (3.9) to be nonsingular for all u^K is that rank $\mathcal{K}(u^K) = N$ for all u^K, where

$$\boldsymbol{\mathcal{K}}(u^K) = [\mathcal{K}_{M\Lambda}(u^K)] = \left[\frac{\partial y_\Lambda(u^K)}{\partial u^M}\right] = [\nabla_{\mathbf{u}}\mathbf{y}(u^K)]^T$$

is a $N\times(N+1)$ matrix.

P r o o f. According to (3.9),

$$\text{(A.4)} \qquad \det\mathcal{M}(u^K) = \det\begin{bmatrix} \mathbf{y}^T(u^K) \\ \boldsymbol{\mathcal{K}}(u^K) \end{bmatrix} = \det\begin{bmatrix} y_\Lambda(u^K) \\ \mathcal{K}_{M\Lambda}(u^K) \end{bmatrix} = \sum_{\Gamma=1}^{N+1}(-1)^{\Gamma+1}y_\Gamma(u^K)\det\overset{(\Gamma)}{\mathbf{H}}(u^K),$$

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where $\overset{(\Gamma)}{\mathbf{H}}(u^K)$ are $N \times N$ minors of $\mathcal{K}(u^K)$. Suppose that rank $\mathcal{K}(u^K) < N$. Then, all $\overset{(\Gamma)}{\mathbf{H}}(u^K)$ are singular and therefore (3.9) is singular according to (A.4).

LEMMA 3. If among $N+1$ components $y_\Lambda(u^K)$ of the solution $\mathbf{y}^T(u^K)$ of the MDR there are N independent functions of u^K, that is if rank $\mathcal{K}(u^K) = N$ for all u^K, then the matrix (3.9) is singular for all u^K if and only if the remaining (dependent) component of $\mathbf{y}^T(u^K)$ is a homogeneous function of degree one of the independent components of $\mathbf{y}^T(u^K)$ (if $y_1(u^K), y_2(u^K), \ldots, y_{\Sigma-1}(u^K), y_{\Sigma+1}(u^K), \ldots, y_{N+1}(u^K)$ are independent components of $\mathbf{y}^T(u^K)$, then the remaining component $y_\Sigma(u^K)$ expressed as $y_\Sigma = \bar{y}_\Sigma(y_1, \ldots, y_{\Sigma-1}, y_{\sigma+1}, \ldots, y_{N+1})$ is a homogeneous function of degree one with respect to all arguments).

P r o o f. Let $y_1(u^K), \ldots, y_{\Sigma-1}(u^K), y_{\Sigma+1}(u^K) \ldots, y_{N+1}(u^K)$ be N independent functions of u^K. Then the $N \times N$ nonsingular minor $\overset{(\Sigma)}{\mathbf{H}}(u^K)$ of $\mathcal{K}(u^K)$ corresponding to deletion of the column Σ is a Jacobian of the invertible mapping $(u^1, \ldots, u^2) \to (y_1, \ldots, y_{\Sigma-1}, y_{\Sigma+1} \cdots, y_{N+1})$. Employing the inverse $u^K = \bar{u}^K(y_1, \ldots, y_{\Sigma-1}, y_{\Sigma+1}, \ldots, y_{N+1})$, $K = 1, 2, \ldots, N$, we may express y_Σ in a form of the following composed function:

$$
\begin{aligned}
\text{(A.5)} \qquad y_\Sigma = y_\Sigma(\bar{u}^K(y_1, \ldots, y_{\Sigma-1}, y_{\Sigma+1}, \ldots, y_{N+1})) \\
= \widetilde{y}_\Sigma(y_1, \ldots, y_{\Sigma-1}, y_{\Sigma+1} \cdots, y_{N+1}) \\
= \widetilde{y}_\Sigma(y_1(u^K), \ldots, y_{\Sigma-1}(u^K), y_{\Sigma+1}(u^K) \ldots, y_{N+1}(u^K)).
\end{aligned}
$$

According to the chain rule,

$$
\text{(A.6)} \qquad \frac{\partial \widetilde{y}_\Sigma}{\partial u^K} = \sum_{\substack{\Delta=1,\\ \Delta\neq\Sigma,\\ \Lambda\neq\Sigma}}^{N+1} \frac{\partial \widetilde{y}_\Sigma(y_\Lambda(u^K))}{\partial y_\Delta} \frac{\partial y_\Delta(u^K)}{\partial u^K}.
$$

Assume that the matrix (3.9), denoted as $\mathcal{M}(u^K)$, is singular for all u^K. Then, there exists a nonzero solution $\mathbf{z}(u^K) = [z^0(u^K), z^1(u^K), \ldots, z^N(u^K)]$ of the equation

$$
\text{(A.7)} \qquad \mathcal{M}(u^K)\mathbf{z}(u^K) = \mathbf{0} \qquad \text{for all} \quad u^K,
$$

which corresponds to the following system of $N+1$ linear homogeneous equations:

$$
\text{(A.8)} \qquad \begin{gathered} z^0(u^K) y_\Lambda(u^K) + \sum_{R=1}^{N} z^R(u^K) \frac{\partial y_\Lambda(u^K)}{\partial u^R} = 0, \\ \Lambda = 1, 2, \ldots, \Sigma-1, \Sigma+1, \ldots, N+1, \end{gathered}
$$

$$
\text{(A.9)} \qquad z^0(u^K) \widetilde{y}_\Sigma(y_\Lambda(u^K)) + \sum_{R=1}^{N} z^R(u^K) \frac{\partial \widetilde{y}(y_\Delta(u^K))}{\partial u^R} = 0, \qquad \Delta \neq \Sigma.
$$

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Substituting (A.6) into (A.9) and taking into account (A.8), we obtain

$$
\text{(A.10)}\qquad z^0\widetilde{y}_\Sigma+\sum_{R=1}^{N}z^R\left[\sum_{\substack{\Delta=1,\\ \Delta\neq\Sigma}}^{N+1}\frac{\partial\widetilde{y}_\Sigma}{\partial y_\Delta}\frac{\partial y_\Delta}{\partial u^R}\right]=z^0\widetilde{y}_\Sigma+\sum_{\substack{\Delta=1,\\ \Delta\neq\Sigma}}^{N+1}\left[\sum_{R=1}^{N}z^R\frac{\partial y_\Delta}{\partial u^R}\right]\frac{\partial\widetilde{y}_\Sigma}{\partial y_\Delta}
$$
$$
= z_0\left(\widetilde{y}_\Sigma-\sum_{\substack{\Delta=1,\\ \Delta\neq\Sigma}}^{N+1}y_\Delta\frac{\partial\widetilde{y}_\Sigma}{\partial y_\Delta}\right)=0.
$$

Suppose that $z^0(u^K)\equiv 0$. Since $\mathbf{z}(u^K)$ is a nonzero vector, among the components $z^1(u^K),\ldots,z^N(u^K)$ at least one cannot identically vanish. It then follows from (A.8) that, in this case, the $N\times N$ minor $\overset{(\Sigma)}{\mathbf{H}}(u^K)$ must be singular. Therefore, the condition $z^0(u^K)\equiv 0$ contradicts the assumptions and (A.10) implies that $\widetilde{y}_\Sigma(y_1,\ldots,y_{\Sigma-1},y_{\Sigma+1},\ldots,y_{N+1})$ is a homogeneous function of degree one with respect to all arguments. Assume that $\widetilde{y}_\Sigma(y_1,\ldots,y_{\Sigma-1},y_{\Sigma+1},\ldots,y_{N+1})$ is a homogeneous function of degree one with respect to all arguments. Then

$$
\text{(A.11)}\qquad \widetilde{y}_\Sigma=\frac{\partial\widetilde{y}_\Sigma}{\partial y_1}y_1+\ldots+\frac{\partial\widetilde{y}_\Sigma}{\partial y_{\Sigma-1}}y_{\Sigma-1}+\frac{\partial\widetilde{y}_\Sigma}{\partial y_{\Sigma+1}}y_{\Sigma+1}+\ldots+\frac{\partial\widetilde{y}_\Sigma}{\partial y_{N+1}}y_{N+1}
$$

and, according to (A.6),

$$
\text{(A.12)}\qquad \frac{\partial\widetilde{y}_\Sigma}{\partial u^K}=\frac{\partial\widetilde{y}_\Sigma}{\partial y_1}\frac{\partial y_1}{\partial u^K}+\ldots+\frac{\partial\widetilde{y}_\Sigma}{\partial y_{\Sigma-1}}\frac{\partial y_{\Sigma-1}}{\partial u^K}
+\frac{\partial\widetilde{y}_\Sigma}{\partial y_{\Sigma+1}}\frac{\partial y_{\Sigma+1}}{\partial u^K}+\ldots+\frac{\partial\widetilde{y}_\Sigma}{\partial y_{N+1}}\frac{\partial y_{N+1}}{\partial u^K},\qquad K=1,2,\ldots,N.
$$

It follows from (A.11), (A.12) that the column Σ of the matrix (4.14) is a linear combination of the remaining N columns, so the matrix is singular.

Appendix B

For the proof we employ the reasoning analogous to that of GODUNOV and SULTANGAZIN [11, 12]. Convexity of $\widehat{\varphi}^0_C(\widehat{\mu},\widehat{z}^\gamma)$ means that the inequality

$$
\text{(B.1)}\qquad \alpha_1\widehat{\varphi}^0_C(\widehat{\mu}^*,\widehat{z}^{*\gamma})+\alpha_2\widehat{\varphi}^0_C(\widehat{\mu}^{**},\widehat{z}^{**\gamma})>\widehat{\varphi}^0_C(\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**},\alpha_1\widehat{z}^{*\gamma}+\alpha_2\widehat{z}^{**\gamma}),
$$

holds for all $\alpha_2\geq 0$, $\alpha_2\geq 0$ such that $\alpha_1+\alpha_2=1$, and for all $\widehat{\mu}^*$, $\widehat{\mu}^{**}$, $\widehat{z}^{*\gamma}$, $\widehat{z}^{**\gamma}$ from a convex domain. Inequality (B.1) can be rearranged to the equivalent form

$$
\text{(B.2)}\qquad \frac{\alpha_1\widehat{\mu}^*}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}\frac{1}{\widehat{\mu}^*}\widehat{\varphi}^0_C(\widehat{\mu}^*,\widehat{z}^{*\gamma})+\frac{\alpha_2\widehat{\mu}^{**}}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}\frac{1}{\widehat{\mu}^{**}}\widehat{\varphi}^0_C(\widehat{\mu}^{**},\widehat{z}^{**\gamma})
> \frac{1}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}\widehat{\varphi}^0_C(\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**},\alpha_1\widehat{z}^{*\gamma}+\alpha_2\widehat{z}^{**\gamma}).
$$

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The right-hand side of (B.2) can be expressed as

$$\text{(B.3)}\qquad \left[\frac{1}{\widehat{\mu}^*}\frac{\alpha_1\widehat{\mu}^*}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}+\frac{1}{\widehat{\mu}^{**}}\frac{\alpha_2\widehat{\mu}^{**}}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}\right] \cdot\widehat{\varphi}^0_C\left(\frac{1}{\dfrac{1}{\widehat{\mu}^*}\dfrac{\alpha_1\widehat{\mu}^*}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}+\dfrac{1}{\widehat{\mu}^{**}}\dfrac{\alpha_2\widehat{\mu}^{**}}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}}, \frac{\dfrac{\widehat{z}^{*\gamma}}{\widehat{\mu}^*}\dfrac{\alpha_2\widehat{\mu}^*}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}+\dfrac{\widehat{z}^{**\gamma}}{\widehat{\mu}^{**}}\dfrac{\alpha_2\widehat{\mu}^{**}}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}}{\dfrac{1}{\widehat{\mu}^*}\dfrac{\alpha_1\widehat{\mu}^*}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}+\dfrac{1}{\widehat{\mu}^{**}}\dfrac{\alpha_2\widehat{\mu}^{**}}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}}\right).$$

Taking into account (B.3), (4.29), (4.30), we obtain from (B.2) the following inequality

$$\text{(B.4)}\qquad \beta_1\check{\lambda}^*\widehat{\varphi}^0_C\left(\frac{1}{\check{\lambda}^*},-\frac{\check{z}^{*\gamma}}{\check{\lambda}^*}\right)+\beta_2\check{\lambda}^{**}\widehat{\varphi}^0_C\left(\frac{1}{\check{\lambda}^{**}},-\frac{\check{z}^{**\gamma}}{\check{\lambda}^{**}}\right) > (\beta_1\check{\lambda}^*+\check{\beta}_2\check{\lambda}^{**})\widehat{\varphi}^0_C\left(\frac{1}{\beta_1\check{\lambda}^*+\beta_2\check{\lambda}^{**}},\ -\frac{\beta_1\check{z}^{*\gamma}+\beta_2\check{z}^{**\gamma}}{\beta_1\check{\lambda}^*+\beta_2\check{\lambda}^{**}}\right),$$

where

$$\text{(B.5)}\qquad \begin{gathered}\beta_1=\frac{\alpha_1\widehat{\mu}^*}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}},\qquad \beta_2=\frac{\alpha_2\widehat{\mu}^{**}}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}},\\ \check{\lambda}^*=\frac{1}{\widehat{\mu}^*},\quad \check{\lambda}^{**}=\frac{1}{\widehat{\mu}^{**}},\quad \check{z}^{*\gamma}=-\frac{\widehat{z}^{*\gamma}}{\widehat{\mu}^*},\quad \check{z}^{**\gamma}=-\frac{\widehat{z}^{**\gamma}}{\widehat{\mu}^{**}}\end{gathered}$$

and $\beta_1\geq 0$, $\beta_2\geq 0$, $\beta_1+\beta_2=1$ since $\widehat{\mu}^*=\theta^*>0$, $\widehat{\mu}^{**}=\theta^{**}>0$. From (5.4), (B.5) and (4.29), (4.30), we finally obtain the inequality

$$\text{(B.6)}\qquad \beta_1\check{\varphi}^0_C(\check{\lambda}^*,\check{z}^{*\gamma})+\check{\beta}_2\check{\varphi}^0_C(\check{\lambda}^{**},\check{z}^{**\gamma})<\check{\varphi}^0_C(\beta_1\check{\lambda}^*+\beta_2\check{\lambda}^{**},\beta_1\check{z}^{*\gamma}+\beta_2\check{z}^{**\gamma}),$$

which implies that $\check{\varphi}^0_C$ is concave since β_1 and β_2 take all admissible values when α_1 and α_2 take all admissible values from the region $\alpha_1\geq 0$, $\alpha_2\geq 0$, $\alpha_1+\alpha_2=1$. In the same way, it can be proved that concavity of $\check{\varphi}^0_C(\check{\lambda},\check{z}^{\gamma})$ implies convexity of $\widehat{\varphi}^0_C(\widehat{\mu},\widehat{z}^{\gamma})$ as well as the opposite situation when $\widehat{\varphi}^0_C(\widehat{\mu},\widehat{z}^{\gamma})$ is concave and $\check{\varphi}^0_C(\check{\lambda},\check{z}^{\gamma})$ is convex.

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The gas flow through the laser-sustained plasmas

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MODELLING of the laser-sustained plasmas (LSP) in a forced convective flow is presented. The emphasis is put on the modelling of the flow through the region of elevated temperature. Theoretical quasi-two-dimensional and full two-dimensional models are compared. Two examples of such flow are considered. In the first, plasma is maintained by cw CO_2 laser with output power of 2 kW and burns in a stream of argon at atmospheric pressure. The results show that quasi-$2D$ model in which the axial flow is given by the relation $\varrho u = \varrho_0 u_0 (\varrho/\varrho_0)^{1/2}$, where ϱ and u are density and velocity of the cold gas, fairly well describes the LSP. The second example is the flow through the argon plasma at 1 atm attached to the surface. In this case the simplified formulae do not exist and hence only the results of the two-dimensional model are presented. They are compared with the results of calculations made for constant temperature $T = 300$ K. It has been found that the presence of the hot region considerably increases the velocity near the surface.

1. Introduction

THE FLOW through the region of elevated temperature was recently studied by several authors [1–5]. The interest in this problem is connected with the laser machining: welding, cutting or cladding. In each of these processes the stream of the shielding gas flows through the plasma that is induced by the laser radiation. The plasma is either attached to the surface of the workpiece and burns in a mixture of the shielding gas and ionized metal vapour, or is detached from the surface and burns in the shielding gas. The first situation corresponds to a jet impinging on a surface after flowing through the region of elevated temperature (plasma). The second – to a jet flowing through the plasma free-burning in the space. Both situations are considered in this paper.

The main difficulty in studying this problem theoretically lies in solving the momentum equation. The most complete description of the laser-sustained plasma (LSP) in a forced convective flow includes the equations of conservation of mass, momentum, and energy [1, 5]. The first paper in which the full $2D$ model was used to calculate laser-sustained argon plasma at 2 atm was due to JENG and KEEFER [1].

However, because of complexity of such a model, simpler models [2–4] can be sometimes favoured. In this paper free-burning laser-sustained argon plasma in a convective flow was studied using different models. The flow has been modelled either in a simplified way or by solving the full set of equations, including momentum equation. The comparison of the results shows the range of validity of simplifying assumptions.

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2. Theoretical model

The flow velocity has a great influence on plasma parameters and intensity of laser radiation needed to maintain a stationary plasma. The theoretical description of experimentally observed phenomena is, however, quite complicated. The quasi-two-dimensional model [6] in which constant axial mass flux is assumed and only energy equation is solved in two dimensions, seems to be realistic only in the situation when plasma is bounded in the radial direction by walls (tube). In the case of unbounded plasma such model overestimates the influence of the flow, which is a result of $\varrho u = \text{const}$.

This problem is not serious for small velocities, few centimeters per second, which are characteristic for natural convection, because then the conduction term is much bigger than the convection term and plays the decisive role. In the case of forced convection, when inlet velocities can be of several meters per second, reliable results can only be expected from full two-dimensional model which solves the momentum equation. The solution of full set of equations consisting of energy, momentum and continuity equation is, however, quite complex. Therefore, in previous paper [4] instead of solving the full set of equations, we made certain assumption concerning the variation of the axial mass flux as a function of plasma density.

When the flow penetrates a region of increasing temperature, its behavior is similar to a flow past an obstacle. The pressure increases when approaching the hot region, the velocity decreases and most of the streamlines bypass the region of high temperature – only a relatively small part of them enter the hot region. When entering the hot region, the velocity of the gas increases with increasing temperature, due to a decrease of gas density, according to the continuity equation.

It has been shown [2, 4] that in the case when the cross-section of a hot plasma S_h is much smaller than the outer cross-section S_0 which constitutes the boundary, $S_0 \gg S_h$, we have

$$\varrho u \cong \varrho_0 u_0 \sqrt{\frac{\varrho}{\varrho_0}}\,. \tag{2.1}$$

Thus, when entering the low density region the mass flux is reduced by the factor $(\varrho/\varrho_0)^{1/2}$. At the other limit there is $S_h = S_0$ and

$$\varrho u = \text{const}. \tag{2.2}$$

Having defined the axial component by Eq. (2.1) (or (2.2)), the radial velocity component can be computed from the continuity equation. Equations (2.1) or (2.2) together with the equations of conservation of mass and energy constitute a quasi-2D model.

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In cylindrical symmetry the equations of conservation of mass, energy and momentum have the form

$$(2.3)\qquad \frac{\partial \varrho}{\partial t} + \frac{1}{r}\,\frac{\partial(\varrho r v)}{\partial r} + \frac{\partial(\varrho u)}{\partial z} = 0,$$

$$(2.4)\qquad \frac{\partial(\varrho h)}{\partial t} + \frac{\partial(\varrho r v h)}{\partial r \partial r} + \frac{\partial(\varrho h)}{\partial z} = \frac{1}{r}\,\frac{\partial}{\partial r}\left(\frac{r k_{\text{eff}}}{c_p}\,\frac{\partial h}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{k_{\text{eff}}}{c_p}\,\frac{\partial h}{\partial z}\right) + \sum_i \kappa_i I_i - \Phi,$$

$$(2.5)\qquad \frac{\partial(\varrho u)}{\partial t} + \frac{\partial(\varrho r v u)}{\partial r \partial r} + \frac{\partial(\varrho u u)}{\partial z} = \frac{\partial}{r\partial r}\left[\mu r\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left(2\frac{\mu \partial u}{\partial z}\right) - \frac{\partial p}{\partial z} - \varrho g,$$

$$(2.6)\qquad \frac{\partial(\varrho v)}{\partial t} + \frac{\partial(\varrho r v v)}{\partial r \partial r} + \frac{\partial(\varrho u v)}{\partial z} = \frac{\partial}{r\partial r}\left(2\frac{r\mu \partial v}{\partial r}\right) + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right)\right] - \frac{\partial p}{\partial r} - \frac{2\mu v}{r^2},$$

and the laser radiation transfer equation is given by

$$(2.7)\qquad \frac{dP_i}{ds_i} = -\kappa_i P_i\,,$$

where u and v are axial and radial velocity components, respectively. h is specific enthalpy $h = h(r, z, t)$; the mass density ϱ, the specific heat at constant pressure c_p, the viscosity μ [7], the effective thermal conductivity k_{eff} [7, 8], the radiation loss function Φ [9], and the absorption coefficient κ (at $10.6\,\mu$m wavelength) depend on h only. I_i is the local laser intensity (evaluated from local laser power P_i, $I_i = P_i/\pi(r_i^2 - r_{i-1}^2)$), and s_i is the local distance along the laser ray path.

The use of the effective thermal conductivity k_{eff} is based on the assumption that the part of radiation which is absorbed in the plasma can be represented as an increase in thermal conductivity $k_{\text{rad}} + k = k_{\text{eff}}$ and included in the diffusion term.

The absorption of the $10.6\,\mu$m radiation in a plasma is described by the absorption coefficient

$$(2.8)\qquad \kappa = \left[\kappa_{ei}^{ff} + \kappa_{ei}^{fb} + \kappa_{ea}\right] \times \left[1 - \exp\left(-\frac{h\nu}{kT}\right)\right],$$

where the first bracket contains the absorption due to the electron-ion inverse bremsstrahlung κ_{ei}^{ff} and photorecombination κ_{ei}^{fb} [10], and the absorption due to the electron-atom inverse bremsstrahlung κ_{ea} [11]. The second bracket contains contribution of the stimulated emission.

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For small Mach numbers (say $M \leq 0.3$), the pressure changes are small and we can assume that ϱ and other thermodynamic quantities depend only on the temperature. It has also been assumed that plasma is in local thermal equilibrium (LTE). At the atmospheric pressure, the deviations from LTE are small and plasma state should be fairly close to LTE.

The relevant properties of argon used for calculations are shown in Fig. 1 in terms of temperature.

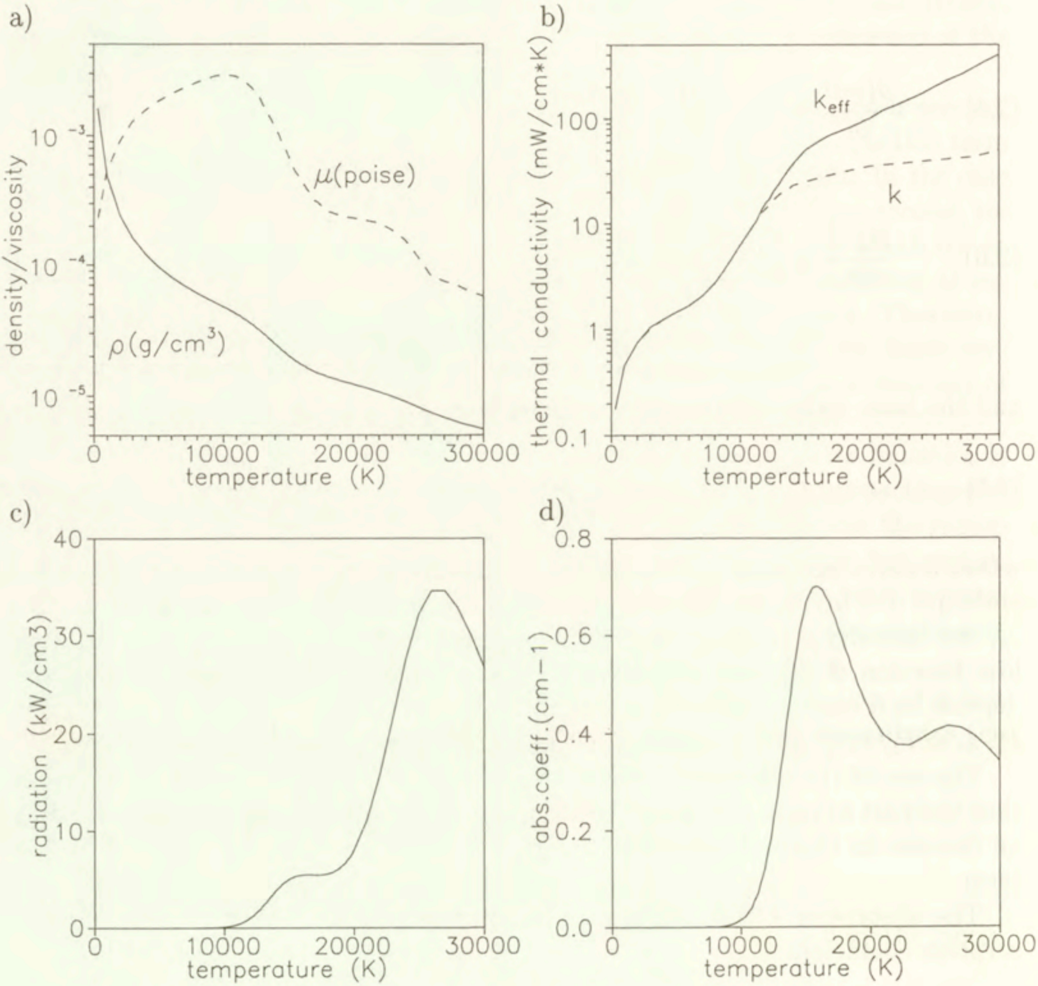

FIG. 1. Properties of argon used in the calculations in terms of temperature; a) the density and visciosity, b) the effective and pure thermal conductivity, c) the radiation power density, d) the absorption coefficient (at 10.6 μm wavelength).

The equations were solved employing the control volume method described by PATANKAR [12] and SIMPLEC algorithm [13]. In most cases a nonuniform grid

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with 40 radial by 80 axial mesh was used. Reasonable results can be obtained already with a 20×40 grid. Asymptotically for large times the solution of the above set of equations leads to the stationary solution which has been used in this paper.

The laser beam was divided into 200 rays along the radial direction and the path of each ray was traced according to geometrical optics. The laser intensity distribution was assumed to be TEM_{01}^{*} mode, and exact power was ascribed to each elementary ray.

Geometrical ray tracing of the laser beam is justified for small values of the f-number because the effect of diffraction can be neglected. In our calculations the f-number, i.e. ratio of the focal length to the laser beam diameter f/D, was ~ 8.8 and hence the spherical aberration was small. Then, the geometrical optics, which does not take into account the diffraction effects, results in an unrealistically small diameter of the laser beam at the focal plane and hence, in an unrealistically high laser beam intensity in the focus spot. In this work the focus spot diameter d_f was assumed to be equal to $d_f = \Theta \times f$, where Θ is the full angle divergence of the laser beam in rad, and f is the focal length of the lens. As estimated from the measured divergence, d_f was about 0.34 mm. In numerical calculations the value of d_f was taken into account assigning the radius $r_f = d_f/2$ to the smallest computational cell. Consequently the highest laser intensity at the focal plane was $I_f = P/\pi r_f^2$ where P was the total laser power.

3. Results

3.1. Free-burning laser-sustained plasma

In the case of the free-burning laser-sustained plasma it has been assumed, according to the situation met in practice, that the laser beam and the stream of the shielding argon gas are directed vertically downwards. The boundary conditions for the system of equations were as follows. We considered discharge in infinite space but it was enough to put $T = T_B = 300$ K at a radial distance $r = 3$ cm. For $r = 0$ the condition of symmetry requires that v and $\partial T/\partial r$ vanish. At the upstream boundary, a constant axial velocity u and uniform temperature $T = T_B$ were assigned. At the outflow boundary, $\partial T/\partial z = 0$ and $\partial(\varrho u)/\partial z = 0$ were assumed. The calculations were made for inlet velocity 5 and 2.5 m/s. The f-number was 8.8.

The temperature distributions for inlet velocity 5 m/s obtained from both quasi-$2D$ and full $2D$ theoretical models are shown in Fig. 2. The total absorption of the laser radiation in the plasma amounts to 74% while the power reradiated by the plasma is 850 W in the case of the full $2D$ model. The maximum temperature is ~ 19500 K. The total absorption obtained from the quasi-$2D$ model is similar but the power reradiated by the plasma is only 680 W. Because the radiation

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power is a strong function of temperature, the larger radiating plasma volume at high temperatures $T > 15\,\mathrm{kK}$ encountered in the case of the full $2D$ model, results in greater radiation power.

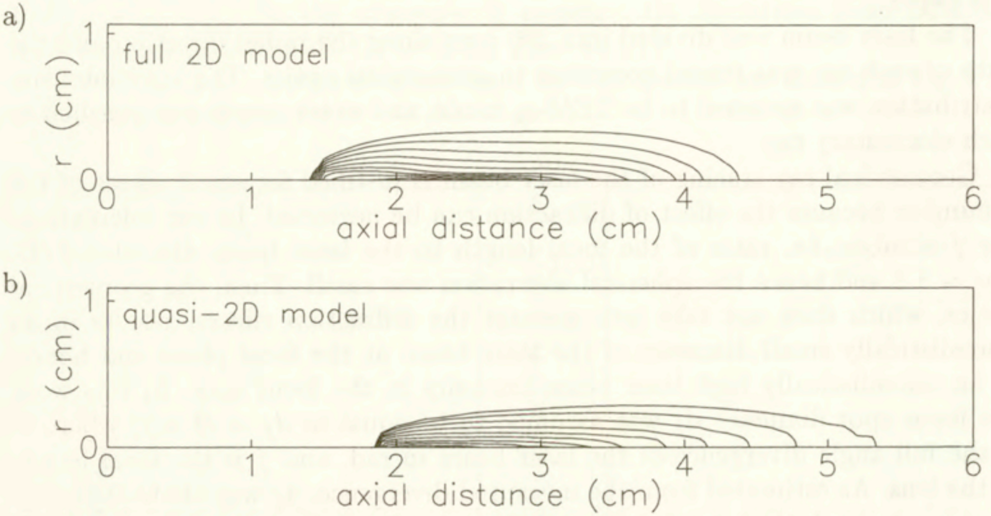

FIG. 2. Temperature distribution in a free-burning plasma; a) quasi-$2D$ model, b) full $2D$ model. Flow and laser beam direction from left to right. Outer isotherm 10000 K. Isotherms interval 1000 K. Laser power 2000 W, inlet velocity 5 m/s. Laser focal plane at $z = 2.5\,\mathrm{cm}$.

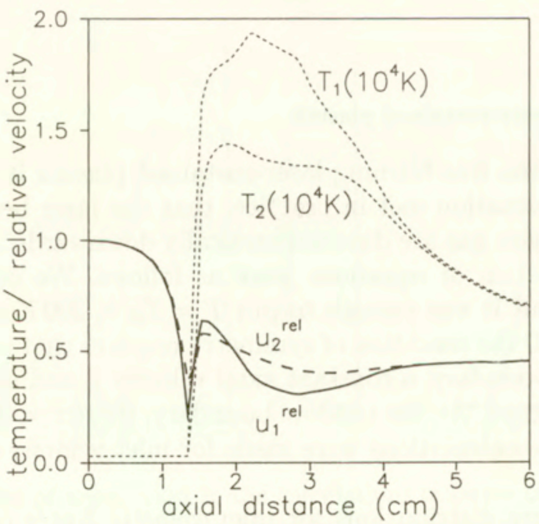

FIG. 3. Axial temperature in full $2D$ model and relative velocities $u^{\mathrm{rel}} = u(2D)/u(\text{quasi-}2D) = u/[u_0(\varrho_0/r)^{1/2}]$. Solid lines $r = 0$, dashed lines $r = 1\,\mathrm{mm}$.

Figure 3 shows the relative axial velocities $u_{\mathrm{rel}} = u_{2D}/u_{\text{quasi-}2D} = u/[u_0(\varrho_0/\varrho)^{1/2}]$ i.e., the ratio of the actual velocity calculated with the use of the full $2D$ model

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and velocity used in the quasi-$2D$ model, for two different radii $r = 0$ and $r = 0.1$ cm. The axial temperatures are taken from the full $2D$ model. The worst agreement is just before the hot region. In the case of $2D$ model the velocity at the axis ($r = 0$) drops to 0.2 of the inlet value. This is the reason why in the case of the full $2D$ model, the hot plasma core is shifted upstream in comparison to the quasi-$2D$ model. In the region of high temperature, the actual velocity and velocity used in quasi-$2D$ model differ by the factor of 2. The overestimation of the axial velocity by formula (2.1) results in more elongated temperature distribution in the case of quasi-$2D$ model.

The velocity vectors and mass flux density vectors in a free-burning plasma are shown in Fig. 4. The velocity increases in the hot region but only part of axial mass flow enters the plasma core. The Reynolds number $\mathrm{Re} = \varrho u D/\mu$ is

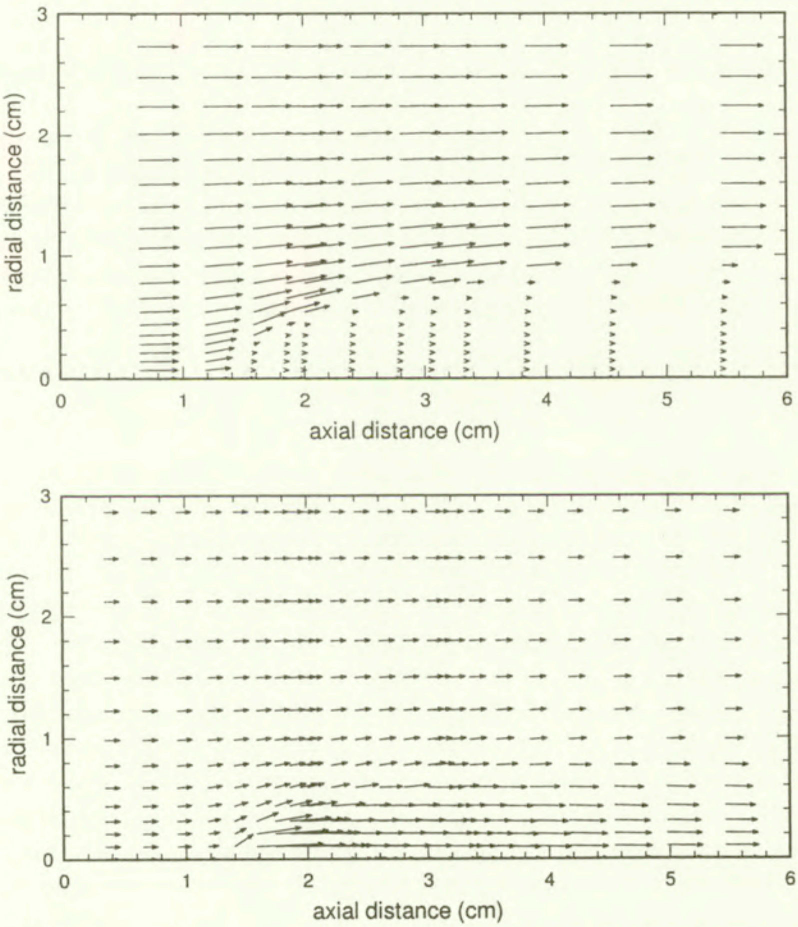

FIG. 4. Velocity vectors and mass flux density vectors in the free-burning plasma. Laser beam direction from left to right. Outer isotherm 10000 K. Isotherms interval 1000 K. Laser power 2 kW, inlet velocity 5 m/s.

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~ 2000. It has been calculated using the velocity, the mass density and viscosity ahead of the hot region ($T = 300\,\mathrm{K}$), and diameter of the hot core D defined by the isotherm $T = 10000\,\mathrm{K}$.

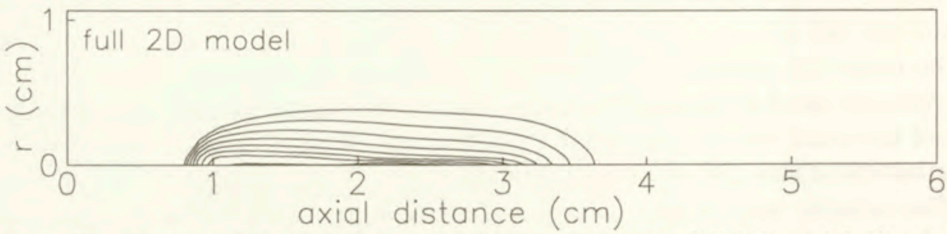

FIG. 5. Temperature distribution in a free-burning plasma – full $2D$ model. Flow and laser beam direction from left to right. Outer isotherm 10000 K. Isotherms interval 1000 K. Laser power 2000 W, inlet velocity 2.5 m/s. Laser focal plane at $z = 2.5\,\mathrm{cm}$.

The temperature distribution in a free-burning plasma (full $2D$ model) for inlet velocity 2.5 m/s is shown in Fig. 5. The decrease of the inlet velocity results in a considerable shift of the plasma relative to the focal plane in the direction of the laser beam. The total absorption of the laser radiation in the plasma amounts to 80% while the power reradiated by the plasma is 1050 W. The enhanced absorption and radiation power as compared to the results obtained for greater inlet velocity are due to different temperature profiles in both cases. The maximum temperature is $\sim 17900\,\mathrm{K}$ and is lower than that in the case of inlet velocity 5 m/s.

3.2. Impinging jet

In the case of an impinging jet it has been assumed that the stream of the shielding gas from a coaxial nozzle flows through the plasma plume attached to the surface and impinges on the workpiece. We considered the flow between two horizontal plates 1 cm apart. Again both the laser beam and the stream of the shielding argon gas are directed vertically downwards. They pass through the round hole in the upper plate and impinge on the lower one. The diameter of the hole is 6 mm. The boundary conditions were as follows. For $r = 0$ the conditions of symmetry were the same as before. At the upstream boundary, constant axial velocity u for $r \leq 3\,\mathrm{mm}$ and $u = 0$ for $r > 3\,\mathrm{mm}$ and uniform temperature $T = T_w = 300\,\mathrm{K}$ were assigned. At the surface of the lower plate both velocity components vanish. At the outflow boundary, $\partial T/\partial r = 0$ and $\partial(\varrho r v)/\partial r = 0$ were assumed. In this case the energy equation was not solved. Instead, in accordance to experimental results it has been assumed that the plasma temperature profile near the lower surface is Gaussian with the maximum temperature of 16000 K. For comparison, the calculations were also made for constant temperature $T = 300\,\mathrm{K}$. The calculations were made for inlet velocity 5 and 25 m/s.

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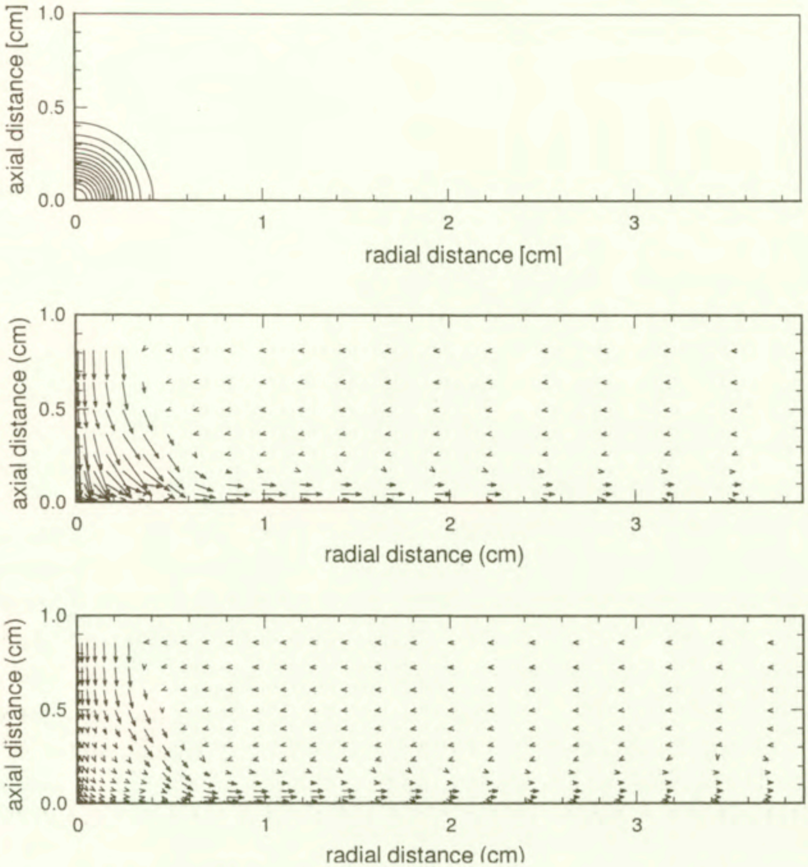

FIG. 6. Temperature distribution, velocity vectors and mass flux density vectors in an impinging jet. Outer isotherm 1000 K. Isotherms interval 2000 K. Stream inlet at the upper, left corner. Stream diameter at the inlet 6 mm, inlet velocity 5 m/s.

The results showing the temperature distribution, velocity vectors and mass flux density vectors for inlet velocity 5 m/s are shown in Fig. 6. Similarly to the case of the free-burning plasma, only part of axial mass flow enters the plasma core. Figure 7 shows the axial velocity $u(r = 0)$ profiles in terms of the axial distance for two inlet velocities 5 m/s and 25 m/s and two different temperature profiles; Gaussian (dotted line Fig. 7 a) and flat ($T = 300$ K).

It is worth noting that in the case of 25 m/s inlet velocity, the presence of the hot region increases the velocity by the factor of 7 in comparison to the solution for constant temperature $u(T_0, r, z)$ at the distance 0.1 cm from the surface. The increase depends on the inlet velocity and amounts to only 3 in the case of 5 m/s inlet velocity. In the case of the 25 m/s inlet velocity the distribution of the axial velocity is quite similar to the $u = [u(T_0, r = 0, z) \times (\varrho_0/\varrho)^{1.2}]$ where $T_0 = 300$ K (Fig. 7 b, dotted line). Such aproximation has been used in [14] where

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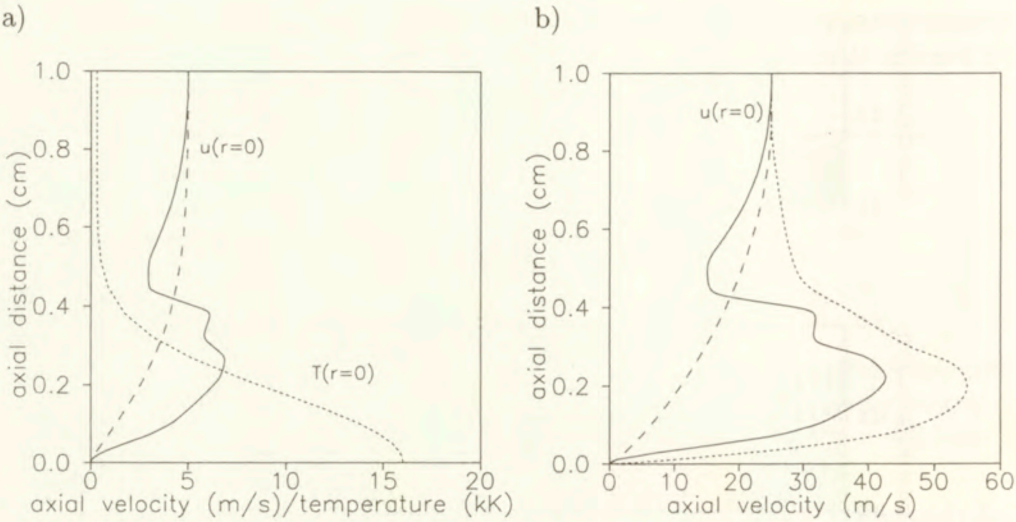

FIG. 7. Axial temperature and velocity profiles for two inlet velocities 5 m/s and 25 m/s. Solid lines: velocities calculated for the temperature profile shown in the Fig. 7 a. Dashed lines: velocities calculated for constant temperature $T_0 = 300\,\mathrm{K}$. Dotted lines: temperature (Fig. 7a), $u = [u(T_0, r = 0, z) \times (\varrho_0/\varrho)^{1/2}]$ (Fig. 7 b).

the authors used analytical solution for $u(T_0, r, z)$. However, for the inlet velocity 5 m/s, similar aproximation differs from the exact value by the factor of 3. The accuracy seems to depend on the Reynolds number and should be investigated in more detail in the future.

4. Conclusions

Numerical modelling of laser-sustained plasma in forced convective flow has been presented and two theoretical models, quasi-two-dimensional and full two-dimensional, have been compared. The results show that in the case of free-burning laser-sustained plasma, the simple formula in which the axial flow is given by the relation $\varrho u = \varrho_0 u_0 (\varrho/\varrho_0)^{1/2}$, where ϱ_0 and u_0 are the density and velocity of the cold gas, gives rough estimation of the mass flow entering the plasma. The quasi-2D model overestimates the axial velocity and as a result, the plasma temperature profile is more elongated and shifted downstream in comparison with full 2D model.

For inlet velocities of 5 and 2.5 m/s, the absorption of the laser radiation amounts to 74% – 80% of the laser beam power, respectively, i.e. about 1500 – 1600 W in the case of 2 kW laser power. More than half of that, 850 – 1050 W is reradiated by plasma. The plasma radiation covers broad range of wavelengths, from the violet to far infrared. In the case of materials strongly reflecting laser radiation, the heating of the surface by this radiation improves thermal coupling

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between the laser beam and a material. The shape of the plasma and its position with respect to the focal plane of the laser beam are slightly different for two models and the agreement with experimental data is better in the case of full $2D$ model. However, the comparison with the plasma photographs shows that the actual axial dimensions of the plasma are still about 25% smaller and at inlet velocity 5 m/s the real plasma front is shifted few mm downwards. The summation of the plasma radiation over the plasma depth showed that 95% of the radiation comes from within the isotherm of 11000 K and the dimension of this isotherm was compared with the plasma photographs. The lack of complete agreement betweem the experiment and theory is probably due to inaccuracies in the argon thermodynamic data used for the calculations. Although we checked that the calculations with the pure thermal conductivity did not change considerably the plasma shape and position of the plasma front with respect to the focal plane of the laser beam, still there could be other sources of errors like the radiation function or the absorption coefficient. Finally the lack of thermal local equilibrium can also result in different temperature profiles. This possibility, however, can only be checked by solving the two-temperature model.

The velocity distribution in the case of a jet impinging on the surface after passing through the region of high temperature, as it is in the case of the laser welding, has also been presented. In the case of the Gaussian temperature profile with the maximum temperature of 16000 K near the surface and the inlet velocity 25 m/s, the velocity increases by the factor of 7 at the distance 0.1 cm from the surface. This increase depends on the inlet velocity and most probably also on the temperature distribution near the surface. As the plasma plume influences the thermal coupling between the laser beam and the workpiece, the problem is interesting from the practical viewpoint and calls for further studies.

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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

An idea of thin-plate thermal mirror
II. Mirror created by a constant heat flux

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and A. MIODUCHOWSKI (EDMONTON)

FOLLOWING Part I, an idea of thermal mirrors created on the surfaces of a simply supported thin plane circular plate of an isotropic thermoelastic solid material by a uniform constant heat flux, which is applied to one of the plate surfaces, is presented. Such a thermal mirror is – within the approximations applied for obtaining the solutions of the heat conduction and thermoelasticity equations – an ideal (aberration-free) optical mirror. The optical properties of the thermal mirror and their time evolution are derived in two extreme cases: a) no energy losses through both plate surfaces, and b) no losses through the perturbed surface and the maximum losses through the opposite surface, and discussed in two asymptotical regimes: the short-time and the long-time ones. Theoretical possibilities of application of the thermal mirror to experimental determination of temperature conductivity of a material are discussed.

1. Introduction

IN PART I OF THE PAPER [3], the idea and the theory of thin-plate thermal mirror created on the surfaces of a simply supported thin plane circular plate of an isotropic thermoelastic solid material by a uniform heat pulse applied to one of the plate surfaces was presented. In the present part a similar problem, but with different heat perturbation, is examined, namely: the heat pulse is replaced by a constant heat flux (also uniform across the perturbed surface) applied in the initial moment.

The aim is to calculate the fundamental optical properties of the mirror (i.e. – its aberration characteristic, optical power, and the focal length), and their time evolution. The goal will be achieved in the same way as in Part I, i.e. the temperature field will be found first, next the deformation of the plate surfaces will be determined, and finally the optical properties of the mirror will be calculated and discussed.

The boundary conditions for the temperature field in the plate are assumed in two extremal versions:

a. All the plate surfaces are adiabatically insulated (i.e. all the losses through the plate surfaces are neglected).

b. The perturbed and the side surfaces are adiabatically insulated, and the temperature of the opposite surface is equal to the the temperature of the plate

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surrounding (i.e. the losses through the perturbed surface are neglected, and the losses through the opposite surface are maximum ones) (1), (2).

All the remaining general assumptions are the same as in Part I. Also identical are the general formulae determining the displacement field in the plate, those determining the deformation of the plate surfaces, and those determining the general optical properties of the mirror. They will be therefore used here without derivation.

2. The thermal problem

Following the specification of the thermal perturbation, the temperature T (counted from the initial (before perturbation) value) in the material is assumed to be dependent on z and t only: $T = T(z,t)$, where z stands for z-coordinate in the cylindrical coordinate system with the origin located in the center of the plate and with z-axis directed perpendicularly from the plate center toward the disturbed surface, and t stands for time. Therefore, according to the general assumptions adopted, the heat conduction equation (in the dimensionless variables) is

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta}{\partial \zeta^2} + \delta\left(\zeta - \frac{1}{2}\right) H(\tau - 0),$$

where

(2.1)
$$\tau = \frac{\kappa}{(2h)^2}\, t, \qquad \zeta = \frac{z}{2h},$$
$$\Theta(\zeta,\tau) = \frac{T[z = z(\zeta), t = t(\tau)]}{w_0}, \qquad w_0 = \frac{(2h)^2}{\kappa \varrho_0 c_p}\, q_0$$

stand, respectively, for: dimensionless time (κ is temperature conductivity of the material (heat conductivity divided by heat capacity per unit volume), $2h$ is the plate thickness); dimensionless z-coordinate as referred to the plate thickness $2h$; and dimensionless temperature (ϱ_0 is density of the material (in unperturbed state), c_p – its specific heat (under constant pressure), $q_0 = \text{const}$ represents the heat source function amplitude); $\delta(x - x_0)$ stands for Dirac delta distribution, and $H(\tau - \tau_0)$ – for unit step function (Heaviside function).

The initial condition is assumed in the form

(2.2)
$$\Theta(\tau = 0) = 0.$$

The boundary conditions are assumed in two alternative versions, according to the assumptions adopted in Sec. 1:

(2.3a)
$$\frac{\partial \Theta^{(a)}}{\partial \zeta}\left(\zeta = \pm\frac{1}{2}\right) = 0,$$

(1) The third extreme case: all the surface losses are maximum ones – is not interesting, because in this case the temperature field inside the plate is not perturbed at all.

(2) The problem of finite surface losses (in the long-time regime) is mentioned in Sec. 7.

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$$\text{(2.3b)}\qquad \begin{cases} \dfrac{\partial \Theta^{(b)}}{\partial \zeta}\left(\zeta = \dfrac{1}{2}\right) = 0, \\ \Theta^{(b)}\left(\zeta = -\dfrac{1}{2}\right) = 0. \end{cases}$$

The solution of the problem expressed by Eqs. (2.1)–(2.3) is found as follows. The Green function to the (one-dimensional) heat conduction problem in the half-space insulated adiabatically is the doubled Green function of that problem in the whole space, and the latter Green function is known [1]. Applying therefore the method of sources and sinks, one may find the Green function to the thermal problem for the plate:

$$\Theta_{Gr}^{(ab)} = \frac{1}{\sqrt{\pi\tau}} \sum_{m=0}^{\infty} (-1)^{mp} \left\{ \exp\left[-\frac{\left(2m + \frac{1}{2} - \zeta\right)^2}{4\tau} \right] + (-1)^p \exp\left[-\frac{\left(2m + \frac{3}{2} + \zeta\right)^2}{4\tau} \right] \right\},$$

where $p = 0$ in case a and $p = 1$ in case b. Next, convoluting the latter Green function and the heat source function, the solution of the problem considered is found:

$$\text{(2.4a)}\qquad \Theta^{(a)} = 2\sqrt{\tau} \sum_{m=0}^{\infty} \left[\operatorname{ierfc} \frac{2m + \frac{1}{2} - \zeta}{2\sqrt{\tau}} + \operatorname{ierfc} \frac{2m + \frac{3}{2} + \zeta}{2\sqrt{\tau}} \right] = \tau + 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{\pi^2 k^2} \left\{ 1 - \exp[-k^2\pi^2\tau] \right\} \cos\left[k\pi \left(\zeta + \frac{1}{2} \right) \right],$$

$$\text{(2.4b)}\qquad \Theta^{(b)} = 2\sqrt{\tau} \sum_{m=0}^{\infty} (-1)^m \left[\operatorname{ierfc} \frac{2m + \frac{1}{2} - \zeta}{2\sqrt{\tau}} - \operatorname{ierfc} \frac{2m + \frac{3}{2} + \zeta}{2\sqrt{\tau}} \right] = 8 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\pi^2 (2k-1)^2} \left\{ 1 - \exp\left[-(2k-1)^2 \frac{\pi^2}{4} \tau \right] \right\} \times \sin\left[(2k-1) \frac{\pi}{2} \left(\zeta + \frac{1}{2} \right) \right],$$

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where the integral complementary error function is

$$\operatorname{ierfc}(x) = \int_x^{\infty} \operatorname{erfc}(y)\,dy, \qquad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp[-y^2]\,dy$$

($\operatorname{erfc}(x)$ stands for the complementary error function), and the first line (in each equation) represents the original solution obtained using the method mentioned ([3]), and the second one – that solution after expansion into Fourier series ([4]) (the function $\Theta^{(a)}$ is symmetrical, and the function $\Theta^{(b)}$ is antisymmetrical with respect to $\zeta + (1/2)$).

3. Fundamental characteristics of the thermal mirror

According to the assumptions adopted, the general formulae determining the displacement U of the plate surfaces with respect to their initial (before perturbation) position (see Fig. 1 in Part I) are the same as in the case considered in Part I. Using these formulae and the solutions of the thermal problem, we have (see Part I, Eqs. (4.2)):

$$\begin{aligned} U^u &= \frac{N_T}{E} + U_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2 \frac{1}{(1+\delta^u)^2}\right] \cong \frac{N_T}{E} + U_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2\right], \\ U_l &= -U_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2 \frac{1}{(1-\delta_l)^2}\right] \cong -U_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2\right], \end{aligned} \tag{3.1}$$

where the superscripts u and l refer to the disturbed and the opposite surfaces of the plate (the upper and the lower surfaces in Fig. 1 in Part I), respectively; r stands for the coordinate of a given point in the cylindrical coordinate system mentioned earlier; r_0 is the plate radius;

$$U_{\max} = \frac{3r_0^2}{4h^3E} M_T\,, \tag{3.2}$$

([3]) The same results are obtainable by applying the Laplace transformation method to solve the following equivalent problems:

$$\frac{\partial\Theta}{\partial\tau} = \frac{\partial^2\Theta}{\partial\zeta^2}, \qquad \Theta(\tau=0)=0, \qquad \frac{\partial\Theta}{\partial\zeta}\left(\zeta=\frac{1}{2}\right) = H(\tau-0),$$

$$\begin{cases} \dfrac{\partial\Theta}{\partial\zeta}\left(\zeta=-\dfrac{1}{2}\right)=0, & \text{in case a,} \\ \Theta\left(\zeta=-\dfrac{1}{2}\right)=0, & \text{in case b.} \end{cases}$$

([4]) The same results are obtainable by applying the Fourier method of separation of independent variables to solve the equivalent problems mentioned in the previous footnote (and expanding the functions $\zeta + (1/2)$ and $(\zeta + (1/2))^2$ into Fourier series).

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(3.3) $$\delta_l^u = \frac{1}{2hE}\left[\pm N_T + \frac{3}{h}M_T\right].$$

Here the upper and the lower signs refer to the perturbed and the opposite surfaces of the plate (the upper and the lower ones in Fig. 1 in Part I), respectively; E stands for the Young modulus (see Part I, Eqs. (4.3) and (4.40)); and (cf. Part I, Eqs. (4.1))

(3.4) $$N_T = E\alpha \int_{-h}^{h} T\,dz,$$

(3.4a) $$N_T^{(a)} = 2hE\alpha w_0 \tau,$$

(3.4b) $$N_T^{(b)} = 2hE\alpha w_0\tau\left[1 - 8\sum_{m=0}^{\infty}(-1)^m\,\mathrm{i}^2\mathrm{erfc}\frac{2m+1}{2\sqrt{\tau}}\right] = hE\alpha w_0\left\{1 - \frac{32}{\pi^3}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^3}\exp\left[-(2k-1)^2\frac{\pi^2}{4}\tau\right]\right\};$$

(3.5) $$M_T = E\alpha\int_{-h}^{h} zT\,dz,$$

(3.5a) $$M_T^{(a)} = 2h^2E\alpha w_0\tau\left[1 - \frac{8}{3\sqrt{\pi}}\sqrt{\tau} + 32\sqrt{\tau}\sum_{m=1}^{\infty}(-1)^{m+1}\,\mathrm{i}^3\mathrm{erfc}\frac{m}{2\sqrt{\tau}}\right] = \frac{1}{6}h^2E\alpha w_0\left\{1 - \frac{96}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}\exp[-(2k-1)^2\pi^2\tau]\right\},$$

(3.5b) $$M_T^{(b)} = 2h^2E\alpha w_0\tau\left\{1 - \frac{8}{3\sqrt{\pi}}\sqrt{\tau} + 8\sum_{m=0}^{\infty}(-1)^m\left[\mathrm{i}^2\mathrm{erfc}\frac{2m+1}{2\sqrt{\tau}} + 4\sqrt{\tau}\,\mathrm{i}^3\mathrm{erfc}\frac{m+1}{\sqrt{\tau}}\right]\right\}$$
$$= \frac{1}{3}h^2E\alpha w_0\left\{1 - \frac{96}{\pi^4}\sum_{k=1}^{\infty}\frac{4-(-1)^{k+1}(2k-1)\pi}{(2k-1)^4}\times\exp\left[-(2k-1)^2\frac{\pi^2}{4}\tau\right]\right\},$$

where, in turn, α stands for (linear) heat expansion coefficient;

$$\mathrm{i}^n\mathrm{erfc}(x) = \int_x^{\infty}\mathrm{i}^{n-1}\mathrm{erfc}(y)\,dy, \qquad n \geq 2,$$

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and ierfc(x) was defined at the end of Sec. 2; and the approximations (with an accuracy to an assumed small number O^*) hold if

$$|\delta_l^u| \leq \frac{1}{2} O^* \tag{3.6}$$

(for detailed argumentation for this criterion see Part I, Appendix).

The general formulae determining the aberration characteristic $\varepsilon = \varepsilon(r)$, the optical power D and the focal length f of the thermal mirror considered are the same as in the case examined in Part I. The deflection angle ε is defined as an angle between incident testing light beam parallel to the symmetry axis and this ray after reflection from the mirror (see Fig. 2 in Part I). The optical power is defined as a reciprocal of the focal length f, and the latter quantity is defined as a distance of the focal point from the mirror along the mirror symmetry axis (see Fig. 2 in Part I). The deflection angle, the optical power and the focal length are understood to be negative in the case of defocusing mirror (the perturbed, or the upper surface in our case), and positive in the case of focusing mirror (the opposite, or the lower surface in our case).

Using the general formulae mentioned above we have (in both cases a and b):

$$\varepsilon_l^u = \mp 2 \arctan \left[\frac{2U_{\max}}{r_0} \frac{r}{r_0} \frac{1}{(1 \pm \delta_l^u)^2} \right] \cong \mp 2 \arctan \left[\frac{2U_{\max}}{r_0} \frac{r}{r_0} \right] \cong \mp \frac{4U_{\max}}{r_0} \frac{r}{r_0}, \tag{3.7}$$

$$D_l^u = \frac{1}{f_l^u} = \mp \frac{4}{r_0^2} U_{\max} \frac{1}{(1 \pm \delta_l^u)^2} \cong \mp \frac{4}{r_0^2} U_{\max}, \tag{3.8}$$

where the upper and the lower signs refer to the disturbed (upper) and the opposite (lower) surfaces, respectively; $U_{\max}$ and δ_l^u are given by Eqs. (3.2) and (3.3), respectively, with Eqs. (3.4) and (3.5); the first approximation in Eqs. (3.7) and (3.8) holds, if Ineq. (3.6) is satisfied; the second approximation in Eq. (3.7) (the so-called paraxial optics approximation) is valid if (in addition)

$$\left(\frac{2U_{\max}}{r_0} \right)^2 \frac{r^2}{r_0^2} \leq \frac{3O^*}{1 + O^*} \cong 3O^*. \tag{3.9}$$

The results expressed by Eqs. (3.7) and (3.8) denote, that the mirrors under considerations (both the upper and the lower one, in both the cases a and b) are – within the approximations applied – the ideal (parabolic) ones, i.e. they are free of optical aberrations (their optical power and focal length are independent of r). In principle, no paraxial optics approximation is therefore needed to idealize them.

As it is seen from the formulae given above, the time evolution of the displacement function U and the optical properties of the thermal mirror is governed by

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the dependence of the functions: N_T (Eqs. (3.4)), $U_{\max}$ (Eqs. (3.2) and (3.5)), and δ (Eqs. (3.3) with (3.4) and (3.5)) – on time. This dependence is complicated and difficult for a simple interpretation. Significant simplification can be obtained for sufficiently short or long time and under the additional condition that the term δ can be neglected in comparison with unity in the suitable formulae.

4. The short-time regime

For sufficiently short time ($\tau \le \tau_{\rm short}$) and for not very strong heat perturbation ($w_0 \le w_{0,\rm short}$), the characteristics of the thermal mirrors (Eqs. (3.2), (3.1), (3.7) and (3.8)) can be approximated (with an accuracy to $(1+O^*)^2 - 1 \cong 2O^*$ in case a, and $(1+O^*)^3 - 1 \cong 3O^*$ in case b, where O^* is an assumed small number) by the following formulae (in both cases a and b; cf. Eqs. (7.5) and (7.6) in Part I):

$$U_{\max} = \dot{U}_{\max}(0)\, t\, \varphi(\tau), \tag{4.1}$$

$$\begin{aligned} U^u &= \dot{U}_{\max}(0)\, t \left\{ \frac{1}{3}\left(\frac{2h}{r_0}\right)^2 + \left[1 - \left(\frac{r}{r_0}\right)^2\right] \varphi(\tau) \right\}, \\ U_l &= -\dot{U}_{\max}(0) \left[1 - \left(\frac{r}{r_0}\right)^2\right] t\, \varphi(\tau), \end{aligned} \tag{4.2}$$

$$\varepsilon_l^u = \mp 2 \arctan\left[\frac{2\dot{U}_{\max}(0)}{r_0}\frac{r}{r_0} t\varphi(\tau)\right] \cong \mp 4 \frac{\dot{U}_{\max}(0)}{r_0}\frac{r}{r_0} t\varphi(\tau), \tag{4.3}$$

$$D_l^u = \frac{1}{f_l^u} = \mp \frac{4}{r_0^2}\dot{U}_{\max}(0) t\varphi(\tau), \tag{4.4}$$

where (see Eqs. $(2.1)_{1,4}$):

$$\dot{U}_{\max}(0) = \frac{3r_0^2}{2h}\alpha w_0 \frac{\kappa}{(2h)^2} = 3\alpha\left(\frac{r_0}{2h}\right)^2 \frac{2hq_0}{\varrho_0 c_p} = 3\alpha\left(\frac{r_0}{2h}\right)^2 \frac{\dot{Q}_{\rm tot}}{\pi r_0^2 \varrho_0 c_p} \tag{4.5}$$

(here, in turn, $\dot{Q}_{\rm tot}$ stands for the total power applied to the perturbed surface),

$$\varphi(\tau) = 1 - \frac{8}{3\sqrt{\pi}}\sqrt{\tau}\,, \tag{4.6}$$

and the approximation in Eq. (4.3) (the paraxial optics approximation) holds, if

$$\frac{6\alpha}{(2h)^2}\frac{2hq_0}{\varrho_0 c_p} r t\varphi(\tau) \le \sqrt{\frac{3O^*}{1+O^*}} \cong \sqrt{3O^*}\,.$$

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Thus, the functions: $U_{\max}$, U_l^u, $\tan(\varepsilon_l^u/2)$, D_l^u and f_l^u divided by t are linear functions of $\sqrt{t}$ (see Eq. $(2.1)_1$) in the short-time regime.

The approximate formulae given above are the same in both cases a and b. The main difference depends on different criteria of applicability of the short-time approximation. These criteria will be therefore deduced and specified separately for each case.

4.1. Criteria of applicability of the short-time approximation – Case a

The analysis will be performed according to the following program:

• first, Ineq. (3.6) is assumed to be satisfied;

• second, a possibility of simplification of the function M_T for simplifying the equations for U_l, ε_l^u, and $D_l^u = 1/f_l^u$ will be analyzed, since – after neglecting the functions δ – these quantities depend on time through the function M_T only (see Eqs. $(3.1)_2$, (3.2), (3.5), (3.7) and (3.8));

• third, a possibility of simplification of equation for the function U^u will be examined;

• fourth, consequence of Ineq. (3.6) will be analyzed, with comments on additional condition(-s), which should be taken into account in connection with this point, and also – in connection with the time approximation discussed.

Thus, at the beginning Ineq. (3.6) is assumed to be satisfied.

Concerning the function M_T let us note that for sufficiently short time, the sum in the brackets in Eq. $(3.5\text{a})_1$ can be truncated after the second term. Because $\mathrm{i}^3\mathrm{erfc}(x)$ is a monotonically decreasing function, therefore for $\tau < \frac{9}{64}\pi \cong 0.44$ this sum represents the Leibniz series ([5]). Then, the sum considered can be approximated by the first two terms only, with an accuracy to O^*, if

$$32\sqrt{\tau}\,\mathrm{i}^3\mathrm{erfc}\frac{1}{2\sqrt{\tau}} \le O^*\left(1 - \frac{8}{3\sqrt{\pi}}\sqrt{\tau}\right). \tag{4.7}$$

([5]) The Leibniz series (LS) is understood to be a converging series of the type

$$LS = \sum_{m=0}^{\infty}(-1)^m a_m, \qquad a_m > a_{m+1} > 0.$$

Such a series can be precisely estimated as follows (Leibniz's theorem):

$$\sum_{m=0}^{2n-1}(-1)^m a_m < LS < \sum_{m=0}^{2n}(-1)^m a_m, \qquad \text{and} \qquad \sum_{m=0}^{2n}(-1)^m a_m > LS > \sum_{m=0}^{2n+1}(-1)^m a_m.$$

In particular case one may obtain:

$$a_0 - a_1 + a_2 > LS > a_0 - a_1,$$

therefore $LS \cong a_0 - a_1$ with an accuracy to O^*, if $a_2 \le O^*(a_0 - a_1)$; also

$$a_0 > LS > a_0 - a_1,$$

therefore $LS \cong a_0$ with an accuracy to O^*, if $a_1 \le \frac{O^*}{1+O^*}a_0$.

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This inequality is satisfied, if

$$\tau \le \tau^{(a)}_{\text{short}} = \frac{1}{4x_a^2}\,, \tag{4.8}$$

where x_a stands for a solution of the equation $\mathrm{i}^3\mathrm{erfc}\,x = O^*\left(\dfrac{x}{16} - \dfrac{1}{12\sqrt{\pi}}\right)$ with respect to x.

Assuming, for example

• $O^* = 0.01$

one may find (⁶)

$$\tau \le \tau^{(a)}_{\text{short}} \cong 0.11. \tag{4.9}$$

Assuming, in addition

• $\kappa \doteq (10^{-7} - 10^{-4})\ \mathrm{m}^2/\mathrm{s}$,

where the sign $\doteq$ reads: "is of the order of", and the first value in the brackets refers to the worst temperature conductors and the second one – to the best ones, one may rewrite the criterion expressed by Ineq. (4.9) in the following dimensional form (using Eq. $(2.1)_1$) (⁷):

$$t \le t^{(a)}_{\text{short}} \cong \begin{cases} 1\cdot(1 - 10^{-3})\,\mathrm{s}, & \text{for} \quad 2h = 10^{-3}\,\mathrm{m}, \\ 1\cdot(10^2 - 10^{-1})\,\mathrm{s}, & \text{for} \quad 2h = 10^{-2}\,\mathrm{m}. \end{cases}$$

Concerning the function U^u let us note – after substituting Eqs. (3.4a) and $(3.5\mathrm{a})_1$ into Eq. $(3.1)_1$ and using Eq. (4.5) – that this function is also the Leibniz series (for $\tau \le \dfrac{9}{64}\pi$) and it can be written in the form:

$$\frac{U^u}{\dot U_{\max}(0)t} = \left[\frac{1}{3}A^2 + \left(1 - \overline{r}^2\right)\right] - \left[\left(1 - \overline{r}^2\right)\frac{8\sqrt{\tau}}{3\sqrt{\pi}}\right] + \left[32\sqrt{\tau}\left(1 - \overline{r}^2\right)\mathrm{i}^3\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}}\right] - \ldots\,,$$

where $A = 2h/r_0$, and $\overline{r} = r/r_0$. The right-hand side of this formula can be truncated after the second term (with an accuracy to O^*), if (cf. Ineq. (4.7))

$$32\sqrt{\tau}\left(1 - \overline{r}^2\right)\mathrm{i}^3\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} \le O^*\left\{\left[\frac{1}{3}A^2 + \left(1 - \overline{r}^2\right)\right] - \left[\left(1 - \overline{r}^2\right)\frac{8\sqrt{\tau}}{3\sqrt{\pi}}\right]\right\}.$$

This inequality is always satisfied for $\overline{r} = 1$. For $\overline{r} < 1$ it can be rewritten in the form:

$$32\sqrt{\tau}\,\mathrm{i}^3\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} \le O^*\left\{\left[1 + \frac{1}{3}\frac{A^2}{1 - \overline{r}^2}\right] - \frac{8\sqrt{\tau}}{3\sqrt{\pi}}\right\}.$$

(⁶) For $O^* = 0.001$ (0.0001) one can find: $\tau^{(a)}_{\text{short}} \cong 0.071$ (0.050).

(⁷) For $O^* = 0.001$ (0.0001) the coefficient 1 before the brackets is replaced by 0.7 (0.5).

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By comparing this inequality and Ineq. (4.7) it is seen, that it is weaker than Ineq. (4.7). If therefore the criterion expressed by Ineq. (4.7) is satisfied, then also the function U^u can be approximated in the same way as the functions U_l, ε_l^u, and $D_l^u = 1/f_l^u$, as it is expressed by Eqs. (4.2)–(4.4).

Concerning the functions δ_l^u let us note that, according to Eq. (3.3), the behaviour of these functions in the case examined is determined by the functions $N_T^{(a)}$ (Eq. (3.4a)) and $M_T^{(a)}$ (Eq. (3.5a)). As it is seen from Eqs. (3.4a) and $(3.5a)_2$, both the latter functions are positive and are monotonically increasing from 0 at $\tau = 0$ ([8]).

The function $\delta^{(a)u}(\tau)$ is therefore also increasing from 0 as $\tau = 0$ to $\delta^{(a)u}(\tau_{\text{short}}^{(a)})$ in the short-time regime; for $\tau_{\text{short}}^{(a)} \cong 0.11$ ($O^* = 0.01$ – see Eq. (4.9)) one can obtain: $0 \le \delta^{(a)u}(\tau) \le \delta^{(a)u}(\tau_{\text{short}}^{a}) \cong 0.28\alpha w_0$ ([9]). The function $\delta_l^{(a)}(\tau)$ is also positive-valued in the short-time regime specified above, however it is not monotonic. It starts from 0 as $\tau = 0$, and approaches a maximum value about $0.059\alpha w_0$ at $\tau = \tau_{l,m}^{(a)} \cong 0.090$.

Thus, the criterion expressed by Ineq. (3.6) for neglecting the functions δ in the short-time regime may be written in the form:

$$
\begin{aligned}
\delta^{u(a)} &\le \delta^u(\tau_{\text{short}}^{(a)}) \le 0.5\,O^* ,\\
\delta_l^{(a)} &\le \delta_l(\tau^*) \qquad \le 0.5\,O^* ,
\end{aligned}
\tag{4.10}
$$

where $\tau^* = \tau_{l,m}^{(a)} \cong 0.090$, if $\tau_{\text{short}}^{(a)} \ge \tau_{l,m}^{(a)}$, or $\tau^* = \tau_{\text{short}}^{(a)}$, if $\tau_{\text{short}}^{(a)} \le \tau_{l,m}^{(a)}$. If $O^* = 0.01$ (and therefore $\tau_{\text{short}}^{(a)} \cong 0.11$), then ([10])

$$
\begin{aligned}
\delta^{u(a)} &\le \delta^u(\tau_{\text{short}}^{(a)}) \cong 0.28\alpha w_0 \le 0.5O^*,\\
\delta_l^{(a)} &\le \delta_l(\tau_{l,m}^{(a)}) \quad \cong 0.059\alpha w_0 \le 0.5O^*.
\end{aligned}
\tag{4.11}
$$

Under the assumption:

- $\alpha \doteq 10^{-5}$ 1/K,

this inequalities read ([11]):

$$
w_0 \le \begin{cases} 18\cdot 10^2\text{K} & \text{for the perturbed (the upper) surface,}\\ 85\cdot 10^2\text{K} & \text{for the opposite (the lower) surface.} \end{cases}
$$

([8]) The series in Eq. $(3.5a)_2$ at $\tau = 0$ is equal to unity (see [2]).

([9]) For $\tau_{\text{short}}^{(a)} = 0.071$ (0.050) ($O^* \cong 0.001$ (0.0001)) the coefficient 0.28 is replaced by 0.20 (0.15).

([10]) If $O^* = 0.001$ (0.0001), then the coefficient 0.28 is replaced by 0.20 (0.15), and the coefficient 0.59 – by 0.057 (0.049).

([11]) For $O^* = 0.001$ (0.0001) the coefficient 18 is replaced by 2.5 (0.33), and the coefficient 85 – by 8.8 (1.0).

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Using Eq. $(2.1)_4$ and assuming (in addition to the assumptions adopted above)

• $\varrho_0 c_p \doteq 5\cdot 10^6$ J/(m^{3}K),

one may rewrite the inequality given above as the following criterion for the disturbing heat flux density (equivalent to the disturbing heat source):

$$2hq_0 \le \begin{cases} B(10^6 - 10^9)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-3}\,\mathrm{m}, \\ B(10^5 - 10^8)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-2}\,\mathrm{m}, \end{cases}$$

where $B \cong 0.9$ for the perturbed (the upper) surface, and $B \cong 4$ for the opposite (the lower) one ([12]).

Concerning the limitation of the heat perturbation, an additional condition should be taken into account. The heat conduction and the mechanical processes are treated in the linear approximation. It is therefore reasonable to require, that the heat perturbation should not significantly change the properties of the material. Let the maximum allowable temperature be T^*. Because the temperature approaches its highest value at the perturbed surface, therefore this requirement may be written in the form $T^{(a)}(z = h) \le T^*$. According to Eqs. $(2.1)_{2,3}$ and $(2.4\mathrm{a})_1$ we have

$$T^{(a)}(z = h, t) = w_0\, \Theta^{(a)}\left(\zeta = \frac{1}{2}, \tau\right) = 2w_0\sqrt{\tau}\left[\frac{1}{\sqrt{\pi}} + 2\sum_{m=1}^{\infty} \operatorname{ierfc}\frac{m}{\sqrt{\tau}}\right].$$

The function $\operatorname{ierfc}(x)$ is a decreasing function of its argument, therefore the temperature of the perturbed surface increases monotonically with time. Under the assumption

• $T^* = 100$ K,

the criterion analyzed may be therefore written in the following approximate form:

$$w_0 \le \frac{\sqrt{\pi}\, T^*}{2\sqrt{\tau^{(a)}_{\mathrm{short}}}} \cong 2.7\cdot 10^2\,\mathrm{K}, \tag{4.12}$$

where $\tau^{(a)}_{\mathrm{short}}$ was assumed to be equal 0.11 ([13]). Using Eq. $(2.1)_4$ one may rewrite this inequality as the following criterion for the disturbing heat flux density (equivalent to the disturbing heat source) ([14]):

$$2hq_0 \le \begin{cases} 0.1\cdot(10^6 - 10^9)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-3}\,\mathrm{m}, \\ 0.1\cdot(10^5 - 10^8)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-2}\,\mathrm{m}. \end{cases}$$

([12]) If $O^* = 0.001$ (0.0001), then $B \cong 0.1$ (0.02) or $B \cong 0.4$ (0.05) for the perturbed (the upper) surface or for the opposite (the lower) one, respectively.

([13]) If $\tau^{(a)}_{\mathrm{short}}$ is put to be equal to 0.071 (0.050), then the coefficient 2.7 is replaced by 3.3 (4.0).

([14]) If $\tau^{(a)}_{\mathrm{short}}$ is put to be equal to 0.071 (0.050), then the coefficient 0.1 before the brackets is replaced by 0.2 (0.2).

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Comparing these results with inequalities for w_0 and q_0 obtained earlier as a criterion for neglecting the functions δ in the short-time regime, one may see that the former inequalities are weaker than the latter ones for $O^* = 0.01$; if therefore the temperature criterion of linearization is satisfied, then the functions δ may be neglected in the suitable formulae in the short-time regime. The reverse situation takes place for $O^* = 0.0001$.

At the end of this subsection two additional conditions should be mentioned. The first concerns the assumption that all the mechanical phenomena are treated in the quasi-static approximation, i.e. the observation time τ can not be too short: $\tau \geq \tau_{\min}$. The suitable criterion of this kind was proposed and commented in Part I.

The second remark concerns the initial condition. The form of this condition indicates that the time, during which the heat perturbation is switched on, should be sufficiently short in the time scale applied.

4.2. Criteria of applicability of the short-time approximation – Case b

The analysis will be performed according to the same program as in Case a (mentioned at the beginning of Subsec. 4.1).

Thus, at the beginning Ineq. (3.6) is assumed to be satisfied.

Concerning the function M_T let us note that for sufficiently short time, the sum in the brackets in Eq. $(3.5\text{b})_1$ can be truncated after the second term. Because the functions $\mathrm{i}^2\mathrm{erfc}(x)$ and $\mathrm{i}^3\mathrm{erfc}(x)$ are both monotonically decreasing ones, therefore the sum examined can be treated as a Leibniz series([5]) for $\tau \leq \dfrac{9\pi}{64} \cong 0.44$, and it can be approximated by its two first terms only with an accuracy to O^*, if

$$\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} + 4\sqrt{\tau}\,\mathrm{i}^3\mathrm{erfc}\,\frac{1}{\sqrt{\tau}} \leq O^* \left(\frac{1}{8} - \frac{\sqrt{\tau}}{3\sqrt{\pi}}\right). \tag{4.13}$$

This inequality is satisfied if

$$\tau \leq \tau^{(b)}_{\text{short}} = \frac{1}{4x_b^2}, \tag{4.14}$$

where x_b stands for a solution of the equation

$$x\,\mathrm{i}^2\mathrm{erfc}\,x + 2\,\mathrm{i}^3\,\mathrm{erfc}\,2x = \frac{O^*}{8}\left(x - \frac{4}{3\sqrt{\pi}}\right)$$

with respect to x.

Assuming, for example, $O^* = 0.01$ one may find ([15]):

$$\tau \leq \tau^{(b)}_{\text{short}} \cong 0.083 \tag{4.15}$$

([15]) For $O^* = 0.001$ (0.0001) one can find $\tau^{(b)}_{\text{short}} \cong 0.053$ (0.038).

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(cf. Ineq. (4.9)). Assuming, in addition $\kappa \doteq (10^{-7} - 10^{-4})\,\mathrm{m}^2/\mathrm{s}$ (where the first value in brackets refers to the worst temperature conductors and the second one – to the best ones), one may rewrite the criterion expressed by Ineq. (4.15) in the following dimensional form (using Eq. $(2.1)_1$) ([16]):

$$t \le t^{(b)}_{\mathrm{short}} \cong \begin{cases} 0.8\cdot(1-10^{-3})\,\mathrm{s}, & \text{for} \quad 2h = 10^{-3}\,\mathrm{m}, \\ 0.8\cdot(10^{2}-10^{-1})\,\mathrm{s}, & \text{for} \quad 2h = 10^{-2}\,\mathrm{m}. \end{cases}$$

Concerning the function U^u let us note – after substituting Eqs. $(3.4\mathrm{b})_1$ and $(3.5\mathrm{b})_1$ into Eq. $(3.1)_1$ and using Eq. (4.5) – that this function also represents the Leibniz series ([5]) (for $\tau \le \frac{9\pi}{64}$), and it can be written in the form:

$$\frac{U^u}{\dot{U}_{\max}(0)\,t} = \left[\frac{1}{3}A^2 + \left(1-\overline{r}^2\right)\right] - \left[\frac{8}{3}A^2 \mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} + \left(1-\overline{r}^2\right)\frac{8}{3\sqrt{\pi}}\sqrt{\tau}\right]$$
$$+ \left[\frac{8}{3}A^2 \mathrm{i}^2\mathrm{erfc}\,\frac{3}{2\sqrt{\tau}} + 8\left(1-\overline{r}^2\right)\left(\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} + 4\sqrt{\tau}\,\mathrm{i}^3\mathrm{erfc}\,\frac{1}{\sqrt{\tau}}\right)\right] - \ldots\,,$$

where $A = 2h/r_0$, and $\overline{r} = r/r_0$. The right-hand side of this formula can be truncated after the second term (with an accuracy to O^*), if

$$\frac{8}{3}A^2\,\mathrm{i}^2\mathrm{erfc}\,\frac{3}{2\sqrt{\tau}} + 8\left(1-\overline{r}^2\right)\left(\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} + 4\sqrt{\tau}\;\mathrm{i}^3\mathrm{erfc}\,\frac{1}{\sqrt{\tau}}\right)$$
$$\le O^*\left\{\left[\frac{1}{3}A^2 + \left(1-\overline{r}^2\right)\right] - \left[\frac{8}{3}A^2\,\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} + \left(1-\overline{r}^2\right)\frac{8}{3\sqrt{\pi}}\sqrt{\tau}\right]\right\}.$$

This inequality is satisfied if the following inequalities are fulfilled (sufficient conditions):

$$\mathrm{i}^2\mathrm{erfc}\,\frac{3}{2\sqrt{\tau}} + O^*\,\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} \le \frac{1}{8}O^*,$$
$$\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} + 4\sqrt{\tau}\,\mathrm{i}^3\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} \le O^*\left(\frac{1}{8} - \frac{\sqrt{\tau}}{3\sqrt{\pi}}\right).$$

The second inequality is identical with Ineq. (4.13). The first inequality is weaker than the second one.

Thus, if Ineq. (4.14) is satisfied, then the function U^u can be approximated by its first two terms (see formula for this function given above). Further approximation depends on neglecting the function $\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}}$. The maximum possible error of the latter approximation does not exceed O^*, if

$$\frac{8}{3}A^2\,\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} \le \frac{O^*}{1+O^*}\left[\frac{1}{3}A^2 + \left(1-\overline{r}^2\right)\left(1 - \frac{8\sqrt{\tau}}{3\sqrt{\pi}}\right)\right].$$

([16]) If $O^* = 0.001$ (0.0001), then the coefficient 0.8 before the brackets is replaced by 0.5 (0.4).

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Because the right-hand side of this inequality is a decreasing function of $\overline{\tau}$ (in the short-time regime), therefore it is fulfilled for each $\overline{\tau}$, if

$$\mathrm{i}^2\mathrm{erfc}\,\frac{1}{2\sqrt{\tau}} \leq \frac{1}{8}\,\frac{O^*}{1+O^*}\,.$$

Because the left-hand side of the latter inequality is an increasing function of τ, therefore it is sufficient to satisfy this inequality for $\tau = \tau^{(b)}_{\mathrm{short}}$. In fact, if Ineq. (4.14) is satisfied, then the inequality considered is satisfied too.

Thus, if the criterion expressed by Ineq. (4.15) is fulfilled, then the function U^u can be approximated in the same way as functions U_l, ε^u_l, and $D^u_l = 1/f^u_l$, as it is expressed by Eqs. (4.2)–(4.4).

Concerning the functions δ^u_l let us note – after substituting Eqs. $(3.4\mathrm{b})_2$ and $(3.5\mathrm{b})_2$ into Eq. (3.3) – that both these functions are positive-valued in the full time interval $(0 \leq \tau \leq \infty)$. The function δ^u is monotonically increasing from 0 (at $\tau = 0$) to αw_0 (at $\tau = \infty$), therefore its maximum value in the short-time regime is equal to $\delta^u(\tau^{(b)}_{\mathrm{short}})$. The function δ_l starts from 0 (at $\tau = 0$), approaches the maximum value about 0.062 at $\tau = \tau^{(b)}_{l,m} \cong 0.115$, and next is monotonically decreasing to 0 (at $\tau = \infty$). The criterion expressed by Ineq. (3.6) can be therefore written in the case under consideration in the same form as in case a (Ineq. (4.10)) with only the subscript (a) replaced by the subscript (b), $\tau^{(a)}_{\mathrm{short}}$ – by $\tau^{(b)}_{\mathrm{short}}$ (see Eqs. (4.14) and (4.15)), and $\tau^{(a)}_{l,m}$ – by $\tau^{(b)}_{l,m} \cong 0.115$. If $O^* = 0.01$ (and therefore $\tau^{(b)}_{\mathrm{short}} \cong 0.083$), then ([17]):

$$\begin{aligned} \delta^{u(b)} &\leq \delta^u(\tau^{(b)}_{\mathrm{short}}) \cong 0.23\alpha w_0 \ \ \leq 0.5O^*\,, \\ \delta^{(b)}_l &\leq \delta_l(\tau^{(b)}_{\mathrm{short}}) \cong 0.060\alpha w_0 \leq 0.5O^*\,. \end{aligned} \tag{4.16}$$

Under assumption $\alpha \doteq 10^{-5}$ 1/K, this inequalities read ([18])

$$w_0 \leq \begin{cases} 22\cdot 10^2\,\mathrm{K} & \text{for the perturbed (the upper) surface,} \\ 83\cdot 10^2\,\mathrm{K} & \text{for the opposite (the lower) surface.} \end{cases} \tag{4.17}$$

Using Eq. $(2.1)_4$ and assuming (in addition to the assumptions adopted above) $\varrho_0 c_p \doteq 5\cdot 10^6\ \mathrm{J/(m^3K)}$, one may rewrite the inequality given above as the following criterion for the disturbing heat flux density (equivalent to the disturbing heat source) ([19]):

$$2hq_0 \leq \begin{cases} C(10^6 - 10^9)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-3}\,\mathrm{m}, \\ C(10^5 - 10^8)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-2}\,\mathrm{m}, \end{cases}$$

([17]) If $\tau^{(b)}_{\mathrm{short}} \cong 0.053$ (0.038) ($O^* = 0.001$ (0.0001)), then the coefficient 0.23 is replaced by 0.16 (0.12), and the coefficient 0.060 – by 0.051 (0.043).

([18]) For $O^* = 0.001$ (0.0001) the coefficient 22 is replaced by 3.2 (0.42), and the coefficient 83 – by 9.8 (1.2).

([19]) If $O^* = 0.001$ (0.0001), then $C \cong 0.2$ (0.02) for the perturbed (the upper) surface, and $C \cong 0.5$ (0.06) for the opposite (the lower) one, respectively.

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where $C \cong 1$ for the perturbed (the upper) surface, and $C \cong 4$ for the opposite (the lower) one.

Concerning the restriction of the heat perturbation, which follows from the requirement for the temperature value not exceeding an assumed value T^*, the criterion has the same general form as in the previous case: $T^{(b)}(z = h) \leq T^*$. According to Eqs. $(2.1)_{2,3}$ and $(2.4\text{b})_1$, we have

$$T^{(b)}(z = h, t) = w_0\, \Theta^{(b)} \left(\zeta = \frac{1}{2}, \tau\right) = 2w_0\sqrt{\tau} \left[\frac{1}{\sqrt{\pi}} - 2\sum_{m=1}^{\infty} (-1)^{m+1} \operatorname{ierfc} \frac{m}{\sqrt{\tau}}\right].$$

This is an increasing function of time. Under the assumption $T^* = 100\,\text{K}$, the criterion analyzed may be written in the identical approximate form as in case a (Ineq. (4.12) for w_0 and the next inequality for $2hq_0$), with only $\tau^{(a)}_{\text{short}}$ replaced by $\tau^{(b)}_{\text{short}}$, the coefficient 2.7 – by 3.1 (for $\tau^{(b)}_{\text{short}} = 0.083$) ([20]), and the coefficient 0.1 – by 0.15 (for $\tau^{(b)}_{\text{short}} = 0.083$) ([21]). Conclusions on the role of this criterion as compared to the criteria obtained earlier for neglecting the functions δ are the same as in the previous case.

At the end of this subsection we note, that two remarks made at the end of the previous subsection concern also the case considered here.

4.3. Conclusions

Estimations given in the two previous subsections show that the short-time approximation seems to be realistic (except for very thin plates with the best temperature conductors) and offering simple interpretation of the time evolution of the properties of the mirror considered. Comparison of the two cases considered shows, that for sufficiently short time there are no significant differences between both the cases examined, i.e. – the energy losses through the lower surface can be neglected.

5. The long-time regime

For sufficiently long time ($\tau \geq \tau_{\text{long}}$) and for not very strong heat perturbation ($w_0 \leq w_{0,\text{long}}$), the characteristics of the thermal mirrors (Eqs. (3.1), (3.2), (3.7) and (3.8)) can be approximated (with an accuracy to $(1+O^*)^2 - 1 \cong 2O^*$ in both cases a and b, where O^* is an assumed small number) by the following formulae (cases a and b are distinguished by the superscript $(j) = (a), (b)$, respectively; cf. Eqs. (8.3) and (8.4) in Part I):

$$U^{(j)}_{\max} = U^{(j)}_{\max}(\infty)\, \phi^{(j)}_M(\tau), \tag{5.1}$$

([20]) Or by 3.8 (4.5) (for $\tau^{(b)}_{\text{short}} = 0.053$ (0.038)).

([21]) Or by 0.2 (0.2) (for $\tau^{(b)}_{\text{short}} = 0.053$ (0.038)).

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$$(5.2)\qquad \begin{aligned} U^{u(j)} &= U^{(j)}_{\max}(\infty)\left\{\left(\frac{2h}{r_0}\right)^2 \phi^{(j)}_N(\tau) + \left[1-\left(\frac{r}{r_0}\right)^2\right]\phi^{(j)}_M(\tau)\right\},\\ U^{(j)}_l &= -U^{(j)}_{\max}(\infty)\left[1-\left(\frac{r}{r_0}\right)^2\right]\phi^{(j)}_M(\tau), \end{aligned}$$

$$(5.3)\qquad \varepsilon^{u(j)}_l = \mp 2\arctan\left[\frac{2U^{(j)}_{\max}(\infty)}{r_0}\,\frac{r}{r_0}\phi^{(j)}_M(\tau)\right] \cong \mp 4\frac{U^{(j)}_{\max}(\infty)}{r_0}\,\frac{r}{r_0}\phi^{(j)}_M(\tau),$$

$$(5.4)\qquad D^{u(j)}_l = \frac{1}{f^{u(j)}_l} = \mp\frac{4}{r_0^2}U^{(j)}_{\max}(\infty)\phi^{(j)}_M(\tau),$$

where (see Eqs. $(2.1)_{1,4}$)

$$(5.5)\qquad U^{(a)}_{\max}(\infty) = \frac{1}{2}U^{(b)}_{\max}(\infty) = \frac{r_0^2}{8h}\alpha w_0 = \alpha\frac{r_0^2}{4}\,\frac{2hq_0}{\kappa\varrho_0 c_p} = \alpha\frac{\dot{Q}_{\text{tot}}}{4\pi\kappa\varrho_0 c_p},$$

$$(5.6\text{a})\qquad \phi^{(a)}_N = 4\tau, \qquad\qquad \phi^{(a)}_M = 1-\frac{96}{\pi^4}\exp[-\pi^2\tau],$$

$$(5.6\text{b})\qquad \phi^{(b)}_N = 1-\frac{32}{\pi^3}\exp[-\frac{1}{4}\pi^2\tau], \qquad \phi^{(b)}_M = 1-\frac{96}{\pi^4}(4-\pi)\exp\left[-\frac{1}{4}\pi^2\tau\right],$$

and the approximation in Eq. (5.3) (the paraxial optics approximation) holds, if

$$\frac{2U^{(j)}_{\max}(\infty)}{r_0}\,\frac{r}{r_0}\phi^{(j)}_M(\tau) \le \sqrt{\frac{3O^*}{1+O^*}} \cong \sqrt{3O^*}\,.$$

The approximate formulae given above are different in each case (a and b). Criteria of applicability of the long-time approximation are also different in each case. These criteria will be therefore deduced and specified separately for each case.

5.1. Criteria of applicability of the long-time approximation – Case a

The analysis will be performed according to the same program as in Case a for the short-time approximation (as it was mentioned at the beginning of Subsec. 4.1).

Thus, at the beginning Ineq. (3.6) is assumed to be satisfied.

Concerning the function M_T let us note that for sufficiently long time, the sum in the brackets in Eq. $(3.5\text{a})_2$ can be truncated after the second term. For this purpose it is sufficient to require:

• the third term of the whole sum to be much smaller than the sum of the first and the second terms in the following sense:

$$\frac{96}{\pi^4}\,\frac{1}{81}\exp[-9\pi^2\tau] \le \frac{0.9O^*}{1+O^*}\left(1-\frac{96}{\pi^4}\exp[-\pi^2\tau]\right),$$

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• and the n-th term of the sum ($(k+1)$-th term of the series), $n \geq 3$ ($k \geq 2$), to be not larger than 0.1 of the $(n-1)$-th term of the sum (the k-th term of the series):

$$\exp[-8\pi^2 k\tau] \leq 0.1 \left(\frac{2k+1}{2k-1}\right)^4 .$$

The first inequality is satisfied for

$$\tau \geq \tau_{\text{long}}^{(a)} = -\frac{1}{\pi^2} \ln x_a , \tag{5.7}$$

where x_a is the suitable solution of the equation $x^9 + 81\dfrac{0.9O^*}{1+O^*}x - \dfrac{81}{96}\pi^4 \dfrac{0.9O^*}{1+O^*} = 0$.

The second inequality is the strongest one for $k = 4$, and then it is much weaker than the former one for $O^* \leq 0.01$.

Thus, assuming (as previously) $O^* = 0.01$ one can obtain ([22]):

$$\tau \geq \tau_{\text{long}}^{(a)} \cong 0.021 ; \tag{5.8}$$

assuming also (as previously) $\kappa \doteq (10^{-7}$–$10^{-4})\ \text{m}^2/\text{s}$, one may rewrite the criterion expressed by Ineq. (5.8) in the following dimensional form (using Eq. $(2.1)_1$) ([23]):

$$t \geq t_{\text{long}}^{(a)} \cong \begin{cases} 0.2\cdot(1 - 10^{-3})\,\text{s}, & \text{for} \quad 2h = 10^{-3}\,\text{m}, \\ 0.2\cdot(10^2 - 10^{-1})\,\text{s}, & \text{for} \quad 2h = 10^{-2}\,\text{m}. \end{cases}$$

Concerning the function U^u let us note – after substituting Eq. (3.4a) and $(3.5a)_2$ into Eq. $(3.1)_1$ and using Eq. (5.5) – that applying analogous criteria for truncation the function U^u after the third term to the criteria used above for truncation of the function $M_T^{(a)}$, we obtain also two inequalities. The first one is always satisfied for $r = r_0$, and it is weaker for $r < r_0$ than the suitable inequality for truncation the function $M_T^{(a)}$. The second inequality is identical with that in the case of the function $M_T^{(a)}$. Thus, if the criteria for truncating the function $M_T^{(a)}$ after the second term are satisfied, than also the suitable sum in the expression for the function U^u can be truncated after the third term.

Concerning the functions δ_l^u in the case under consideration let us note that – according to Eqs. (3.3), (3.4a) and (3.5a) – the function δ^u monotonically increases with time from 0 at $\tau = 0$ to infinity as $\tau \to \infty$, being asymptotically limited as follows: $\delta^u(\tau) \leq \alpha w_0 \left((1/4) + \tau\right)$ – for arbitrary time. Because of exponential dependence on time, it seems to be reasonable to limit the long-time interval by the values: $\tau_{\text{long}}^{(a)} - 10\tau_{\text{long}}^{(a)}$. Then Ineq. (3.6) can be rewritten in the form $\delta^{u(a)}(\tau) \leq \delta^{u(a)}(10\,\tau_{\text{long}}^{(a)}) \leq \dfrac{1}{2} O^*$.

([22]) For $O^* = 0.001$ (0.0001) $\tau_{\text{long}}^{(a)} \cong 0.041$ (0.064).

([23]) For $O^* = 0.001$ (0.0001) the coefficient 0.2 before the brackets is replaced by 0.4 (0.6).

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The function δ_l initially increases from 0 at $\tau = 0$ to the maximum value of about $0.059\alpha w_0$ at $\tau = \tau_{l,m}^{(a)} \cong 0.090$, and next monotonically decreases to minus infinity as $\tau \to \infty$ (asymptotically as $\alpha w_0\,((1/4) - \tau)$). Inequality (3.6) can be therefore rewritten in the long-time interval bounded as above in the form (for both functions $\delta_l^{u(a)}$ and $\delta_l^{(a)}$):

$$
\begin{aligned}
&\delta^{u(a)} \le \delta^{u(a)}(10\tau_{\mathrm{long}}^{(a)}) \le 0.5O^* ,\\
&|\delta_l^{(a)}| \le |\delta_l(\tau^*)| \qquad\quad \le 0.5O^* ,
\end{aligned}
\tag{5.9}
$$

where $\tau^* = \tau_{l,m}^{(a)} \cong 0.090$, if $\delta_l(\tau_{l,m}^{(a)}) \cong 0.059\alpha w_0 > |\delta_l^{(a)}(10\tau_{\mathrm{long}}^{(a)})|$, or $\tau^* = 10\tau_{\mathrm{long}}^{(a)}$ in the opposite case. If $O^* = 0.01$ (and therefore $\tau_{\mathrm{long}}^{(a)} \cong 0.021$), then approximately (cf. Ineq. (4.11)) ([24])

$$
\begin{aligned}
&\delta^{u(a)} \le 0.44\alpha w_0 \;\; \le 0.5O^* ,\\
&|\delta_l^{(a)}| \le 0.059\alpha w_0 \le 0.5O^* .
\end{aligned}
\tag{5.10}
$$

Under the assumption $\alpha \doteq 10^{-5}\,1/\mathrm{K}$ this inequality reads ([25])

$$
w_0 \le \begin{cases} 11\cdot 10^2\mathrm{K} \text{ (for the perturbed (the upper) surface)},\\ 85\cdot 10^2\mathrm{K} \text{ (for the opposite (the lower) surface)}.\end{cases}
$$

Using Eq. $(2.1)_4$ and assuming: $\varrho_0 c_p \doteq 5\cdot 10^6\ \mathrm{J/(m^3K)}$, one may rewrite this inequality as the following criterion for the disturbing heat flux density (equivalent to the disturbing heat source):

$$
2hq_0 \le \begin{cases} D(10^6 - 10^9)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-3}\,\mathrm{m},\\ D(10^5 - 10^8)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-2}\,\mathrm{m},\end{cases}
$$

where $D \cong 0.6$ for the perturbed (the upper) surface, and $D \cong 4$ for the opposite (the lower) one ([26]).

As far as the limitation of the heat perturbation in the long-time regime is concerned, two additional conditions should be taken into account. The first such condition follows from the requirement, that the temperature cannot exceed a certain value T^*, as it was mentioned in Subsecs. 4.1 and 4.2. The second condition follows from the requirement that τ cannot be too large, thus allowing

([24]) If $O^* = 0.001$ (0.0001) ($\tau_{\mathrm{long}}^{(a)} \cong 0.041$ (0.064)), then the coefficient 0.44 is replaced by 0.66 (0.89), and the coefficient 0.059 – by 0.17 (0.39).

([25]) For $O^* = 0.001$ (0.0001) the coefficient 11 is replaced by 0.76 (0.056), and the coefficient 85 – by 3.0 (0.13).

([26]) If $O^* = 0.001$ (0.0001), then $D \cong 0.04$ (0.003) for the perturbed (the upper) surface, and $D \cong 0.15$ (0.006) for the opposite (the lower) one, respectively.

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to neglect thermal losses through the surfaces (first of all – the radiation losses), as it was mentioned in Part I.

The discussion of the first condition, performed in an analogous way as it was done in Subsecs. 4.1 and 4.2 taking into account the previously adopted limits of the long-time regime interval, leads to following approximate criterion:

$$w_0 \leq \frac{T^*}{\frac{1}{4} + 10\tau_{\text{long}}^{(a)} - \frac{2}{\pi^2}\exp[-\pi^2 10\tau_{\text{long}}^{(a)}]} \cong \frac{T^*}{\frac{1}{4} + 10\tau_{\text{long}}^{(a)}} \cong 2.2\cdot 10^2\,\text{K}, \tag{5.11}$$

where T^* was assumed to be equal to 100 K, and $\tau_{\text{long}}^{(a)}$ – to be equal to 0.021 ([27]).

Using Eq. $(2.1)_4$ one may rewrite this inequality as the following criterion for the disturbing heat flux density (equivalent to the disturbing heat source) ([28]):

$$2hq_0 \leq \begin{cases} 0.1\cdot(10^6 - 10^9)\,\text{W/m}^2, & \text{for} \quad 2h = 10^{-3}\,\text{m}, \\ 0.1\cdot(10^5 - 10^8)\,\text{W/m}^2, & \text{for} \quad 2h = 10^{-2}\,\text{m}. \end{cases}$$

Comparing these results with inequalities for w_0 and q_0 obtained earlier as a criterion for neglecting the functions δ in the long-time regime, one may see that the former inequalities are weaker than the latter ones for $O^* = 0.01$; if therefore the temperature criterion of linearization is satisfied, then the functions δ may be neglected in the suitable formulae in the long-time regime. The reverse situation takes place for $O^* = 0.0001$.

The second condition mentioned above concerns the assumption concerning the adiabatic insulation of the plate. In fact, the plate loses its energy at least by thermal radiation through the perturbed and the opposite surfaces. These radiation losses can be neglected, if τ is not too large. The discussion of this problem, performed in an analogous way as it was done in Part I (Sec. 8) ([29]),

([27]) If $\tau_{\text{long}}^{(a)}$ is put to be equal to 0.041 (0.064), then the coefficient 2.2 is replaced by 1.5 (1.1).

([28]) If $\tau_{\text{long}}^{(a)}$ is put to be equal to 0.041 (0.064), then the coefficient 0.1 before the brackets is replaced by 0.07 (0.05).

([29]) For the problem:

$$\frac{\partial\Theta}{\partial\tau} = \frac{\partial^2\Theta}{\partial\zeta^2}, \qquad \Theta(\tau=0)=0, \qquad \frac{\partial\Theta}{\partial\zeta}\left(\zeta=\frac{1}{2}\right) = H(\tau-0) - \beta\Theta\left(\zeta=\frac{1}{2}\right),$$

$$\frac{\partial\Theta}{\partial\zeta}\left(\zeta=-\frac{1}{2}\right) = \beta\Theta\left(\zeta=-\frac{1}{2}\right),$$

where

$$\beta = \frac{2h}{\kappa}\,\frac{4b\sigma_{SB}T_0^3}{\varrho_0 c_p}$$

stands for dimensionless coefficient of radiation losses as obtained from the linearized Stefan–Boltzmann law (b stands here for a correction factor for a real body as compared with the perfectly black one, σ_{SB} – for the Stefan–Boltzmann constant, and T_0 – for the absolute initial temperature (before the perturbation)).

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leads to the following criterion of neglecting the radiation losses through the surfaces [30]:

$$\tau \leq \tau_{\max} := O^* \, \tau_{\text{rad}} := \frac{O^*}{4\beta} = O^* \frac{\kappa}{2h} \frac{\varrho_0 c_p}{16 b \sigma_{SB} T_0^3},$$

where β and the remaining symbols used here are defined in footnote 29. Assuming $O^* = 0.01$, $\kappa \doteq (10^{-7} - 10^{-4})\,\text{m/s}^2$, $\varrho_0 c_p \doteq 5 \cdot 10^6\,\text{J/(m}^3\text{K)}$, and

- $b = 0.1$,
- $\sigma_{SB} \cong 5.67 \cdot 10^{-8}\,\text{J/(m}^2\text{sK}^4)$,
- $T_0 = 300\,\text{K}$,

we have (in dimensionless and in dimensional forms) [31]:

(5.12)
$$\tau \leq \tau_{\max} \doteq \begin{cases} 2 \cdot (1 - 10^3), & \text{for} \quad 2h = 10^{-3}\,\text{m}, \\ 2 \cdot (10^{-1} - 10^2), & \text{for} \quad 2h = 10^{-2}\,\text{m}, \end{cases}$$
$$t \leq t_{\max} \doteq \begin{cases} 2 \cdot 10\,\text{s}, & \text{for} \quad 2h = 10^{-3}\,\text{m}, \\ 2 \cdot 10^2\,\text{s}, & \text{for} \quad 2h = 10^{-2}\,\text{m}. \end{cases}$$

Comparing this result and Ineq. (5.8) one can see, that the criterion expressed by Ineq. (5.12) can significantly restrict the applicability of the long-time approximation in the case under consideration, especially if high accuracy of the approximation is required for thin plates and bad temperature conductors.

At the end of this subsection we note, that two remarks done at the end of Subsec. 4.1 concern the case considered here too, although in the long-time regime they are not so important as previously.

5.2. Criteria of applicability of the long-time approximation – Case b

The analysis will be performed according to the same programe as in Case a for the short-time approximation (as it was mentioned at the beginning of Subsec. 4.1).

Thus, at the beginning Ineq. (3.6) is assumed to be satisfied.

In connection with the function M_T let us note that, for sufficiently long time, the sum in the brackets in Eq. $(3.5\text{b})_2$ can be truncated after the second term. In fact, after merging the unity and the first term of the series one obtains a new series, which can be treated as the Leibniz series. Then, the sum under consideration can be truncated after the second term with an error not exceeding O^*, if [5]

(5.13)
$$\tau \geq \tau_{\text{long}}^{(b)} = -\frac{4}{\pi^2} \ln x_b,$$

[30] Note that in Part I there is a numerical mistake: number 8 in the formulae for τ_{rad}, $\tau_{\max}$ and $t_{\max}$ should be replaced by the number 16, and number 4 in Eqs. (8.5) – by the number 2.

[31] For $O^* = 0.001$ (0.0001) the coefficient 2 is replaced by 0.2 (0.02).

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where x_b is the suitable solution of the equation

$$\frac{1+O^*}{O^*}x^9 + 81\frac{4-\pi}{3\pi+4}x - \frac{81}{3\pi+4}\frac{\pi^4}{96} = 0.$$

Assuming (as previously) $O^* = 0.01$ one can obtain ([32])

$$\tau \geq \tau_{\text{long}}^{(b)} \cong 0.16\,; \tag{5.14}$$

assuming also (as previously) $\kappa \doteq (10^{-7} - 10^{-4})\,\text{m}^2/\text{s}$, one may rewrite the criterion exepressed by Ineq. (5.14) in the following dimensional form (using Eq. (2.1)) ([33]):

$$t \geq t_{\text{long}}^{(b)} \cong \begin{cases} 2\cdot(1-10^{-3})\,\text{s}, & \text{for} \quad 2h = 10^{-3}\,\text{m}, \\ 2\cdot(10^{2}-10^{-1})\,\text{s}, & \text{for} \quad 2h = 10^{-2}\,\text{m}. \end{cases}$$

Concerning the function U^u let us note, that the sum in the brackets in Eq. $(3.4\text{b})_2$ can be written in the form:

$$S_N = 1 - a_1 + a_2 - \dots\,,$$

where a_k stands for the absolute value of the k-th term of the series in these brackets, and the sum in Eq. $(3.5\text{b})_2$ – in the form:

$$S_M = [1-b_1] - b_2 + \dots\,,$$

where b_k stands for the absolute value of the k-th term of the series in these brackets, and the term $[1 - b_1]$ is treated as the first term of the series S_M. The function $U^{u(b)}$ can therefore be written in the form (after neglecting the function δ):

$$U^{u(b)} = U_{\max}^{(b)}(\infty)S, \qquad S = A^2 S_N + (1-\overline{r}^2)S_M\,,$$

where $A = 2h/r_0$, and $\overline{r} = r/r_0$.

Both series S_N and S_M represent the Leibniz series ([5]) in the long-time regime specified above. They are limited as follows:

$$\begin{gathered} 1 - a_1 \leq S_N \leq 1 - a_1 + a_2\,, \\ [1-b_1] - b_2 \leq S_M \leq [1-b_1], \end{gathered}$$

and therefore the series S may be estimated as follows:

$$A^2(1-a_1) + (1-\overline{r}^2)([1-b_1]-b_2) \leq S \leq A^2(1-a_1+a_2) + (1-\overline{r}^2)[1-b_1].$$

([32]) If $O^* = 0.001$ (0.0001), then $\tau_{\text{long}}^{(b)} \cong 0.26$ (0.35).

([33]) If $O^* = 0.001$ (0.0001), then the coefficient 2 before the brackets is replaced by 3 (4).

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The series S_M is approximated (with an accuracy to O^*) by its first term $S_M \cong [1-b_1]$, which specifies the long-time regime and requires b_2 to satisfy the condition:

$$b_2 \leq \frac{O^*}{1+O^*}[1-b_1].$$

The series S_N is approximated (with an accuracy to O_1^*) by its first two terms $S_N \cong 1-a_1$, which requires a_2 to satisfy the condition $a_2 \leq O_1^*(1-a_1)$; one may verify that $O_1^* < O^*$ in the long-time regime specified above, therefore the condition for a_2 can be written in the form:

$$a_2 < O^*(1-a_1).$$

The series S can be approximated as follows:

$$S \cong A^2(1-a_1) + (1-\overline{r}^2)(1-b_1)$$

with the maximum possible (relative) error not exceeding

$$O^*_{\text{tot}} = \frac{1}{S}[A^2 a_2 + (1-\overline{r}^2)\, b_2];$$

one can prove, using inequalities for S, a_2, b_2 given above, that

$$O^*_{\text{tot}} < O^*$$

in the long-time regime.

Thus, if the functions: $U_l^{(b)}$, $\varepsilon_l^{u(b)}$, $D_l^{u(b)}$, $f_l^{u(b)}$ can be approximated (with an accuracy to O^*) as it is expressed by Eqs. (5.1), $(5.2)_2$, (5.3), (5.4), then also the function $U^{u(b)}$ can be approximated as it is expressed by Eq. $(5.2)_1$ (with an accuracy not worse than O^*).

In turn, because of the time evolution of the functions δ, as it was mentioned in Subsec. 4.2, Ineq. (3.6) in the case examined in the long-time regime specified above may be written in the form analogous to Ineq. (4.10), with the superscript (a) replaced by (b), $\tau^{(a)}_{\text{short}}$ – by $10\,\tau^{(b)}_{\text{long}}$, and τ^* – by $\tau^{(b)}_{\text{long}}$. If $O^* = 0.01$ (and therefore $\tau^{(b)}_{\text{long}} \cong 0.16$), then [34]

$$\begin{aligned} \delta^{u(b)} &\leq \delta^u(10\tau^{(b)}_{\text{long}}) \cong 0.98\alpha w_0 \leq 0.5 O^*\,, \\ \delta_l^{(b)} &\leq \delta_l(\tau^{(b)}_{\text{long}}) \cong 0.059\alpha w_0 \leq 0.5 O^*\,. \end{aligned} \tag{5.15}$$

Under the assumptions $O^* = 0.01$, and $\alpha \doteq 10^{-5}\,1/\text{K}$, these inequalities read [35]:

$$w_0 \leq \begin{cases} 5.0\cdot 10^2\,\text{K}, & \text{for the perturbed (the upper) surface,} \\ 84\cdot 10^2\,\text{K}, & \text{for the opposite (the lower) surface.} \end{cases}$$

[34] For $O^* = 0.001$ (0.0001) ($\tau^{(b)}_{\text{long}} \cong 0.26$ (0.35)) the coefficient 0.98 is replaced by 1.0 (1.0), and the coefficient 0.059 – by 0.049 (0.039).

[35] For $O^* = 0.001$ (0.0001) the coefficient 5.0 is replaced by 0.50 (0.050), and the coefficient 84 – by 10 (1.3).

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Using Eq. $(2.1)_4$ and assuming $\varrho_0 c_p \doteq 5\cdot 10^6\,\mathrm{J/(m^3K)}$, one may rewrite the inequality given above as the following criterion for the disturbing heat flux density (equivalent to the disturbing heat source):

$$2hq_0 \le \begin{cases} E(10^6 - 10^9)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-3}\,\mathrm{m}, \\ E(10^5 - 10^8)\,\mathrm{W/m^2}, & \text{for} \quad 2h = 10^{-2}\,\mathrm{m}, \end{cases}$$

where $E \cong 0.25$ for the perturbed (the upper) surface, and $E \cong 4$ for the opposite (the lower) one ([36]).

As far as the limitation of the heat perturbation is concerned in the long-time regime, two additional conditions mentioned in Subsec. 5.1 should be taken into account in the case under consideration. The first such condition (following from the requirement that the temperature can not exceed a certain value T^*) examined in an analogous way as it was done in Subsecs. 4.1, 4.2 and 5.1 along with the fact that the temperature of the perturbed (the upper) surface is a limited function of time (as it is seen in Eq. $(2.4\mathrm{b})_2$ for $\zeta = 1/2$ ([37])), leads to the following approximate criterion:

$$w_0 \le \frac{T^*}{1 - \dfrac{8}{\pi^2}\exp\left[-\dfrac{\pi^2}{4}10\tau_{\text{long}}^{(b)}\right]} \cong 1.0\cdot 10^2\,\mathrm{K}, \tag{5.16}$$

where T^* was assumed to be equal to 100 K, and $\tau_{\text{long}}^{(b)}$ to be equal to 0.16 (or more). Using Eq. $(2.1)_4$ one may rewrite this inequality as the criterion for the disturbing heat flux density (equivalent to the disturbing heat source), obtaining the same result as expressed by the inequality following Ineq. (5.11) with the coefficient 0.1 replaced by 0.05. Comparison of these results with inequalities for w_0 and q_0 obtained earlier as the criterion for neglecting the functions δ in the long-time regime, leads to the same conclusions as those in the previous case (see Subsec. 5.1).

The second condition concerning the limitation of the heat perturbation in the long-time regime represents the criterion for neglecting the radiation losses in the case considered. The discussion of this problem, performed in an analogous way as it was done in Part I (Sec. 8) ([38]), leads to the criterion which is weaker

([36]) If $O^* = 0.001$ (0.0001), then $E \cong 0.025$ (0.0025) or $E \cong 0.5$ (0.06) for the perturbed (the upper) surface or for the opposite (the lower) one, respectively.

([37]) For the value of the number series occurring here – see [2].

([38]) For the problem:

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta}{\partial \zeta^2}, \qquad \Theta(\tau = 0) = 0,$$

$$\frac{\partial \Theta}{\partial \zeta}\left(\zeta = \frac{1}{2}\right) = H(\tau - 0) - \beta\,\Theta\left(\zeta = \frac{1}{2}\right), \qquad \Theta\left(\zeta = -\frac{1}{2}\right) = 0,$$

where β is explained in footnote 29.

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(by the factor of 2) than the analogous criterion in case a (i.e. the numbers 4 and 16 in the denominators in the inequality precedeing Ineqs. (5.12) are replaced by the numbers 2 and 8, respectively, and the coefficient 2 in Ineqs. (5.12) – by 4). The conclusion concerning the role of this criterion in limiting the applicablity of the long-time approximation is also the same as that applied to the previous case.

At the end of this subsection we note, that two remarks made at the end of Subsec. 4.1 concern the case considered here too, although in the long-time regime they are not so important as in the case of the short-time regime.

5.3. Conclusions

Estimates given in the two previous subsections show that the long-time regime seems to be a realistic and useful (except the cases of thin plates with bad temperature conductors, when higher accuracy is required, especially in Case b – see and compare estimates of $\tau_{\max}$ in Cases a and b). It starts relatively quickly (especially in Case a). The function $U_{\max}$ increases significantly with time (at least twice when τ increases from τ_{long} to $10\,\tau_{\text{long}}$ – in both the cases). By comparing Ineqs. (4.9) and (5.8) one may see that in Case a for $O^* = 0.01$ and 0.001, both regimes – the short- and the long-time ones – cover the full time range from $\tau_{\min}$ to $\tau_{\max}$; for $O^* = 0.0001$ the gap between the two regimes is relatively small. Comparison of Ineqs. (4.15) and (5.14) shows that in Case b situation is not so comfortable – the gap between the two regimes is quite large (the smaller O^*, the larger the gap).

6. Estimates for possible experiments

The thermal mirror considered may be experimentally studied by investigating at least one of the funtions $F = \{U, \varepsilon, D = 1/f\}$. Each of these functions can be experimentally investigated and interpreted using the theoretical scheme presented, if some conditions are fulfilled. Almost all such general conditions were mentioned in the previous sections. Here the last such condition will be noticed and shortly discussed. This condition is rather obvious: each of the functions F can be observable in a given time τ and at least on an assumed level $F^* = \{U^*, \epsilon^*, D^* = 1/f^*\}$, i.e. $|F| \geq F^*$, if the heat perturbation is sufficiently strong.

The observability conditions for the functions F will be examined in terms of the quantities (see Eqs. (3.1), (3.7), (3.8), (2.1) and (3.2) with Eqs. (3.5)): $U_{\max}$ (the functions U), $4U_{\max}/r_0$ (the functions ε), and $4U_{\max}/r_0^2$ (the functions $D = 1/f$). The heat perturbation level will be characterized alternatively by one of the quantities $G^F = \{(w_0)^F, (2hq_0)^F, (\dot{Q}_{\text{tot}})^F, (Q_{\text{tot}})^F)\}$ for each of the functions F, where: $2hq_0 = \kappa \varrho_0 c_p/(2h)$ (see Eq. $(2.1)_4$) stands for the perturbing heat flux density (equivalent to the perturbing heat source) applied to the perturbed

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surface, $\dot{Q}_{\text{tot}} = \pi r_0^2 2hq_0$ (see Eq. (4.5)) – for the total perturbing power applied to the perturbed surface, and $Q_{\text{tot}} = \dot{Q}_{\text{tot}}\, t$ – for the total perturbing energy applied to the perturbed surface during the time $t = \tau\,(2h)^2/\kappa$.

The observability conditions considered will be formulated separately in the short- and long-time regimes, and – also separately – in Cases a and b. Thus, using Eqs. (4.1) with Eq. (4.6) and Eqs. (5.1) with Eqs. $(5.6)_2$, one can write down the inequalities: $|F| \geq F^*$ in the form of 12 inequalities (3 functions F in 2 time regimes in two cases) in 4 alternative versions (for 4 quantities G). All these 48 inequalities can be written in the following form:

$$G^F = \begin{cases} \left.\begin{matrix} (w_0)^F \\ (2hq_0)^F \\ (\dot{Q}_{\text{tot}})^F \end{matrix}\right\} \geq F^*\psi(\tau)S^F_{G'}\,, \\ (Q_{\text{tot}})^F \qquad \geq F^*\psi(\tau)S^F_{Q_{\text{tot}}}\tau, \end{cases} \tag{6.1}$$

where

$$\psi(\tau) = \begin{cases} [\tau\varphi(\tau)]^{-1} \quad \text{(in Cases a and b)} \ \text{– in the short-time regime,} \\ \left.\begin{matrix} 12\left[\phi_M^{(a)}(\tau)\right]^{-1} & \text{(in Case a)} \\ 6\left[\phi_M^{(b)}(\tau)\right]^{-1} & \text{(in Case b)} \end{matrix}\right\} \text{– in the long-time regime} \end{cases} \tag{6.2}$$

(the functions: $\varphi(\tau)$ and $\phi_M^{(j)}(\tau)$ are defined by Eqs. (4.6) and $(5.6)_2$, respectively), and

$$\left\{ \begin{aligned} S^U_{w_0} &= \frac{2h}{3\alpha r_0^2}\,, \\ S^U_{2hq_0} &= \frac{\kappa\varrho_0 c_p}{3\alpha r_0^2}\,, \\ S^U_{\dot{Q}_{\text{tot}}} &= \frac{\pi\kappa\varrho_0 c_p}{3\alpha}\,, \\ S^U_{Q_{\text{tot}}} &= \frac{(2h)^2\pi\varrho_0 c_p}{3\alpha}\,, \end{aligned} \qquad S^\varepsilon_G = \frac{r_0}{4}S^U_G\,, \qquad S^{1/f}_G = \frac{r_0^2}{4}S^U_G \right\}. \tag{6.3}$$

For the numerical exemplification of the criteria expressed by Ineqs. (6.1), it is assumed that

$$\tau = \tau_{\text{short}}\,.$$

This assumption gives sufficient conditions for $|F| \geq F^*$ at the end of the short-time regime and at least in a part of the long-time regime in Case a or in the

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whole this regime in Case b (39). Now, assuming (as previously): $\alpha \doteq 10^{-5}$ 1/K; $\kappa \doteq (10^{-7} - 10^{-4})\,\mathrm{m^2/s}$, where the first value in the brackets concerns the worst and the second one – the best temperature conductors; $\varrho_0 c_p \doteq 5\cdot10^6\,\mathrm{J/(m^3K)}$; $r_0 \doteq 10\cdot 2h$; and

- $U^* \doteq 10^{-6}$ m;
- $\varepsilon^* \doteq 10^{-4}$ rad;
- $f^* \doteq 40$ m;
- $\tau^{(a)}_{\mathrm{short}} = 0.11$, $\tau^{(b)}_{\mathrm{short}} = 0.083$ (which corresponds to $O^* \cong 0.01$ (39));

one may obtain the minimum perturbation power (or energy) as it is given in the table in Case a (the minimum values of w_0, $2hq_0$ and $\dot{Q}_{\mathrm{tot}}$ are by a dozen percent larger, and that value of Q_{tot} is by a dozen percent smaller in Case b) (40).

	for $U_{\max} \geq U^*$	for $\dfrac{4U_{\max}}{r_0} \geq \epsilon^*$	for $\dfrac{4U^{\max}}{r_0^2} \geq \dfrac{1}{f^*}$	for $2h \doteq$
$\dfrac{w_0}{\mathrm{K}} \geq \dfrac{w_{0,\min}}{\mathrm{K}} \doteq$	$\begin{cases} 6 \\ 0.6 \end{cases}$	$\begin{cases} 1.5 \\ 1.5 \end{cases}$	$\begin{cases} 4 \\ 40 \end{cases}$	10^{-3} m 10^{-2} m
$\dfrac{2hqo}{\mathrm{W/m^2}} \geq \dfrac{2hq_{0,\min}}{\mathrm{W/m^2}} \doteq$	$\begin{cases} 3\cdot(10^3 - 10^6) \\ 3\cdot(10 - 10^4) \end{cases}$	$\begin{cases} 0.8\cdot(10^3 - 10^6) \\ 8\cdot(10 - 10^4) \end{cases}$	$\begin{cases} 2\cdot(10^3 - 10^6) \\ 200\cdot(10 - 10^4) \end{cases}$	10^{-3} m 10^{-2} m
$\dfrac{\dot{Q}_{\mathrm{tot}}}{\mathrm{W}} \geq \dfrac{\dot{Q}_{\mathrm{tot,min}}}{\mathrm{W}} \doteq$	$\begin{cases} 1\cdot(1 - 10^3) \\ 1\cdot(1 - 10^3) \end{cases}$	$\begin{cases} 0.25\cdot(1 - 10^3) \\ 2.5\cdot(1 - 10^3) \end{cases}$	$\begin{cases} 0.6\cdot(1 - 10^3) \\ 60\cdot(1 - 10^3) \end{cases}$	10^{-3} m 10^{-2} m
$\dfrac{Q_{\mathrm{tot}}}{\mathrm{J}} \geq \dfrac{Q_{\mathrm{tot,min}}}{\mathrm{J}} \doteq$	$\begin{cases} 1 \\ 10^2 \end{cases}$	$\begin{cases} 0.25 \\ 2.5\cdot10^2 \end{cases}$	$\begin{cases} 0.6 \\ 60\cdot10^2 \end{cases}$	10^{-3} m 10^{-2} m

Comparison of the values of $Q_{\mathrm{tot,min}}$ given above with analogous values given in Part I (Eqs. $(9.3)_1$, $(9.4)_1$, $(9.5)_1$) shows, that the observability conditions (in terms of the total perturbation energy) in the cases considered here are nearly the same as those in the case considered in Part I. However, the continuous heat flux perturbation may be more convenient for experimental organization than the pulsed one.

(39) Note that: $U_{\max}$ is a monotonically increasing function (and $\psi(\tau)$ is a decreasing function) of time; and τ_{short} decreases and τ_{long} inreases as the quantity O^* decreases (see also the remark on the relation between both the regimes given in Subsec. 5.3).

(40) Let us note that $w_{0,\min}$, $2hq_{0,\min}$ and $\dot{Q}_{\mathrm{tot,min}}$ decrease with τ, increase as O^* decreases in the short-time regime and decrease as O^* decreases in the long-time regime, and $Q_{\mathrm{tot,min}}$ – inversely. The observability conditions in terms of $w_{0,\min}$, $2hq_{0,\min}$ and $\dot{Q}_{\mathrm{tot,min}}$ become therefore stronger (weaker), and those in terms of $Q_{\mathrm{tot,min}}$ become weaker (stronger) as τ decreases (increases, resp.).

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7. Possible applications to determination of the temperature conductivity (and the surface losses coefficients)

As it is seen from the suitable formulae given above (after returning back to dimensional time $t = \tau\,(2h)^2/\kappa$), the time evolution of the thermal mirror depends, among others, on the temperature conductivity κ of the material. Measuring suitable properties of the mirror it is therefore possible to determine κ. However, as it is seen from the formulae mentioned, such a procedure performed in an arbitrary conditions may require some additional information (which should be known or measured), and may prove to be complicated for interpretation.

The problem simplifies in the short-time and the long-time regimes. In fact, as it follows from Eqs. (4.2), (4.3), (4.4) and (4.6), in the short-time regime the quantities $[U^u(r=0) - U^u(r)]/t$, U_l/t, $[tan(\varepsilon_l^u/2)]/t$ and $D_l^u/t = 1/(f_l^u\,t)$, as referred to their values at $t = 0$ (which may be determined by an extrapolation of the suitable experimental data to $t = 0$) are linear functions of $\sqrt{t}$ with the coefficient (at $\sqrt{t}$) equal to $4\sqrt{\kappa}/(2h\sqrt{\pi})$. Measuring the evolution of these quantities one may therefore determine this coefficient and, knowing it and the plate thickness $2h$ of the plate – find the values of κ of a given material.

Analogously, as it follows from Eqs. (5.2), (5.3), (5.4) and (5.6), logarithms of the absolute values of the time derivatives of the quantities $U^u(r=0) - U^u(r)$, U_l, $tan(\varepsilon_l^u/2)$ and $D_l^u = 1/f_l^u$ in the long-time regime are linear functions of t with the coefficient (at t) equal to $\pi^2\kappa/(2h)^2$ (in Case a) or $\pi^2\kappa/[4(2h)^2]$ (in Case b). Measuring the evolution of these quantities one may therefore determine this coefficient and, knowing it and the plate thickness $2h$ – determine κ of a given material.

Additionally let us briefly note that one may think also on applying the thermal mirror considered to experimental determination of the surface losses, if the temperature conductivity of a given material is known. In such a case the thermal problem is formulated as follows:

$$\frac{\partial\Theta}{\partial\tau} = \frac{\partial^2\Theta}{\partial\zeta^2}, \qquad \Theta(\tau=0), \qquad \frac{\partial\Theta}{\partial\zeta}\left(\zeta = \frac{1}{2}\right) = H(\tau - 0) - \beta_1\,\Theta\left(\zeta = \frac{1}{2}\right),$$

$$\begin{cases} \dfrac{\partial\Theta}{\partial\zeta}\left(\zeta = -\dfrac{1}{2}\right) = \beta_2\,\Theta\left(\zeta = -\frac{1}{2}\right) & \text{in Case a,} \\ \Theta\left(\zeta = -\frac{1}{2}\right) = 0 & \text{in Case b.} \end{cases}$$

Here $H(\tau = 0)$ stands for the Heaviside step function, and β – for the dimensionless surface losses coefficients (in particular case, when the plate loses its energy through its main surfaces by heat radiation only, $\beta_1 = \beta_2 = \beta$, where, in turn, β is defined in Footnote 29). Solving this problem (by applying the Fourier method of separating the independent variables) and calculating the optical characteristics by means of the general formulae given earlier, one may

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conclude that for a sufficiently long time, logarithms of the absolute values of the time derivatives of the quantities $U^u(r=0) - U^u(r)$, U_l, $\tan(\varepsilon_l^u/2)$ and $D_l^u = 1/f_l^u$ are linear functions of t with the coefficient (at t) equal to $\mu_1^2\,\kappa/(2h)^2$, where μ_1 is the first positive solution of the following characteristic equation: $\tan\mu = \mu\,(\beta_1+\beta_2)/(\mu^2 - \beta_1\,\beta_2)$ (in case a) or: $\tan\mu = -\mu/\beta_1$ (in case b). From measurements of the time evolution of the quantities mentioned one may therefore determine the quantity μ_1. Then from the characteristic equation for μ one may determine $\beta_2 = \mu_1\,\tan\mu_1$, if $\beta_1 = 0$ (an ideal thermal insulation on the perturbed surface), or $\beta_1 = \mu_1\,\tan\mu_1$, if $\beta_2 = 0$ (an ideal thermal insulation on the opposite surface), or $\beta_1 = -\mu_1/\tan\mu_1$, if $\beta_2 = \infty$ (ideal losses on the opposite surface, realized for instance by a thermostate, as it was assumed in case b).

8. Final remarks

Remark on the criteria for neglecting the distortion of properties of optical mirrors due to absorption of light, as well as general conclusions are analogous to those presented in Part I.

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A self-consistent model of rate-dependent plasticity of polycrystals

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THE HILL–HUTCHINSON self-consistent scheme for polycrystals is extended to rate-dependent plasticity with work-hardening. A computational version of the model requires three material parameters for a single crystal which are approximately identified from the experimental stress – plastic strain curve for a polycrystalline material. Contours of the macroscopic plastic potential are calculated after different plastic prestrains in uniaxial tension.

1. Introduction

IN THE CLASS of "self-consistent" models of a polycrystalline aggregate, an auxiliary problem is formulated for a single crystal (inclusion) embedded in a homogeneous medium (matrix) of some different material (HILL [1]). For path-dependent plastic materials, the typical problem is that of *incremental* equilibrium, and a deformation process is analyzed step by step. To take advantage of ESHELBY'S [2] solution, which is uniform within an ellipsoidal inclusion, the auxiliary problem must be posed as *linear*, that is, at the current step of computations all relevant parameters of the matrix material must be treated as constants. On the other hand, for adequate modeling of the interaction between a crystal and the surrounding aggregate of grains, it is proper to assign to the matrix the actual overall properties of the polycrystal. It is not evident how to reconcile both requirements when the grains undergo elastic-plastic deformation so that the overall response of the polycrystal is not incrementally linear.

In a number of papers, the matrix material in the auxiliary problem was characterized by the constants of linear elasticity. The aforementioned interaction was in effect modeled as purely elastic, irrespectively of the average plastic strain attributed to the matrix. That approach was originally developed for small-strain rate-independent plasticity (KRÖNER [3], BUDIANSKY and WU [4]) and extended to creep (BROWN [5]) and rate-dependent plasticity in the framework of small strain (WENG [6]) and of large strain (NEMAT-NASSER and OBATA [7], HARREN [8]). During advanced plastic flow, the flexibility of the constraints of a grain is most likely underestimated in that way with respect to the actual constraints within the aggregate.

That deficiency was removed in HILL'S [1] formulation who proposed to use, in the auxiliary incremental problem for rate-independent plasticity, the instantaneous compliances that connect the *actual* rates of overall stress and strain

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of the polycrystal. This can be justified by appealing to the homogeneity of degree zero of the compliances, and offers the possibility to incorporate plastic anisotropy of the polycrystal to the definition of the matrix material. Since the compliances are unknown in advance, the increased accuracy of that approach is connected with a greater computational effort in comparison with the use of elastic moduli. Hutchinson [9] demonstrated that the approach is numerically applicable in the case of transversal isotropy at small strain. An extension to finite strain was formulated by Iwakuma and Nemat - Nasser [10] and applied to two-dimensional problems. The behaviour of a three-dimensional self-consistent model of a time-independent elastic-plastic polycrystal was analysed numerically by Lipiński *et al.* [11].

An intermediate approach was proposed by Berveiller and Zaoui [12] who retained the formulae for an elastically isotropic matrix but with the material stiffness parameter modified, e.g. in order to correspond to the secant modulus. A similar concept was also applied to viscoplastic polycrystals (Cailletaud [13]).

Rate-dependent elasto-plastic polycrystals were also investigated by using the self-consistent scheme with approximately defined inelastic properties of the matrix. Nemat - Nasser and Obata [7] proposed the matrix moduli derived from certain local moduli dependent explicitly on the assumed time step. Molinari *et al.* [14] defined the moduli by linearizing around the current strain-rate the nonlinear relationship between the plastic part of strain-rate and stress. Another approach based on the adaptation of the self-consistent scheme developed for linear viscoelasticity was recently proposed by Navidi *et al.* [15] and illustrated by the example of an isotropic multiphase material.

Hutchinson [16] showed that Hill's method can be applied to examining steady creep without the need of approximating the actual nonlinear constitutive relation between the strain-rate and stress. As the basic step in the absence of elasticity effects, the linear auxiliary problem was posed in a natural way in terms of the *differentials* of strain-rate and stress, related to each other in a given state by the tensor of creep compliances independent of those differentials([1]). Hutchinson's analysis was limited to steady deformations of non-hardening materials.

The aim of this paper([2]) is to extend the Hill - Hutchinson self-consistent model of polycrystals to rate-dependent plasticity with work-hardening. Certain additional terms will appear in the basic equations since Hill's [1] theory did not deal with rate-dependence effects, and Hutchinson's [16] model did not account for work-hardening. To retain consistency, elastic compliances are neglected at the outset. Accordingly, the model is suited for simulation of developed plastic flow rather than of the range where the aggregate is predominantly elastic, just

([1]) The restriction to a non-hardening material obeying a power crep law allowed the system of incremental equation to be integrated to a total form.

([2]) Based on the former author's Ph.D. Thesis [17].

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contrary to the models which use elastic compliances for defining the matrix material.

A three-parameter computational version of the model is applied here to simulate uniaxial tension of an aluminum alloy. An approximate procedure is proposed for identifying the material parameters for a single crystal from the standard uniaxial tension test of a polycrystalline specimen. The calculated stress-strain curve and contours of a visco-plastic potential at various stages of the deformation serve as examples of the predictions of the model. A comparison is made with the experimental stress-strain curve and conventional yield surfaces.

Throughout this paper any changes in geometry during deformation are disregarded, i.e. the small strain formulation is used and lattice rotations are neglected. Bold-face small letters, Roman or Greek, denote second-order symmetric tensors, and bold-face capital letters denote fourth-order tensors possessing the minor symmetries, with the respective unit tensor denoted by $\mathbf{I}$ and the transpose indicated by a superscript T. A juxtaposition of two tensor symbols denotes double contraction. A superimposed dot over a symbol denotes the material time derivative, understood as a forward rate.

2. General equations of the model

2.1. Constitutive framework of rate-dependent plasticity

Constitutive equations of isothermal rate-dependent plasticity are assumed in the following general form

$$
\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^p + \dot{\boldsymbol{\varepsilon}}^e, \qquad \dot{\boldsymbol{\varepsilon}}^e = \mathbf{M}^e\,\dot{\boldsymbol{\sigma}},
$$
$$
\dot{\boldsymbol{\varepsilon}}^p = \mathbf{f}(\boldsymbol{\sigma}, g_K), \qquad \dot{g}_K = \mathfrak{g}_K(\boldsymbol{\sigma}, g_L), \tag{2.1}
$$

where $\mathbf{f}$ and $\mathfrak{g}_K$ are sufficiently smooth, given functions. In the small strain formulation adopted, $\dot{\boldsymbol{\varepsilon}}^e$ and $\dot{\boldsymbol{\varepsilon}}^p$ are the elastic and plastic parts, respectively, of the small-strain rate, and $\dot{\boldsymbol{\sigma}}$ is the rate of the Cauchy stress. The elastic strain-rate $\dot{\boldsymbol{\varepsilon}}^e$ is related to $\dot{\boldsymbol{\sigma}}$ by the fourth-order tensor $\mathbf{M}^e$ of elastic compliances, while $\dot{\boldsymbol{\varepsilon}}^p$ is a function of the current material state, represented here by the stress $\boldsymbol{\sigma}$ and a certain number P of material parameters g_K $(K = 1, ..., P)$. The latter need not be interpreted as internal state variables, and for anisotropic materials they can be regarded as either scalars or components of tensor variables for a fixed orientation of a material element relative to a given reference frame. In the computational version of the model and at the level of a single crystal, g_K will be identified with the current critical resolved shear stress on the K-th slip system.

Elasticity effects during plastic flow will be neglected, so that we will substitute

$$
\mathbf{M}^e \cong \mathbf{0}, \qquad \dot{\boldsymbol{\varepsilon}} \cong \dot{\boldsymbol{\varepsilon}}^p, \tag{2.2}
$$

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but *conceptually* the elastic compliances will be treated as vanishingly small rather than as being exactly zero. The distinction will become important in the presence of discontinuous changes in stress, in particular in the calculations of an instantaneous overall plastic potential.

Assuming that $\mathbf{f}$ is differentiable with respect to its arguments, we can write

$$\ddot{\boldsymbol{\varepsilon}} = \mathbf{M}\dot{\boldsymbol{\sigma}} + \ddot{\boldsymbol{\varepsilon}}^r, \tag{2.3}$$

where $\mathbf{M}$ and $\ddot{\boldsymbol{\varepsilon}}^r$ are functions only of the material state, viz.

$$\mathbf{M} = \mathbf{M}(\boldsymbol{\sigma}, g_K) \equiv \mathbf{f}_{,\boldsymbol{\sigma}}(\boldsymbol{\sigma}, g_K), \qquad \ddot{\boldsymbol{\varepsilon}}^r \equiv \sum_K \mathbf{f}_{,g_K}(\boldsymbol{\sigma}, g_J)\mathrm{g}_K(\boldsymbol{\sigma}, g_L); \tag{2.4}$$

a comma followed by an index denotes partial differentiation. It follows that in a given state of the material, the *second* rate of strain is a *linear*, although inhomogeneous, function of stress-rate. At the moment we do not assume that a plastic potential exists so that $\mathbf{M}$ need not be symmetric in general.

Under the assumption (2.2) of negligible elastic compliances, the relationship (2.3) characterizes a rate-dependent plastic response at either level of constitutive description. By using a subscript c for constitutive quantities at the level of a single crystal (grain), we shall write

$$\ddot{\boldsymbol{\varepsilon}}_c = \mathbf{M}_c\dot{\boldsymbol{\sigma}}_c + \ddot{\boldsymbol{\varepsilon}}_c^r. \tag{2.5}$$

Overall (macroscopic) quantities will be distinguished by a superimposed bar, while unweighted volume averaging will be denoted by curly brackets. By defining the overall stress $\overline{\boldsymbol{\sigma}}$ and strain $\overline{\boldsymbol{\varepsilon}}$ as

$$\overline{\boldsymbol{\sigma}} = \{\boldsymbol{\sigma}_c\}, \qquad \overline{\boldsymbol{\varepsilon}} = \{\boldsymbol{\varepsilon}_c\}, \tag{2.6}$$

the constitutive relationship (2.3) for a polycrystal reads

$$\ddot{\overline{\boldsymbol{\varepsilon}}} = \overline{\mathbf{M}}\dot{\overline{\boldsymbol{\sigma}}} + \ddot{\overline{\boldsymbol{\varepsilon}}}^r. \tag{2.7}$$

Our primary task is now to express $\overline{\mathbf{M}}$ and $\ddot{\overline{\boldsymbol{\varepsilon}}}^r$ in terms of $\mathbf{M}_c$ and $\ddot{\boldsymbol{\varepsilon}}_c^r$.

2.2. Self-consistent method

Following HILL [1], consider the auxiliary incremental problem for an ellipsoidal inclusion embedded in an infinite homogeneous matrix. The inclusion represents a single grain and the matrix replaces the polycrystalline material surrounding the grain. A uniform stress $\boldsymbol{\sigma}_c$ within each grain and the average stress in the matrix, taken equal to $\overline{\boldsymbol{\sigma}}$ by assumption, are regarded as known, along with the current values of all material parameters. Then, in contrast to Hill's work concerned with rate-independent plasticity, the (plastic) strain rate $\dot{\boldsymbol{\varepsilon}}_c^p = \dot{\boldsymbol{\varepsilon}}_c$ in

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each grain (and also the average strain rate $\dot{\bar{\boldsymbol{\varepsilon}}}$) are also known from the constitutive equations of rate-dependent plasticity at the micro-level. The auxiliary problem is posed here in terms of the differentials $d\boldsymbol{\sigma}_c$, $d\dot{\boldsymbol{\varepsilon}}$ and $d\bar{\boldsymbol{\sigma}}$, $d\dot{\bar{\boldsymbol{\varepsilon}}}$, or equivalently, in terms of stress-rates $\dot{\boldsymbol{\sigma}}_c$, $\dot{\bar{\boldsymbol{\sigma}}}$ and of the *second-order* rates of strain, $\ddot{\boldsymbol{\varepsilon}}$, $\ddot{\bar{\boldsymbol{\varepsilon}}}$. From (2.5) and (2.7) it follows that the problem is *linear* but inhomogeneous; an analogous problem but without the additive non-homogeneous term was examined by HUTCHINSON [16]. Here, $d\bar{\boldsymbol{\sigma}} = \mathbf{0}$ does not imply $d\boldsymbol{\sigma}_c = \mathbf{0}$ so that stress redistribution in the polycrystal takes place also at a constant overall stress. The Hill–Hutchinson self-consistent scheme has thus to be modified(³).

For this purpose, we observe first that the relationship between a *difference* $\Delta\dot{\boldsymbol{\sigma}}$ in two uniform fields of stress-rate within an ellipsoidal hole in the matrix and a difference $\Delta\ddot{\boldsymbol{\varepsilon}}$ in the associated second-order rates of straining of the ellipsoid is still linear homogeneous,

$$\Delta\ddot{\boldsymbol{\varepsilon}} = -\mathbf{M}^*\Delta\dot{\boldsymbol{\sigma}}, \tag{2.8}$$

where $\mathbf{M}^*$ can be connected with the Eshelby tensor $\mathbf{S}$ [2] for the matrix with a constant compliance tensor $\mathbf{M}$ by the equation [1]

$$(\mathbf{I} - \mathbf{S})\mathbf{M}^* = \mathbf{S}\,\mathbf{M}. \tag{2.9}$$

It should be noted that in the derivation of (2.8), the matrix is treated as being in a uniform stress state at the instant under consideration, which is an additional assumption [16].

By identifying $\Delta\dot{\boldsymbol{\sigma}}$ with $\dot{\boldsymbol{\sigma}}_c - \dot{\bar{\boldsymbol{\sigma}}}$ and $\Delta\ddot{\boldsymbol{\varepsilon}}$ with $\ddot{\boldsymbol{\varepsilon}}_c - \ddot{\bar{\boldsymbol{\varepsilon}}}$, and substituting (2.5) and (2.7), from (2.8) we obtain

$$(\mathbf{M}^* + \mathbf{M}_c)\dot{\boldsymbol{\sigma}}_c = (\mathbf{M}^* + \mathbf{M})\dot{\bar{\boldsymbol{\sigma}}} - \left(\ddot{\boldsymbol{\varepsilon}}_c^r - \ddot{\bar{\boldsymbol{\varepsilon}}}^r\right). \tag{2.10}$$

This relationship differs from an analogous equation given by HILL [1] merely by the last additive term which is independent of the overall stress-rate. We may thus define the "concentration-factor tensor" $\mathbf{B}_c$ by the unchanged formula(⁴)

$$\mathbf{B}_c = (\mathbf{M}^* + \mathbf{M}_c)^{-1}(\mathbf{M}^* + \mathbf{M}). \tag{2.11}$$

By introducing the following expression for a relaxation stress-rate

$$\dot{\boldsymbol{\sigma}}_c^r = -(\mathbf{M}^* + \mathbf{M}_c)^{-1}\left(\ddot{\boldsymbol{\varepsilon}}_c^r - \ddot{\bar{\boldsymbol{\varepsilon}}}^r\right), \tag{2.12}$$

we arrive at

$$\dot{\boldsymbol{\sigma}}_c = \mathbf{B}_c\dot{\bar{\boldsymbol{\sigma}}} + \dot{\boldsymbol{\sigma}}_c^r. \tag{2.13}$$

(³) Similar modification was done in earlier papers in the context different from the present one, without appealing to the second-order rate of strain.

(⁴) All matrices are implicity assumed to be invertible if needed.

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Note that $\mathbf{B}_c$ and $\dot{\boldsymbol{\sigma}}_c^r$ are uniquely defined in the current state, independently of the overall stress-rate.

By substituting (2.13) into (2.5), we obtain

$$\dot{\boldsymbol{\varepsilon}}_c = \mathbf{M}_c \mathbf{B}_c \dot{\overline{\boldsymbol{\sigma}}} + \mathbf{M}_c \dot{\boldsymbol{\sigma}}_c^r + \dot{\boldsymbol{\varepsilon}}_c^r . \tag{2.14}$$

Now, by taking the volume averages of (2.13), (2.14) and comparing with (2.6), we obtain the known pair of self-consistency conditions

$$\{\mathbf{B}_c\} = \mathbf{I}, \qquad \{\mathbf{M}_c \mathbf{B}_c\} = \mathbf{M} \tag{2.15}$$

along with another pair

$$\{\dot{\boldsymbol{\sigma}}_c^r\} = \mathbf{0}, \qquad \{\mathbf{M}_c \dot{\boldsymbol{\sigma}}_c^r + \dot{\boldsymbol{\varepsilon}}_c^r\} = \dot{\overline{\boldsymbol{\varepsilon}}}^r. \tag{2.16}$$

For spherical grains when $\mathbf{M}^*$ depends only on $\mathbf{M}$, the conditions in (2.16) are equivalent to each other, as can be seen by taking the average of both sides of (2.12) multiplied beforehand by $(\mathbf{M}^* + \mathbf{M}_c)$.

The conditions (2.15) and (2.16) can further be transformed to a form which will appear more convenient in calculations. From (2.15) we immediately obtain

$$\{(\mathbf{M} - \mathbf{M}_c)\mathbf{B}_c\} = \mathbf{0}, \tag{2.17}$$

with an advantage that the averaged expression possesses diagonal symmetry if $\mathbf{M}_c$ and $\mathbf{M}$ do [16]. For spherical grains, or for $\mathbf{M}^*$ dependent only on $\mathbf{M}$, the condition $(2.16)_1$ rearranged with the help of (2.12), (2.11) and $(2.15)_1$ yields

$$\dot{\overline{\boldsymbol{\varepsilon}}}^r = \{\dot{\boldsymbol{\varepsilon}}_c^r \widehat{\mathbf{B}}_c\}, \qquad \widehat{\mathbf{B}}_c = \left(\mathbf{M}^{*T} + \mathbf{M}_c^T\right)^{-1} \left(\mathbf{M}^{*T} + \mathbf{M}^T\right), \tag{2.18}$$

where $\widehat{\mathbf{B}}_c$ reduces to $\mathbf{B}_c$ if $\mathbf{M}_c$ and $\mathbf{M}$ possess diagonal symmetry.

2.3. Plastic potential

Suppose now that the relationship between $\dot{\boldsymbol{\varepsilon}}_c^p$ and $\boldsymbol{\sigma}_c$ admits a potential, viz.

$$\dot{\boldsymbol{\varepsilon}}_c^p = \frac{\partial \omega_c(\boldsymbol{\sigma}_c, g_K)}{\partial \boldsymbol{\sigma}_c} . \tag{2.19}$$

For instance, let $\dot{\boldsymbol{\varepsilon}}^p$ in a given grain be a sum of the strain rates due to slipping on N individual slip systems defined by the two unit vectors: the slip direction $\mathbf{m}^K$ and the normal $\mathbf{n}^K$ to the slip plane, so that

$$\dot{\boldsymbol{\varepsilon}}_c^p = \sum_K \dot{\gamma}^K \boldsymbol{\alpha}^K, \qquad \boldsymbol{\alpha}^K = \frac{1}{2}\left(\mathbf{m}^K \otimes \mathbf{n}^K + \mathbf{n}^K \otimes \mathbf{m}^K\right), \qquad K = 1, ..., N, \tag{2.20}$$

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where $\dot{\gamma}^K$ is a slip rate on the K-th system. If any $\dot{\gamma}^K$ depends on $\boldsymbol{\sigma}_c$ only through the shear stress resolved on the K-th slip system, that is, if

$$\dot{\gamma}^K = \dot{\gamma}^K(\tau^K, g_L), \qquad \tau^K = \boldsymbol{\sigma}_c \boldsymbol{\alpha}^K \tag{2.21}$$

then (2.19) is satisfied (Kestin and Rice [18]); this can be verified by substituting

$$\omega_c(\boldsymbol{\sigma}_c, g_L) = \int\limits_0^{\boldsymbol{\sigma}_c} \mathbf{f}(\boldsymbol{\sigma}, g_L)\, d\boldsymbol{\sigma} = \sum_K \int\limits_0^{\boldsymbol{\tau}^K} \dot{\gamma}^K(\boldsymbol{\tau}, g_L)\, d\boldsymbol{\tau}. \tag{2.22}$$

If (2.19) holds then $\mathbf{M}_c = \partial^2\omega_c/\partial\boldsymbol{\sigma}_c\partial\boldsymbol{\sigma}_c$ is diagonally symmetric. Then, we can conclude (cf. HILL [19], HILL and RICE [20]) about diagonal symmetry of $\mathbf{M}$ from the assumed equality of work differentials at the micro and macro-levels expressed in the *incremental* form as

$$\dot{\overline{\boldsymbol{\varepsilon}}}\, \delta\overline{\boldsymbol{\sigma}} = \{\dot{\boldsymbol{\varepsilon}}\, \delta\boldsymbol{\sigma}\}, \tag{2.23}$$

where $\delta\boldsymbol{\sigma}$ is a statically admissible field of a stress increment such that $\{\delta\boldsymbol{\sigma}\} = \delta\overline{\boldsymbol{\sigma}}$, which need not be related to a (compatible) field $\dot{\boldsymbol{\varepsilon}}$. In particular, (2.23) holds if the prefix δ refers to a purely *elastic* change of the stress field within the aggregate; this is not in contradiction with our assumption that elastic compliances are negligibly small (but *not* exactly zero) since the proportional scaling down of elastic compliances does not influence the distribution of $\delta\boldsymbol{\sigma}$ obtained for given $\delta\overline{\boldsymbol{\sigma}}$. Following RICE [21], or simply by substituting (2.2) and (2.19) into the equality (2.23), the constitutive relationship for a polycrystal can be expressed in terms of a macroscopic potential Ω, viz.

$$\Omega = \{\omega_c\}, \qquad \dot{\overline{\boldsymbol{\varepsilon}}}^p = \frac{\partial\Omega}{\partial\overline{\boldsymbol{\sigma}}}, \qquad \mathbf{M} = \frac{\partial^2\Omega}{\partial\overline{\boldsymbol{\sigma}}\,\partial\overline{\boldsymbol{\sigma}}}. \tag{2.24}$$

3. Three-parameter version of the model

For computational purposes we assume the constitutive equations for $\dot{\boldsymbol{\varepsilon}}^p$ in the form (2.20) and, following many other authors (e.g. BROWN [5], PAN and RICE [22], ASARO and NEEDLEMAN [23]), specify the rate-dependence expression $(2.21)_1$ as a power law

$$\dot{\gamma}_K = \dot{\gamma}^0 \left|\frac{\tau_K}{g_K}\right|^{1/m} \operatorname{sgn}(\tau_K) \tag{3.1}$$

with $K = 1, ..., 12$ ($P = N = 12$) and $\boldsymbol{\alpha}_K$ corresponding to fundamental slip systems in fcc crystals. $\dot{\gamma}^0$ is not an independent material parameter but plays the role of a given time-scale factor. Evolution equations for the parameters g_K

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(for $0 < m \ll 1$ interpreted as critical values of the resolved shear stress) are assumed in the usual rate form

$$g_K = \sum_L h_{KL} |\dot{\gamma}_L|. \tag{3.2}$$

In the simplest version of the model, we will assume linear hardening obeying the Taylor hypothesis, $h_{KL} = h = \text{const}$. As the initial condition for (3.2) in a virgin state of a macroscopically isotropic polycrystal we take $g_K = \tau^0$, where τ^0 is the initial critical value for τ_K, the same for all slip systems.

The expression (2.22) for the local plastic potential reduces to

$$\omega_c = \frac{m}{1+m} \sum_K \tau_K \dot{\gamma}_K = \frac{m}{1+m} \boldsymbol{\sigma}_c \, \dot{\boldsymbol{\varepsilon}}_c^p . \tag{3.3}$$

To calculate the macroscopic plastic potential Ω, we need a relationship between purely elastic stress increments at the micro and macro-levels. The simplest assumption is to neglect elastic heterogeneity (and thus *elastic* anisotropy of crystals), which means that $\delta\boldsymbol{\sigma} \equiv \delta\overline{\boldsymbol{\sigma}}$. Then, at a fixed distribution of g_K within the aggregate, we obtain

$$\Omega \left(\overline{\boldsymbol{\sigma}} + \Delta\overline{\boldsymbol{\sigma}}\right) = \frac{m}{1+m} \left\{ \left(\boldsymbol{\sigma}_c + \Delta\overline{\boldsymbol{\sigma}}\right) \dot{\boldsymbol{\varepsilon}}_c^p \left(\boldsymbol{\sigma}_c + \Delta\overline{\boldsymbol{\sigma}}\right) \right\}, \tag{3.4}$$

where $\overline{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma}_c$ are the current macro and micro-stresses, respectively, and $\Delta\overline{\boldsymbol{\sigma}}$ stands for an instantaneously applied finite increment of the macroscopic stress, corresponding to purely elastic response of the aggregate.

To summarize, in the simplest computational version of the proposed model there are three independent material parameters assumed at the micro-level: m, τ^0 and h. In the next section, an approximate procedure is proposed for the identification of these parameters from the standard uniaxial tension test performed on a *polycrystalline* specimen.

The numerical implementation of the self-consistent scheme (more details are given in [17]) has followed closely that described by HUTCHINSON [16] and employed the (corrected) formulae given by KNEER [24] for a spherical inclusion in a transversally isotropic matrix. Volume averaging in the simulation of uniaxial tension was replaced by the averaging over 36 orientations of fcc crystals corresponding to different values of *two* Euler angles. In the calculations of the plastic potential, this has been complemented by additional averaging over 12 values of the third Euler angle that defines rotation about the tensile axis.

4. An approximate identification procedure

Suppose that the stress – plastic strain curve obtained experimentally from the standard uniaxial tension test, within some strain interval of developed

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plastic flow, can be approximated by a linear segment whose slope defines the macroscopic hardening modulus, say H. A sample result of that type is presented in Fig. 1 which has been obtained for an aluminum alloy PA6 at the room temperature([5]). To eliminate the influence of an initial period of a constrained plastic flow in the grains, the macroscopic initial yield stress, say Σ^0, is defined by backward extrapolation of the linear segment up to the zero plastic strain offset. By performing at least two experiments at a different rate of stress or of plastic strain, in the standard manner we can determine the exponent $\overline{m}$ of strain-rate sensitivity of the polycrystalline material.

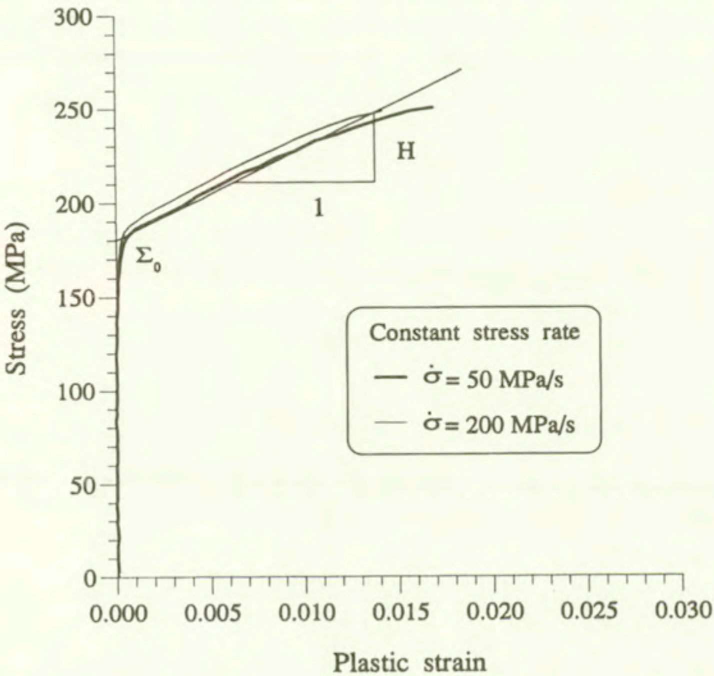

FIG. 1. Linear approximation of an initial segment of the uniaxial stress – plastic strain curve for an aluminum alloy PA6.

In that way, an approximation of the experimental relationship between the stress and plastic strain-rate in uniaxial tension of a polycrystalline specimen in a certain range of strain is constructed as

$$\dot{\overline{\varepsilon}}^p = \dot{\overline{\varepsilon}}^0 \left(\frac{\overline{\sigma}}{\Sigma^0 + H\overline{\varepsilon}^p} \right)^{1/\overline{m}} . \tag{4.1}$$

The reference strain-rate $\dot{\overline{\varepsilon}}^0$ is identified with the plastic strain rate corresponding to stress-strain curve from which the parameters Σ^0 and H have been determined.

([5]) All experimental data presented below are taken from the Thesis [17].

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We proceed now to identification of the material parameters at the micro-level. The comparison of Eqs. (4.1) and (3.1) shows that a natural assumption is to take $\overline{m}$ as the rate sensitivity exponent also for a single crystal,

$$m \cong \overline{m}. \tag{4.2}$$

For many metallic materials in the room temperature, the value of the rate sensitivity parameter m is much smaller than unity. If $m \ll 1$ and if the hardening is neglected then the macroscopic stress in the range of steady viscoplastic flow under uniaxial tension is slightly smaller than τ^0 multiplied by the Taylor factor 3.06 (HUTCHINSON [16]). A similar value of that factor was found in the course of present computations, including hardening. Guided by these results, we may assume that

$$\tau^0 \cong \frac{1}{3}\Sigma^0 \tag{4.3}$$

as an approximate value of the initial yield stress for a single crystal.

From HILL'S lemma [19] applied to the uniaxial tension with elastic heterogeneity neglected, we have

$$\overline{\sigma}\,\dot{\overline{\varepsilon}}^p = \{\boldsymbol{\sigma}\,\dot{\boldsymbol{\varepsilon}}^p\} = \left\{\sum_K \tau_K \dot{\gamma}_K\right\}. \tag{4.4}$$

At small strains with $\overline{\sigma} \cong \Sigma^0$ and $\tau_K \dot{\gamma}_K \cong \tau^0 |\dot{\gamma}_K|$ for $m \ll 1$, from (4.4) and (4.3) we obtain

$$\dot{\overline{\varepsilon}}^p \cong \frac{1}{3}\left\{\sum_K |\dot{\gamma}_K|\right\}. \tag{4.5}$$

Let, as mentioned above, $\dot{\overline{\varepsilon}}^p$ be taken as the reference strain-rate $\dot{\overline{\varepsilon}}^0$, and define $\dot{\gamma}^0$ as the *mean* value of $|\dot{\gamma}_K|$. Since the number of slip systems $N = 12$, (4.5) then gives

$$\dot{\gamma}^0 \cong \frac{1}{4}\dot{\overline{\varepsilon}}^0. \tag{4.6}$$

In turn, on multiplying both sides of (4.5) by h and using the Taylor hypothesis, we obtain

$$h\dot{\overline{\varepsilon}}^p \cong \frac{1}{3}\{\dot{g}_K\} \tag{4.7}$$

for any K. Now, on the basis of (4.3) and of numerical tests, the *mean* rate $\{\dot{g}_K\}$ of the critical shear stress in the range of stabilized plastic flow at $m \ll 1$ is

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postulated to be close to the macroscopic uniaxial stress-rate divided by 3. With H as the macroscopic hardening modulus, from (4.7) we finally arrive at

$$h \cong \frac{1}{9}H. \tag{4.8}$$

The material parameters m, τ_0 and h at the micro-level may can be estimated from the simple formulae (4.2), (4.3) and (4.8) if the uniaxial macroscopic law (4.1) has been given, with the relationship (4.6) between the respective time-scale factors. The identification formulae might also be applied, as they stand, to a nonlinear hardening law. A satisfactory agreement is shown below between the experimental macroscopic curve and that calculated by using the self-consistent model with the material parameters for a single crystal estimated as above. However, the applicability of the proposed identification procedure remains yet to be verified by other examples.

5. Example

In Fig. 2 the comparison is made between experimental stress – plastic strain curves for uniaxial tension of a polycrystalline aluminum alloy PA6 and those calculated numerically by using the micromechanical model described in Sec. 2. The constitutive equations have been implemented in the version specified in Sec. 3, and the material parameters have been determined according to the identification procedure described in Sec. 4. The parameter values are listed in the figure, with $n = 1/m$(6). It can be seen that the model predicts in a satisfactory manner the character of the macroscopic stress-strain curve up to $\sim 1.5\%$ of the plastic strain. This is not just an effect of curve-fitting since the three material parameters for a single crystal have been determined beforehand and in an indirect manner. For larger strains the assumption of linear hardening has turned out to be an oversimplification and should be replaced by a nonlinear law.

Figure 3 shows the evolution of the plastic potential during the uniaxial tension of the polycrystal model in x_3-direction. The subsequent pictures correspond to the initial state and to the plastic prestrain 5% and 7%. Characteristic shape changes of the potential contours are observed: while the initial contours are ellipsoidal, subsequent contours have a rounded-off nose in vicinity of the (tensile) loading point(7) and are flattened on the opposite (compressive) side. Simultaneously, a translation of the contours of the plastic potential towards the loading point can be observed, accompanied first by lateral contraction and then by expansion. The qualitative changes closely resemble those observed in

(6) h_1/h_2 denotes the ratio of the diagonal to off-diagonal components of the hardening matrix for a single crystal, equal to 1 by the Taylor hypothesis.

(7) The sharp corners that appear in the figures are due to the adopted way of graphical presentation where calculated points have been connected by straight lines.

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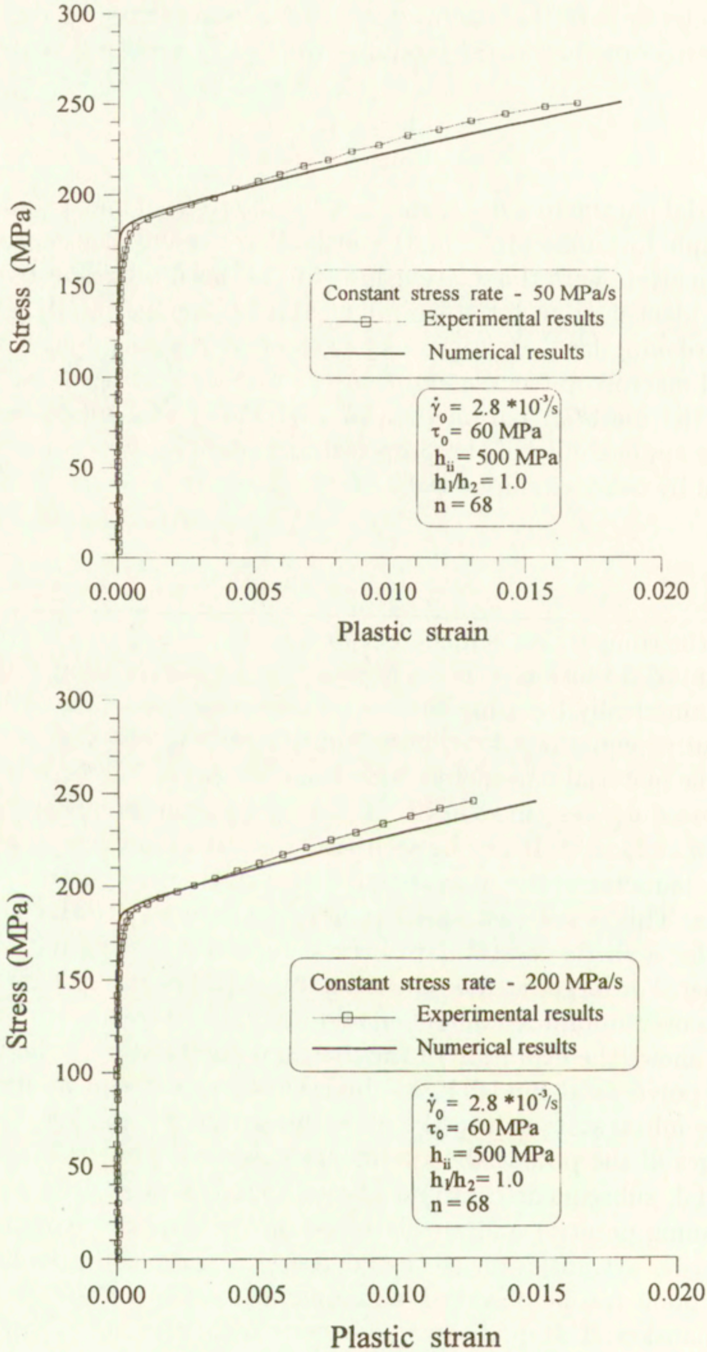

FIG. 2. Comparison of the calculated and experimental uniaxial stress – plastic strain curves at two prescribed macroscopic stress rates.

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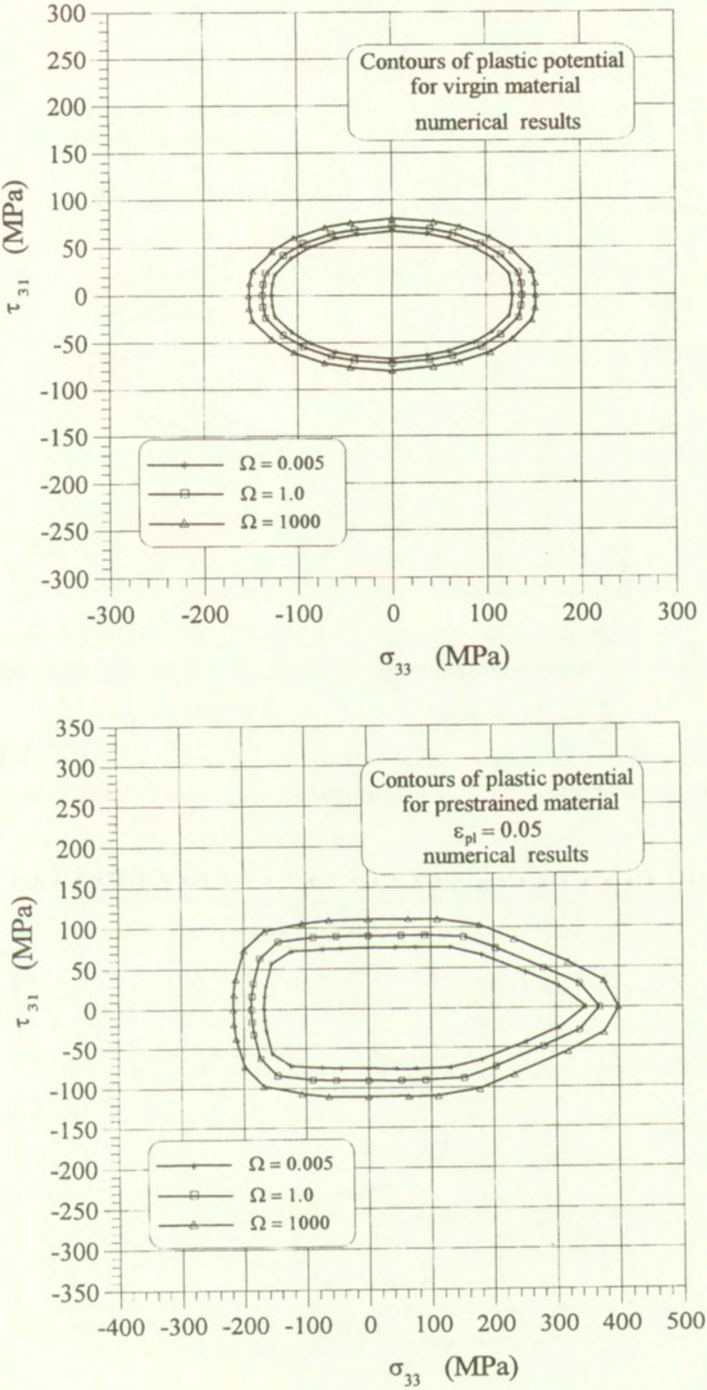

[FIG. 3 a, b]

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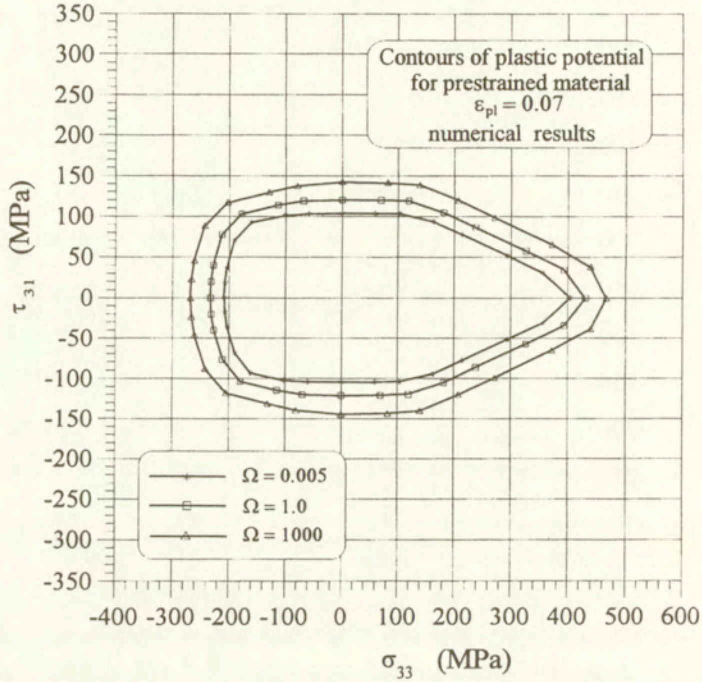
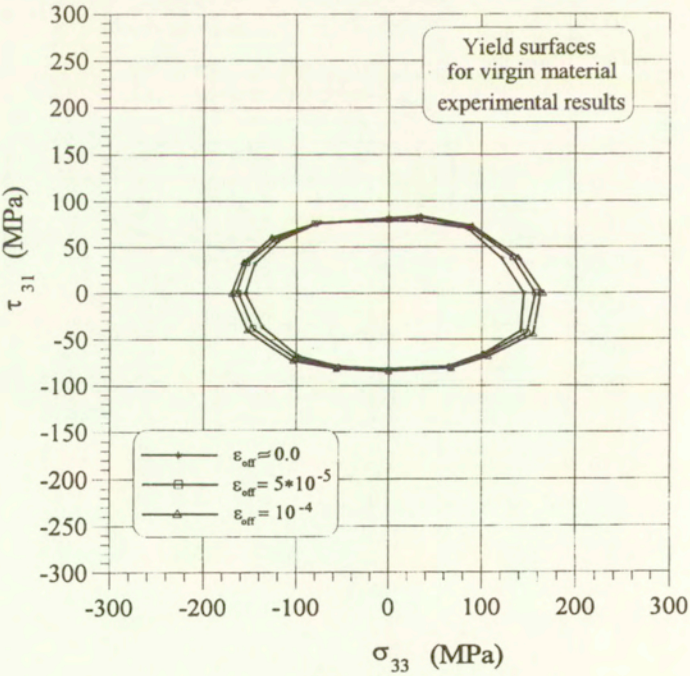


FIG. 3. Calculated contours of the macroscopic plastic potential (MPA/s) for a polycrystalline material after plastic prestrain in uniaxial tension in x_3-direction.



[FIG. 4 a]

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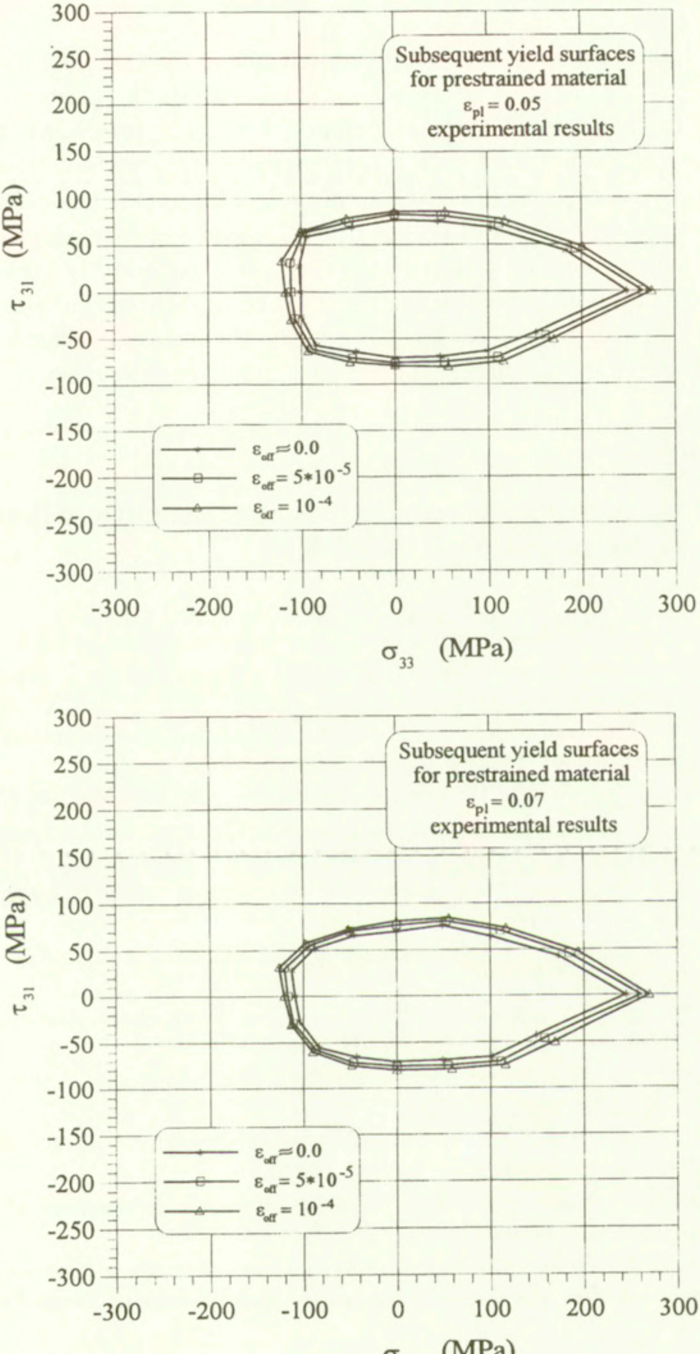

FIG. 4. Experimental yield surfaces for an aluminum alloy PA6 for various definitions of the yield offset after plastic prestrain in uniaxial tension in x_3-direction.

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experimental yield surfaces for a small yield offset for the polycrystalline alloy tested, as shown in Fig. 4. This is not surprising since the contours of the plastic potential in the layer in the stress space where the potential starts to grow very rapidly, can be identified with a conventional yield surface (RICE [21]). The qualitative agreement between the calculated contours of the plastic potential for the micromechanical model shown in Fig. 3, and the respective experimental yield surfaces shown in Fig. 4, is perhaps unexpectedly good in view of the small number of material parameters in the model. This indicates that the assumptions used are likely to reflect, to a reasonable extent, the nature of plastic deformation of a polycrystalline metal under the selected monotonic loading.

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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Focusing a shock wave; microscopic structure of the phenomenon

Z. A. WALENTA and J. ORZEŃSKI (WARSZAWA)

THE PROBLEM OF FOCUSING a shock wave after its reflection from a concave wall is considered experimentally (in a shock tube) and numerically (employing the Direct Simulation Monte–Carlo technique). Rarefied flow conditions make it possible to clarify both the size of the focus and the structure of its neighbourhood.

1. Introduction

THE PROCESS OF FOCUSING of a shock wave was investigated by a number of researchers for many years (see e.g. [1, 2, 4] and the papers cited there). The interest was related mainly to its possible applications (military, medical, industrial etc.), however the phenomenon itself can offer some advantages of a more fundamental nature (possibility of producing the medium in the state very far from thermodynamic equilibrium).

There are several methods of focusing the shock wave. The one used in the first (to our knowledge) experiment on this subject, was proposed by PERRY and KANTROWITZ [1]. A plane shock wave, generated in a shock tube of circular cross-section, is transformed into a ring-shaped one by an axisymmetric inner body, placed at the axis of the tube. Having reached the gap between the inner body and the end plate of the tube, the shock turns towards the axis, becomes cylindrical and eventually, focuses.

This method allows in principle to produce a nearly perfect focus, where the parameters of the gas (pressure, temperature) are very high. Unfortunately it is plagued by the inherent instabilities of the converging shock. Moreover, because of the geometrical complexity, this method cannot be applied to many problems of practical interest.

The method most frequently used, is based on reflection of a plane, or spherical, divergent shock wave from a suitably shaped concave reflector (paraboloidal or ellipsoidal, respectively) [2, 3, 4]. This method is simpler in realization, but produces "foci" of larger size.

The present study is concerned with investigation of the mechanism of focusing of the plane shock after reflection from a concave wall. Such configuration was extensively studied in the past, however the experiments were always performed at relatively high densities and the description was based on the model of continuous medium, in which the shock was assumed to be infinitely thin.

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Therefore the problem of the structure of the focusing shock and the focus itself – the main aim of the present paper – was never considered.

In their fundamental paper "The focusing of weak shock waves" STURTEVANT and KULKARNY [2] distinguish between the case of very weak shocks, for which "the wavefronts emerge from focus crossed and folded, in accordance with the predictions of geometrical acoustics theory" and the strong-shock case, for which "the fronts beyond focus are uncrossed, as predicted by the theory of shock dynamics". In the present study we shall concentrate only on the strong shock case because detection of weak shock waves at low gas densities faces great experimental difficulties.

2. Experiment

2.1. Apparatus and procedure

The problem formulated above required the apparatus suitable for low-density experiments. The shock tube of the Institute of Fundamental Technological Research, Polish Academy of Sciences, Warsaw, was selected for this purpose. This tube, [5], especially designed for work in the rarefied gas regime, is of 250 mm internal diameter and about 17 meters length. Such dimensions are needed to work at densities, corresponding to mean free paths of the gas particles of the order of 1 mm, without too strong, disturbing influence of the boundary layer, generated at the shock tube walls.

Inside the test section of the tube a plane, parabolic reflector was placed (Fig. 1). The reflector was made of aluminum. Its span was equal to 210 mm, its depth – 45 mm, which produced the "geometrical focal length" 61.25 mm. The width of the reflector (in the direction perpendicular to the picture) was equal to 136 mm. To maintain the planarity of the flow, the reflector was placed between two aluminum plates, extending 75 centimeters upstream of the test section.

After interaction with reflector of such a shape, the shock wave converged and produced a "linear focus" at the plane of symmetry of the reflector.

The measurements were performed with the standard, electron beam attenuation technique [6]. The beam (of about 0.5 mm thickness) was perpendicular to the tube axis and parallel to the reflecting surface. Position of the beam with respect to the reflector could be varied. The maximum distance of the beam from the reflector in the direction of the tube axis was equal to about 60 mm; the minimum distance was about 2.4 mm (as the distance of the electron beam from the solid wall must be large enough to avoid the disturbing influence of the wall on the measurement). The maximum distance of the beam from the plane of symmetry was 10 mm. The field of observation defined in this way was large enough to investigate the structure of the shock focus.

For one experiment, a number of runs at the same flow conditions and different locations of the beam relative to the wedge was done. In a single run

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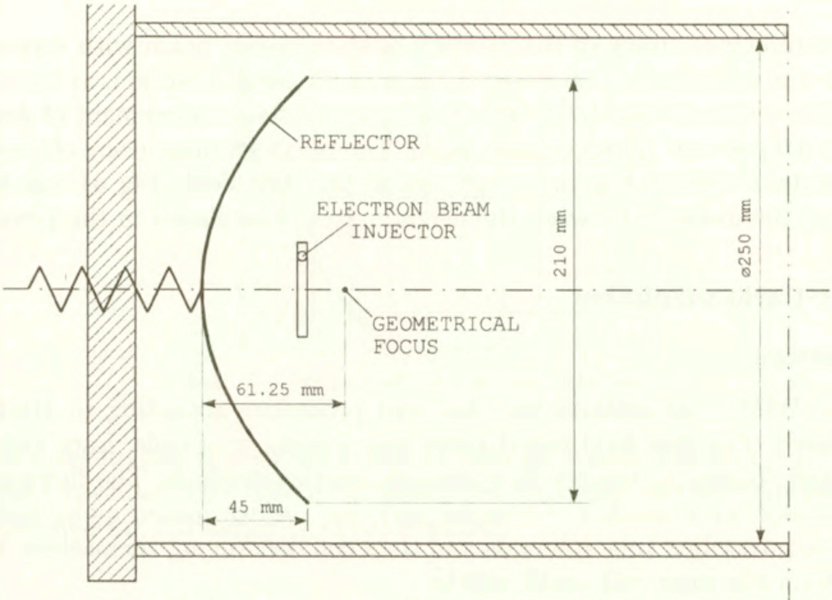

FIG. 1. Shock tube with a parabolic reflector mounted inside.

the density history at one point in the flow was recorded (compare Fig. 3). To obtain the momentary picture of the density field, the results were then recalculated with suitable numerical procedure. The reference time instant, necessary for superposition of the results, was provided in each run by a laser differential interferometer, placed at a fixed point in front of the reflector.

Meaningful results were obtained when the neighbouring locations of the beam were, at the most, one shock wave thickness apart. A special care was needed to maintain good repeatability of the flow conditions. The obtained scatter of shock speeds, ±2.5 per cent of the mean value, was sufficiently small from this point of view.

2.2. Conditions of the experiment

The conditions of experiment were selected as follows:

- argon (spectrally pure) was used as the test gas;
- the shock Mach number (Ms) was 1.60 ± 0.03;
- the initial pressure – 7.33 Pa;
- the initial temperature (equal to room temperature) – 298 ± 1 K.

For such conditions the mean free path of the gas atoms was about 0.95 mm, which enabled us to investigate the shock structure with the electron beam of 0.5 mm diameter (the maximum slope thickness of the incident shock was equal to about 7 mm [7]).

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2.3. Accuracy

The possible inaccuracy of the position of the electron beam with respect to the wedge was estimated to be about 0.1 mm in the axial direction, as well as in the direction perpendicular to it. The inaccuracy of the measurement of density was about ±5 per cent of the current value. The resulting inaccuracy of position of any constant density line in the picture of the flow field (Fig. 4) was lower than half of the distance between the neighbouring lines shown in the picture.

3. Monte–Carlo simulation

3.1. The method

For the DSMC calculations the standard procedure according to BIRD [8] was employed. The flow field was divided into a number of cells. Each cell was chosen small enough to neglect flow nonuniformities inside it. The flowing gas was represented by a number of "model particles", which moved along straight lines during prescribed time intervals and then collided among themselves. Only particles from the same cell could collide.

To simulate the particles the Hard Sphere (HS) model was used. Selection of particles for collision was performed with the ballot-box scheme, proposed by YANITSKIY [9]. Interactions of the particles with physical boundaries were simulated, following MAXWELL [10], with the concept of accommodation coefficient:

$$\alpha = \frac{n_d}{n_d + n_s},$$

where n_d – number of molecules reflected diffusely at the surface, n_s – number of molecules reflected specularly, $n_d + n_s$ – total number of reflected molecules.

If the particle hit the boundary on its way, a random number of molecules was generated. In case when its value was smaller than α, the particle was alowed to reflect specularly, otherwise – diffusely. Total number of cells in the calculations never exceeded 32400; total number of the model particles could not be larger than 160000. The results were averaged over 100 to 500 calculation runs and then smoothed, following the procedure suggested by HONMA *et al.* [11].

3.2. Details of calculation

The matrix of cells employed in the calculations was rectangular. All cells, except those neighbouring the reflector, had the same dimensions: 0.5λ in the direction of wave propagation and 0.8λ in the direction perpendicular to it (λ is the mean free path in the undisturbed gas). The cells neighbouring the reflector were smaller than that; their shapes and sizes resulted from cutting off a part of the original rectangle by the curved reflecting surface (Fig. 2).

The accommodation coefficient at the reflecting surface was assumed to be equal to α = 0.3. Such a value was selected on the basis of the previously

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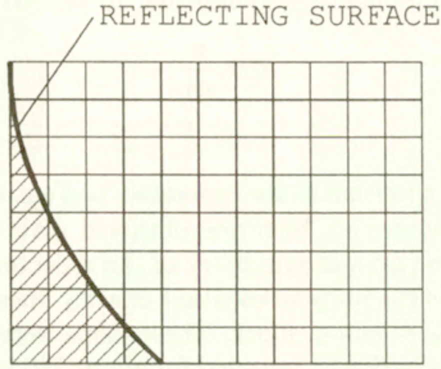

FIG. 2. Cell pattern for DSMC calculations.

obtained results for regular reflection of the shock from the wedge [12]. At the sidewalls, specular reflection of the molecules (accommodation coefficient $\alpha = 0$) was assumed for the majority of calculations.

4. Results

4.1. Form of presentation

The results are presented in the form of isolines of density, temperature, pressure and velocity at the front side of the reflector, for several subsequent time instants. The diagrams of the shock wave structure, showing the distributions of gas density inside the shock, are also shown.

The gas parameters are expressed in non-dimensional form:

Density

$$\overline{\varrho} = \frac{\varrho - \varrho_1}{\varrho_2 - \varrho_1},$$

where ϱ is the current density value, subscripts 1 and 2 correspond to the values in front and behind the incident shock, respectively.

Temperature

$$\overline{T} = \frac{T - T_1}{T_2 - T_1}.$$

Pressure

$$\overline{p} = \frac{p - p_1}{p_2 - p_1}.$$

Horizontal velocity component (parallel to the axis of symmetry)

$$\overline{u} = \frac{u}{u_2}$$

(the velocity in front of the incident shock equals zero).

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Vertical velocity component (perpendicular to the axis)

$$\overline{v} = \frac{v}{u_2}.$$

4.2. Experiment

The results of the experiment are presented in Figs. 3–6. Figure 3 presents five examples of the raw density histories, obtained with the electron beam densitometer, placed in the plane of symmetry of the reflector at five distances from its apex. Figure 4 shows the maps of constant density lines for five (evenly spaced in time) positions of the reflected shock. These maps were obtained from density traces, like those shown in Fig. 3, recorded for about 100 positions of the densitometer with respect to the reflector. Figure 5 (solid line) presents the diagram of the reflected shock trajectory and, finally, Figure 6 shows the density diagrams inside the shock (shock wave structures) for the same positions as Fig. 4.

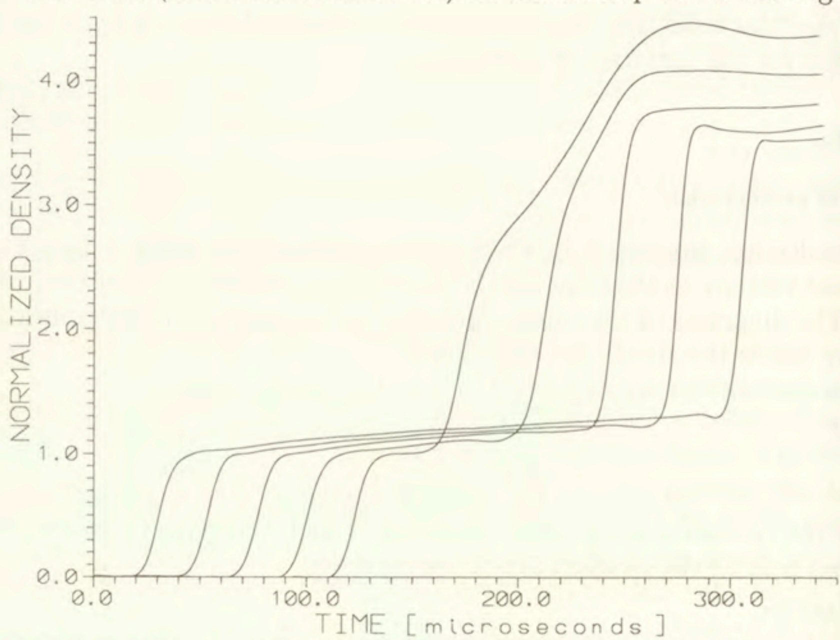

FIG. 3. Density histories at five points on the axis of the reflector (from experiment).

The constant density lines (Fig. 4) do not exhibit strong curvature. The reflected shock wave velocity varies only slightly, as one can infer from the fact, that the shock wave trajectory (Fig. 5, solid line) is close to a straight line. Similarly, the variation of the density increase across the reflected shock (Fig. 6) is not strong.

These findings could not be understood without the information on the whole flow field, not only the area close to the plane of symmetry of the reflector. To gain this information we used the Monte–Carlo numerical calculations.

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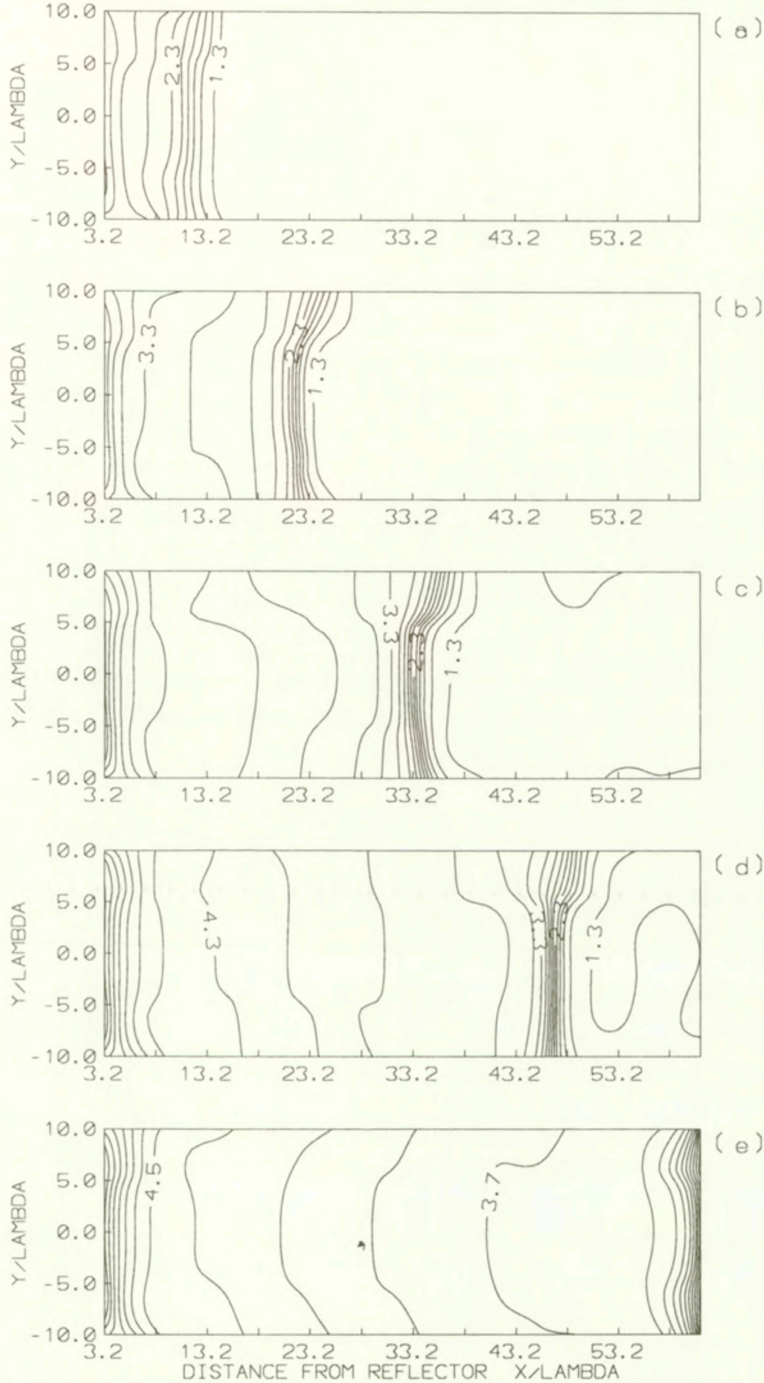

FIG. 4. Constant normalized density contours ($\overline{\varrho}$) for five positions of the reflected shock (from experiment). Distances from the apex of the reflector to the centre of the reflected shock equal to: a) 11.8λ, b) 21.8λ, c) 33.2λ, d) 46.9λ, e) 60.8λ (λ – mean free path of the particles in the undisturbed gas).

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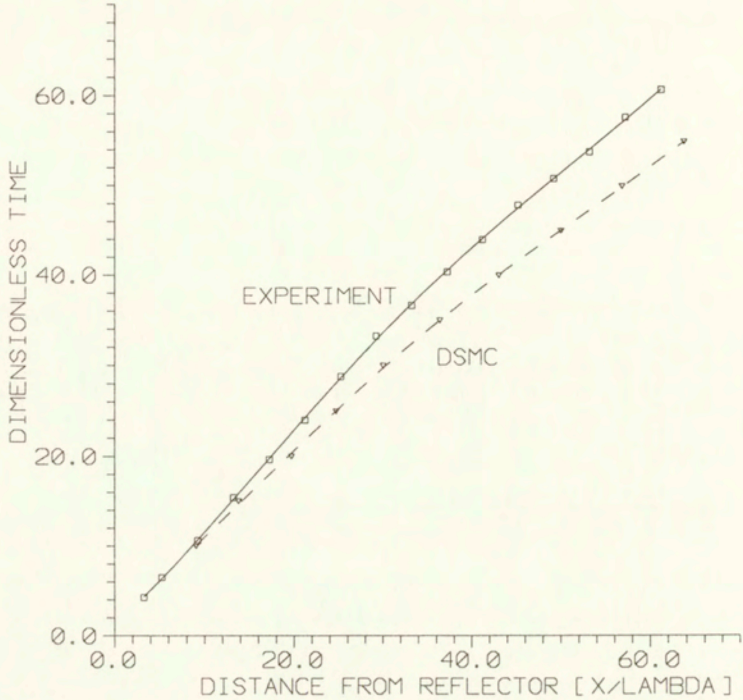

FIG. 5. Reflected shock trajectories from experiment and DSMC calculations.

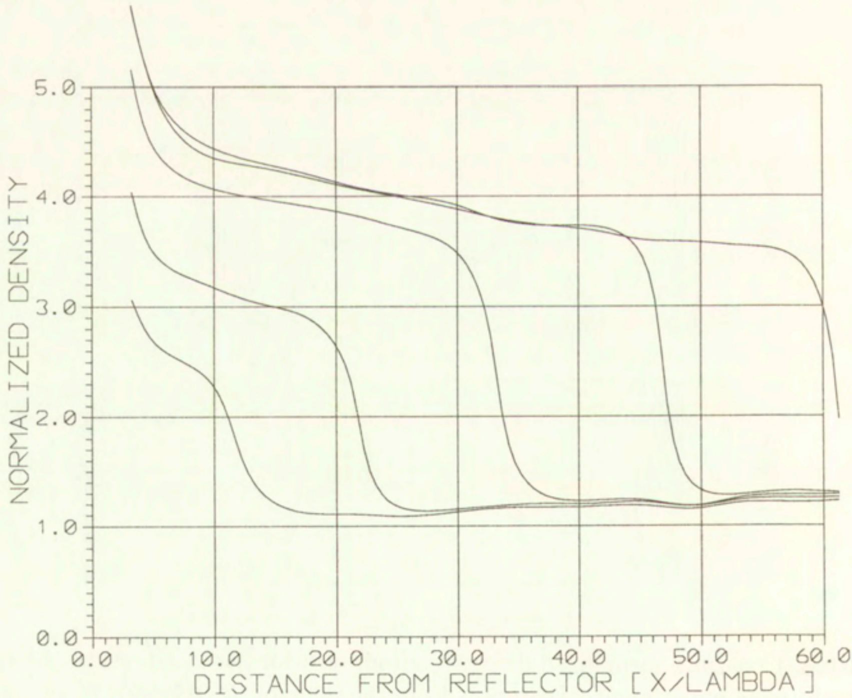

FIG. 6. Density distributions inside the reflected shock for shock positions specified in Fig. 4 (from experiment).

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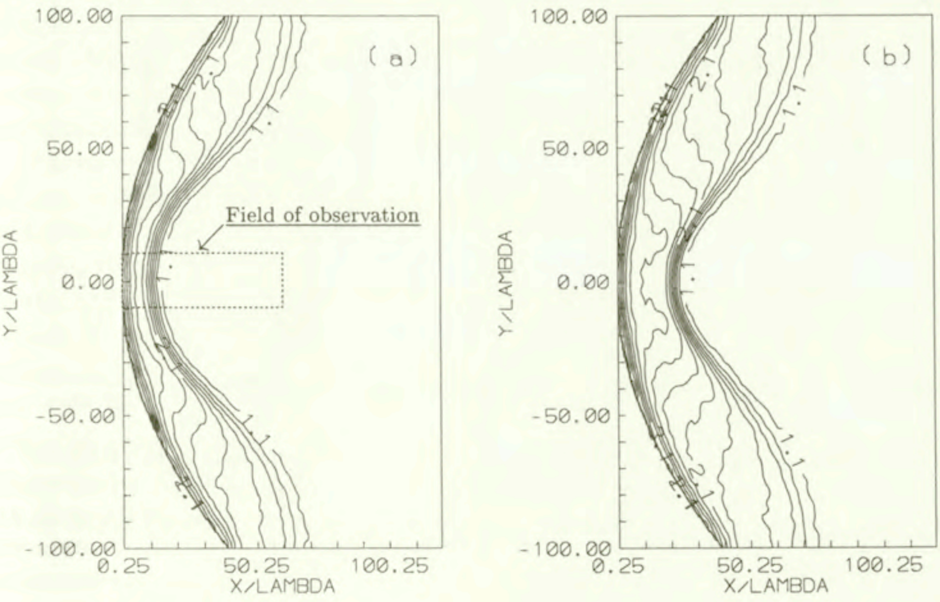
4.3. Monte–Carlo simulation

4.3.1. Parabolic reflector – results and comparison with experiment. The DSMC calculations of the reflected shock focusing were performed for the same flow conditions as the described experiment. The results are shown in the following figures. Figure 7 shows six instantaneous maps of the constant density isolines, five of them, a – e, correspond approximately to the situations shown in Fig. 4. Figure 5 (dashed line) presents the respective diagram of the reflected shock trajectory, while Fig. 8 the corresponding density distributions along the axis of the reflector (reflected shock wave structures at the axis). Figure 9 shows maps of isotherms, isobars and lines of constant values of horizontal and vertical (directed towards the plane of symmetry) velocity components for shock position corresponding to Fig. 7 d (maximum intensity of the shock at the axis).

The above figures allow us to understand the phenomena occurring in the neighbourhood of the shock focus. The gas, flowing in axial direction in the area behind the incident shock, turns towards the plane of symmetry when passing through the reflected one (compare Fig. 9 d, showing areas where gas velocity has vertical component directed towards the axis). Thus, behind the reflected shock two streams flow in opposite directions, meeting at the plane of symmetry. The highest pressure, temperature, density are produced there. In consequence, the reflected shock at the plane of symmetry is stronger than far from it. As the stronger shock moves faster with respect to the gas in front of it, the central part of the reflected shock becomes plane (Fig. 7 e) and then convex (Fig. 7 f). This takes place before the shock reaches the geometrical focus of the mirror and thus limits the focusing process.



[FIG. 7 a, b]

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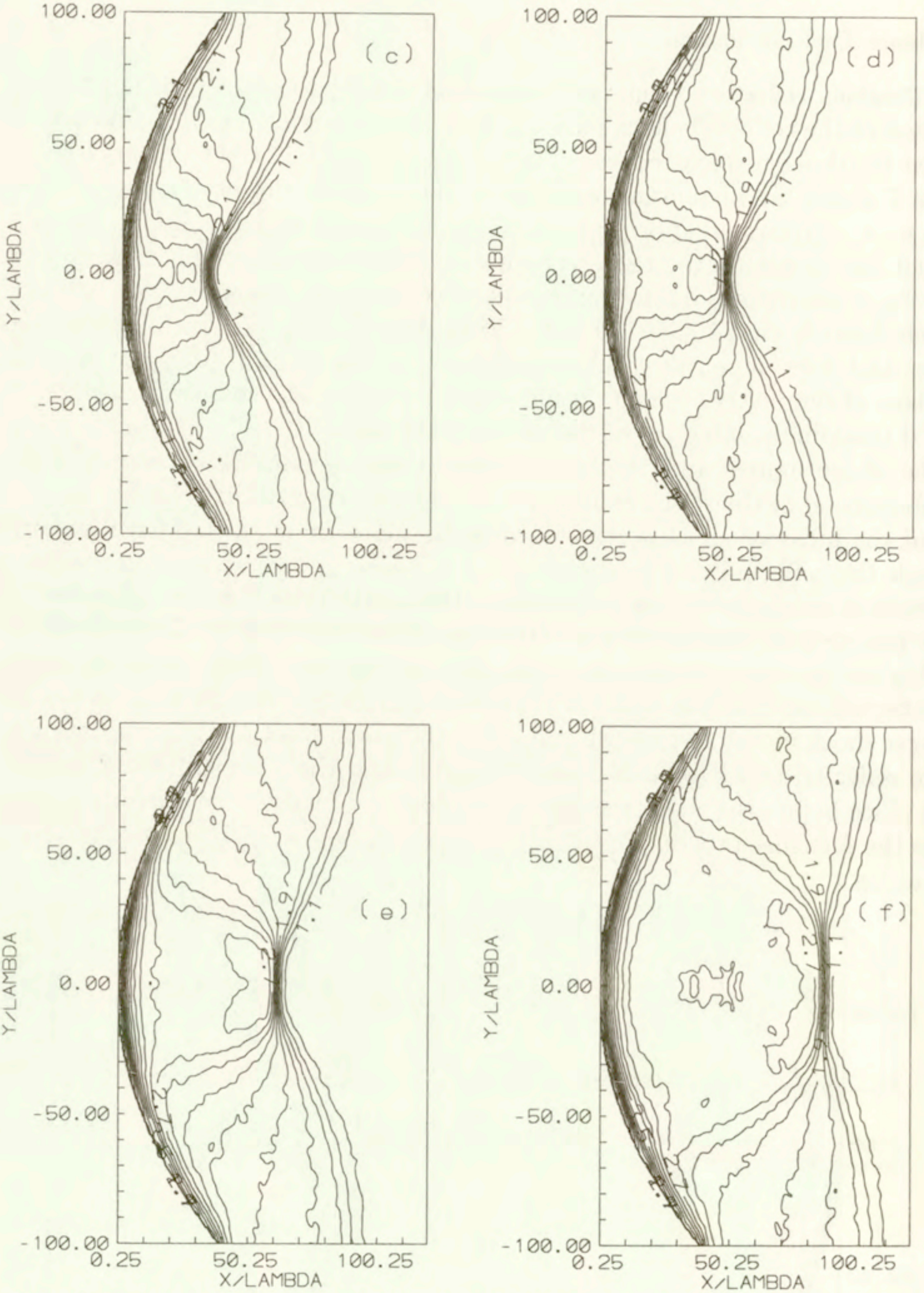

FIG. 7. Constant normalized density contours ($\overline{\varrho}$) for six positions of the reflected shock (from DSMC calculations). Distances from the apex of the reflector to the centre of the reflected shock equal to: a) 11.1λ, b) 21.7λ, c) 33.4λ, d) 47.3λ, e) 61.0λ, f) 86.2λ.

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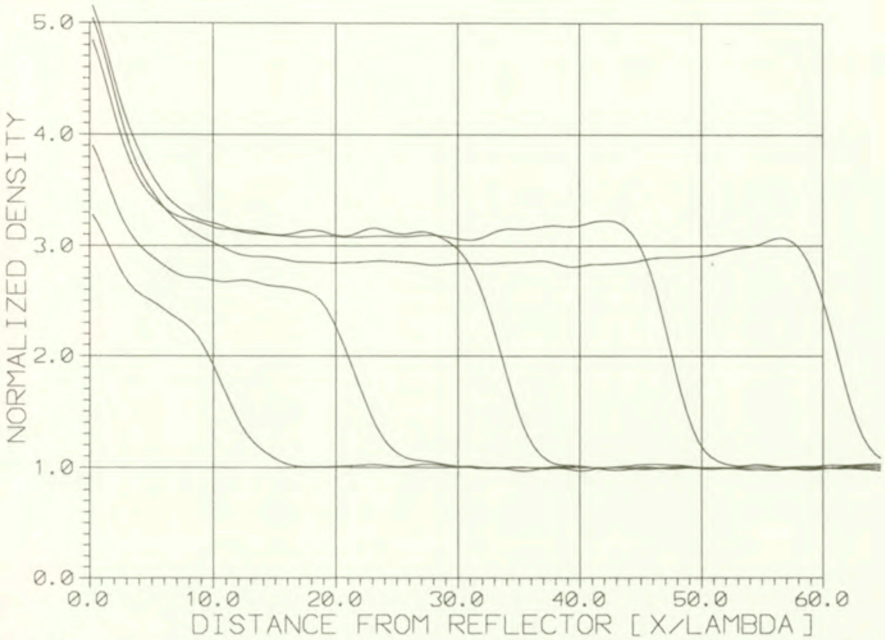

FIG. 8. Density distributions inside the reflected shock at the axis of the reflector, for five shock positions specified in Fig. 7 a – e (from DSMC calculations).

Similar physical explanation is mentioned briefly by STURTEVANT and KULKARNY [2] when they describe the differences between linear (geometrical) and nonlinear approaches to the focusing of the shock wave. The behaviour of the gasdynamic focus was briefly described by NISHIDA *et al.* [13], and the effects of the changing shape (from the concave to the convex) could be seen on the isobars shown by NISHIDA and KISHIGE [14] (their Figs. 3 – 5), but no physical explanation was offered in these papers.

There is a good agreement between the Monte–Carlo simulation and experimental results. The shapes of the constant density lines from experiment (Fig. 4) and DSMC calculations (Fig. 7) show close resemblance, provided that one keeps in mind that Fig. 4 presents only the narrow part of the flow field, corresponding to the central part of Fig. 7 (as marked there by the dashed line). Actually, the "shock focus" takes nearly the whole width of the experimental field of observation (Fig. 4). It may therefore be concluded, that dimension of the "focus" in the direction perpendicular to the tube axis is equal to about 10 – 15 mean free paths of the gas particles in front of the incident shock. Similar estimation can be made on the basis of Fig. 7.

The shapes of the reflected shock trajectories from experiment and DSMC simulation, as shown in Fig. 5, are also similar, although in the experiment the reflected shock is definitely slower. The density distributions along the axis of the reflector (Figs. 6 and 8) exhibit perhaps the most marked differences: the

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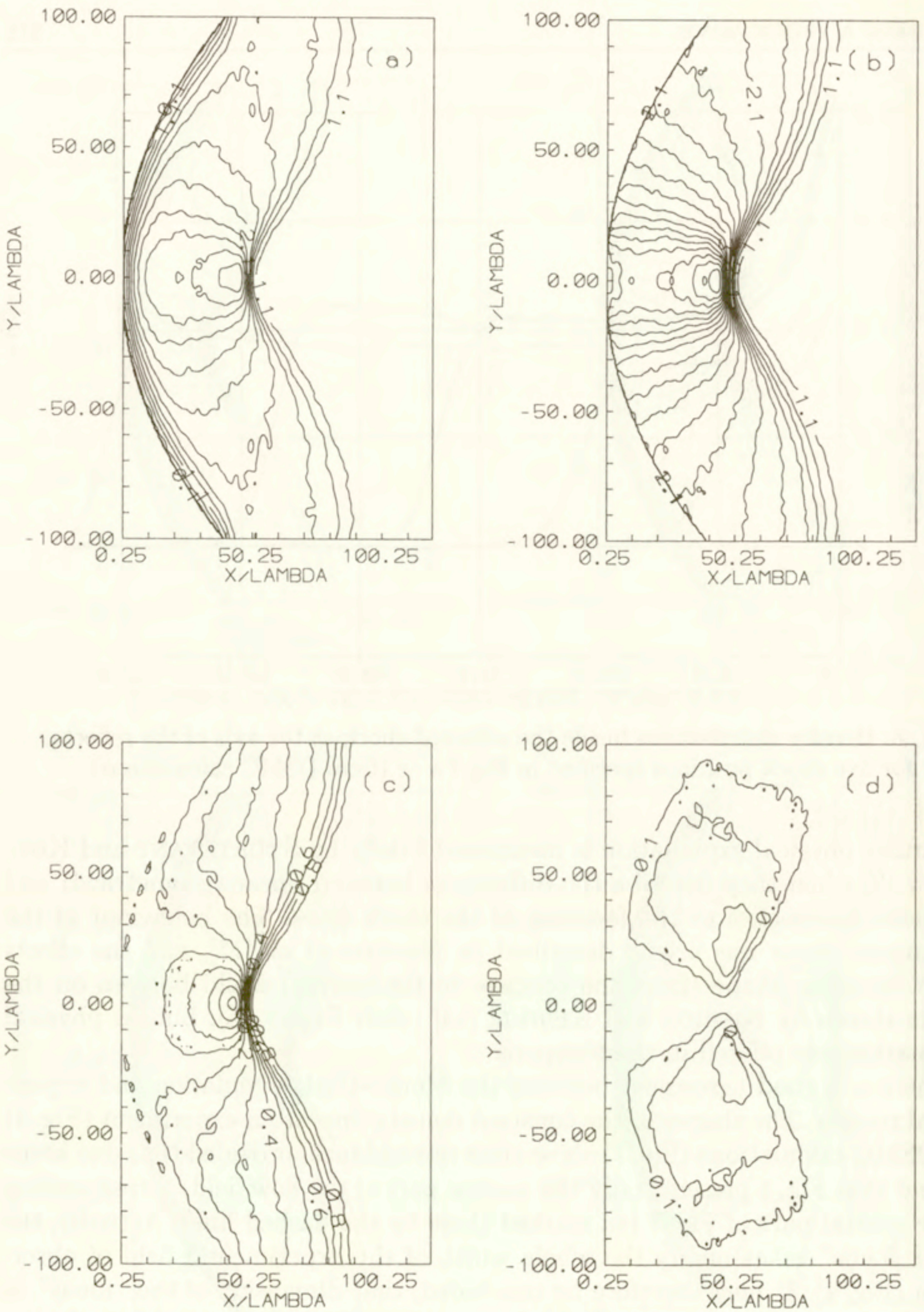

FIG. 9. Contours of constant normalized: a) temperature ($\overline{T}$), b) pressure ($\overline{p}$), c) horizontal ($\overline{u}$) and d) vertical ($\overline{v}$) velocity components for shock position specified in Fig. 7 d (from DSMC calculations).

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density increase across the reflected shock is substantially higher in the experiment than in the simulation and, moreover, behind the reflected shock the gas density further increases, while in the simulation there is a definite minimum of density in this area.

The described differences result from flow nonuniformity in front of the reflected shock wave, clearly visible in Figs. 3 and 6 in the form of density variation in this area. This nonuniformity is due to the influence of boundary layer at the sidewalls of the shock tube. To verify this supposition, a relatively simple calculation was made of a plane shock wave, moving in a narrow channel with nonzero accommodation coefficient at the walls, reflecting from a plane endwall of the channel. The results indicate both the increase of gas density behind the incident shock, and the increase of density behind the reflected shock. Similar effect has also been obtained experimentally by PIVA [15].

One more point should be mentioned here: in the reported experiment the reflector did not span the whole distance between the walls of the shock tube (Fig. 1), as it did in the simulation (Fig. 2). To check what influence it could have upon the focusing process, additional calculations were performed for suitable geometry. The obtained results indicate, that noticeable differences are visible only in the vicinity of the edges of the reflector. The density distributions along the axis of the reflector for this geometry are nearly identical with those for the standard case, shown in Fig. 8. It should be pointed out here, that STURTEVANT and KULKARNY [2] obtained a similar result experimentally.

4.3.2. Other shapes of the reflector. Calculations described up to this point were done for parabolic reflectors. This shape, however, was obtained from geometrical acoustics (linear theory); it is not necessarily the best one for reflecting shock waves (nonlinear phenomena). It was therefore interesting to see the shock focusing with reflectors of other shapes. Apart from that, as no technologically produced shape is perfect, it was important from the practical point of view to know the effect of the possible inaccuracy of the shape of the reflector.

The calculations were performed for reflectors of the same depth as before, with shapes described by the equation:

$$x = cy^{\beta},$$

where x – coordinate along the shock tube axis, y – distance from the plane of symmetry. Exponent β was taken equal to 1.5 and 2.5, constant c was calculated to obtain the required depth of the reflector.

The results are quite similar to those for a parabolic reflector. To explain it we inspected carefully the considered shapes. We found, that for the same distances from the plane of symmetry (y-coordinate), the differences between the corresponding x-coordinates were smaller than the size of the "focus". Most probably then, in order to influence the phenomenon more substantially, the

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disturbance of the shape of the reflector would have to be much larger than the dimension of the focus.

5. Comparison with high-density results

The results presented here, both experimental and numerical, agree qualitatively with those of other authors for high gas densities. The shapes of the waves from Monte–Carlo simulation (Fig. 7) are similar to those obtained for strong shocks experimentally by STURTEVANT and KULKARNY ([2] – Fig. 4 of that paper), or numerically by NISHIDA and KISHIGE ([14] – Fig. 4 there). Similarly, our experimental results (Fig. 4) agree with corresponding parts of their pictures.

Exact quantitative comparison is, unfortunately, impossible because of different geometrical and flow conditions (depth of the reflector, shock Mach number, gas density). The most visible difference is, of course, that at low density all waves are of finite thickness, "more widely spread". Still, the waves visible at high-density pictures of the flow can also be recognized at low densities (unless they are too weak to be detected). Hence, it may be stated, that the present investigation in the rarefied flow regime confirms the results known from research at high gas density.

6. Conclusions

1. The presented experimental results supply the information about the process of formation, size and structure of the focus of the "strong" shock after its reflection from a concave wall.

2. The results of the DSMC simulation are in good agreement with experiment. They make it also possible to understand the mechanism of focusing a shock after reflection, with all its inherent limitations. In particular, the fact that the shock after reflection from a concave wall is never uniformly strong, puts the lower bound upon the size of the focus which, in turn, limits the maximum intensity of the focused shock.

3. The present results agree qualitatively with results of other authors, both experimental and numerical, obtained for high density gases.

Acknowledgments

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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

On anisotropic functions of vectors and second order tensors – all subgroups of the transverse isotropy group $C_{\infty h}$

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A UNIFIED PROCEDURE for constructing both the generating sets and the functional bases is suggested, which reduces the representation problem for anisotropic functions of any finite number of vector and second order tensor variables under any subgroup $g \subset C_{\infty h}$, to that for the same types of anisotropic functions of not more than two vector and/or second order tensor variables. By using this procedure and new results for isotropic extension of anisotropic functions, simple irreducible generating sets and functional bases in unified forms are presented to determine general reduced forms of scalar-, vector-, and second order tensor-valued anisotropic functions of any finite number of vectors and second order tensors, under all kinds of subgroups of the transverse isotropy group $C_{\infty h}$. The results given are derived in the sense of nonpolynomial representation.

1. Introduction

IN CONTINUUM PHYSICS, scalar-, vector- and second order tensor-valued functions of vector and second order tensor[1] variables serve as mathematical models of macroscopic physical behaviours of materials. Such tensor functions are required to possess form-invariance under the action of the material symmetry group due to material objectivity and material symmetry. Let $f(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$, $\mathbf{h}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ and $\mathbf{F}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ be, respectively, scalar-, vector- and second order tensor-valued functions of the a vector variables $\mathbf{u}_i$, the b skewsymmetric second order tensor variables $\mathbf{W}_\sigma$, and the c symmetric second order tensor variables $\mathbf{A}_L$, where $i = 1, \ldots, a$; $\sigma = 1, \ldots, b$ and $L = 1, \ldots, c$. Moreover, let g be a subgroup of the full orthogonal group Orth, which may serve as a material symmetry group for solid materials. The tensor functions f, $\mathbf{h}$ and $\mathbf{F}$ are invariant or form-invariant under the group g, respectively, if for every orthogonal tensor $\mathbf{Q} \in g$,

$$
\begin{aligned}
f(\mathbf{Q}\mathbf{u}_i, \mathbf{Q}\mathbf{W}_\sigma\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_L\mathbf{Q}^{\mathrm{T}}) &= f(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L), \\
\mathbf{h}(\mathbf{Q}\mathbf{u}_i, \mathbf{Q}\mathbf{W}_\sigma\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_L\mathbf{Q}^{\mathrm{T}}) &= \mathbf{Q}\mathbf{h}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L), \\
\mathbf{F}(\mathbf{Q}\mathbf{u}_i, \mathbf{Q}\mathbf{W}_\sigma\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_L\mathbf{Q}^{\mathrm{T}}) &= \mathbf{Q}\mathbf{F}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)\mathbf{Q}^{\mathrm{T}}.
\end{aligned}
$$

General reduced forms of the tensor functions f, $\mathbf{h}$ and $\mathbf{F}$ under the above invariance restrictions are called *representations* for f, $\mathbf{h}$ and $\mathbf{F}$ under the group

(1) Throughout the paper, vector and tensor mean three-dimensional vectors and tensors.

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g. It has been known (see PIPKIN and WINEMAN [15], WINEMAN and PIPKIN [30]) that finding representations for the tensor functions f, $\mathbf{h}$ and $\mathbf{F}$ under the group g is equivalent to determining *functional bases* (for f) and *generating sets* (for $\mathbf{h}$ and $\mathbf{F}$) under the group g. Moreover, both the functional bases and the generating sets to be used are further required to be *irreducible* in order to arrive at compact representations. For detail, refer to the definitions given later.

Proper subgroups of the full orthogonal group Orth include the proper orthogonal group Orth^+, the five classes of cubic crystal groups, the two classes of icosahedral groups, the five classes of transverse isotropy groups and the denumerably infinitely many classes of subgroups of the latter (see, e.g., SPENCER [26], VAINSHTEIN [28]). Of them, the 32 classes of crystallographic point groups, the five classes of transverse isotropy groups and Orth (see SPENCER [26]) are related to common solid materials in engineering, while the others are associated with quasi-crystalline solids and texture materials (see VAINSHTEIN [28]), etc. In the past decades, many efforts were devoted to finding representations for various kinds of tensor functions under the aforementioned orthogonal subgroups, and many significant results were obtained. Here, we would not reproduce the large number of the related references. For details, refer to TRUESDELL and NOLL [27], SPENCER [26], KIRAL and ERINGEN [10], SMITH [21] and the recent reviews by BETTEN [4], RYCHLEWSKI and ZHANG [17], and ZHENG [43] *et al.*, and to the references therein. Although now many results in many cases, mainly in the sense of polynomial representation, are available (see SPENCER [26], KIRAL and ERINGEN [10], and SMITH [21]), general aspect of representation problems, mainly in the sense of nonpolynomial representation, remains open, except for some particular cases such as isotropic, orthotropic and transversely isotropic functions etc. (see ADKINS [1–3], PIPKIN and RIVLIN [14], WANG [29], SMITH [19], BOEHLER [5], and PENNISI and TROVATO [13], SMITH [20], ZHENG [41], JEMIOŁO and TELEGA [8], *et al.*). Applying the isotropic extension method for anisotropic functions, initiated by LOKHIN and Sedov [12] and BOEHLER [6, 7] and LIU [11] (see also RYCHLEWSKI [16], ZHANG and RYCHLEWSKI [40], ZHENG and SPENCER [44]) and further developed recently by the author (see XIAO and GUO [31], XIAO [35, 39]), as well as the general representation theorems given in XIAO [33, 34], we shall derive general irreducible representations for scalar-, vector- and second order tensor-valued anisotropic functions of any finite number of vectors and second order tensors, under various kinds of orthogonal subgroups, in a series of works. In this paper, we shall confine ourselves to anisotropic functions under subgroups of the transverse isotropy group $C_{\infty h}$ (see PIPKIN and RIVLIN [14], SMITH and RIVLIN [23, 24], SMITH, SMITH and RIVLIN [25], SMITH and KIRAL [22], SMITH [18, 20, 21], SPENCER [26], KIRAL and SMITH [9], ZHENG [41, 42, 43], XIAO [32, 36, 37], *et al.* for related results in some cases; in particular, see ZHENG [42] for the counterpart of the two-dimmentional case of the problem considered here).

At the end of this introduction, we state some facts that will be used.

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Throughout the paper, V, Skw and Sym are used to represent the vector space, the skewsymmetric and symmetric second order tensor spaces, respectively. $\mathbf{u}$, $\mathbf{v}$, $\mathbf{r}$ etc., $\mathbf{W}$, $\mathbf{H}$ etc., $\mathbf{A}$, $\mathbf{B}$ etc. are used to denote vectors, skewsymmetric second order tensors and symmetric second order tensors, respectively. On the other hand, $g(X)$ is the symmetry group of the set X of vectors and tensors (see XIAO [33]) and for any subgroup $g \subset \mathrm{Orth}$, $M(g)$ is the g-subspace of the space $M \in \{V, \mathrm{Skw}, \mathrm{Sym}\}$ (see XIAO [32, 33]). The former consists of the orthogonal tensors preserving X, and the latter includes the elements in M that are invariant under the action of the group g.

A finite set of scalar-valued functions that are invariant under the group $g \subset \mathrm{Orth}$, i.e. invariants under g, is a functional basis under g if each invariant under g is expressible as a real single-valued function of these invariants. On the other hand, a finite set of vector-valued (resp. second order tensor-valued) functions that are form-invariant under the group $g \subset \mathrm{Orth}$, i.e. vector (resp. second order tensor) generators under g, is a generating set under g if each vector-valued (resp. second order tensor-valued) function that is form-invariant under g is expressible as a linear combination of these generators whose coefficients are invariants under g. Furthermore, a functinal basis (resp. a generating set) under the group $g \subset \mathrm{Orth}$ is irreducible if none of its proper subsets is again a functional basis (resp. a generating set) under the group g. A criterion for generating set is as follows (see XIAO [33]).

The vector-valued or skewsymmetric tensor-valued or symmetric tensor-valued functions $\boldsymbol{\psi}_1, \cdots, \boldsymbol{\psi}_r$ that are form-invariant under the group $g \subset \mathrm{Orth}$ form a generating set under g iff

$$\forall X \in V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c : \ \mathrm{rank}\{\boldsymbol{\psi}_1(X), \cdots, \boldsymbol{\psi}_r(X)\} \geq \dim M(g \cap g(X)), \tag{1.1}$$

where $M = V$, Skw, Sym, respectively, when $\boldsymbol{\psi}_i$ is vector-valued, skewsymmetric tensor-valued and symmetric tensor-valued, respectively. Here, for any set S of vectors or tensors and any subspace L, the notation $\mathrm{rank}S$ and $\dim L$ are used to represent the number of the linearly independent elements in the set S and the dimension of the subspace L, respectively.

To facilitate the application of the above criterion, we list the following facts for the subgroups $g \subset C_{\infty h}$.

$$\dim V(g) = \begin{cases} 1, & g \subset \mathrm{Orth}^+, \quad g \neq C_1 , \\ 2, & g = C_{1h} , \\ 3, & g = C_1 , \\ 0, & \text{otherwise} ; \end{cases} \tag{1.2}$$

$$\dim \mathrm{Skw}(g) = \begin{cases} 3, & g = C_1, S_2 , \\ 1, & \text{otherwise} ; \end{cases} \tag{1.3}$$

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$$\dim \mathrm{Sym}(g) = \begin{cases} 6, & g = C_1, S_2\,, \\ 4, & g = C_2, C_{1h}, C_{2h}\,, \\ 2, & \text{otherwise}\,. \end{cases} \tag{1.4}$$

The groups appearing above will be given later. On the other hand, a criterion for functional bases is as follows (see PIPKIN and WINEMAN [15, 30]; see also XIAO [33]).

The invariants $f_1, \cdots, f_r$ under the group $g \subset \mathrm{Orth}$ form a functional basis under g iff

$$f_1(\overline{X}) = f_1(X), \cdots, f_r(\overline{X}) = f_r(X) \Longrightarrow \exists\, \mathbf{Q} \in g:\ \overline{X} = \mathbf{Q} \star X\,, \tag{1.5}$$

for $\overline{X}$, $X \in V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$. The latter means that $\overline{X}$ and X pertain to the same g-orbit. Thus, the variable X is determined to within an orthogonal tensor pertaining to the group g.

To check the irreducibility of a given functional basis, we shall use the following fact.

A functional basis I under the group $g \subset \mathrm{Orth}$ is irreducible iff for any given element $f_0 \in I$ there exist X, $X' \in V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$, which belong to two different g-orbits, such that

$$f_0(X) \neq f_0(X') \ \ \&\ \ f(X) = f(X') \quad \text{for all } f \in I/\{f_0\}\,. \tag{1.6}$$

In fact, for any given element f_0, the proper subset $I/\{f_0\}$ can not be a functional basis under g if (1.6) holds, or else according to the aforementioned criterion for functional bases, X and X' must pertain to the same g-orbit (see (1.5)) and hence $f_0(X) = f_0(X')$, which contradicts $(1.6)_1$.

Let S be a functional basis or generating set under the group g. We shall speak of the irreducibility of an element in S. By this we mean the following fact: an invariant or a generator $\chi \in S$ is irreducible if the proper subset $S/\{\chi\}$ fails to supply a functional basis or generating set under g. Obviously, a functional basis or generating set is irreducible iff each element of it is irreducible.

By means of the Schoenflies symbol we list the transverse isotropy groups $C_{\infty h}$ and all its finite subgroups as follows.

$$\begin{aligned}
C_{\infty h}(\mathbf{n}) &= \{\pm \mathbf{R}_{\mathbf{n}}^{\theta} \mid \theta \in R\}, \\
C_{\infty}(\mathbf{n}) &= C_{\infty h}(\mathbf{n}) \cap \mathrm{Orth}^+; \\
C_1 &= \{\mathbf{I}\}, \qquad S_2 = \{\pm \mathbf{I}\}, \\
C_{1h}(\mathbf{n}) &= \{\mathbf{I}, -\mathbf{R}_{\mathbf{n}}^{\pi}\}, \qquad C_2(\mathbf{n}) = \{\mathbf{I}, \mathbf{R}_{\mathbf{n}}^{\pi}\}, \qquad C_{2h}(\mathbf{n}) = \{\pm \mathbf{I}, \pm \mathbf{R}_{\mathbf{n}}^{\pi}\}; \\
S_{4m+2}(\mathbf{n}) &= \{\pm \mathbf{R}_{\mathbf{n}}^{2k\pi/2m+1} \mid k = 1, \cdots, 2m+1\}, \\
C_{2mh}(\mathbf{n}) &= \{\pm \mathbf{R}_{\mathbf{n}}^{k\pi/m} \mid k = 1, \cdots, 2m\}, \\
C_{2m+1}(\mathbf{n}) &= S_{4m+2}(\mathbf{n}) \cap \mathrm{Orth}^+, \qquad C_{2m}(\mathbf{n}) = C_{2mh}(\mathbf{n}) \cap \mathrm{Orth}^+;
\end{aligned}$$

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$$S_{4m}(\mathbf{n}) = \{(-\mathbf{R}_{\mathbf{n}}^{\pi/2m})^k \mid k = 1, \cdots, 4m\},$$
$$C_{2m+1h}(\mathbf{n}) = \{(-\mathbf{R}_{\mathbf{n}}^{\pi/2m+1})^k \mid k = 1, \cdots, 4m+2\}.$$

Here and hereafter $\mathbf{n}$ is a given unit vector and $\mathbf{I}$ is the identity tensor. Henceforth, we shall quote the above subgroups by dropping out the defining vector $\mathbf{n}$ when no confusion arises.

For $\mathbf{u}, \mathbf{v}, \mathbf{r} \in V$ and $\mathbf{A} \in \mathrm{Sym}$, we introduce the following notations.

$$\begin{aligned} \overset{\circ}{\mathbf{u}} &= \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \\ \overset{\circ}{\mathbf{A}} &= \mathbf{A} - (\mathbf{n}\cdot\mathbf{A}\mathbf{n})\mathbf{n}\otimes\mathbf{n}; \end{aligned} \tag{1.7}$$

$$\mathbf{q}(\mathbf{A}) = \frac{1}{2}(\mathbf{e}\cdot\mathbf{A}\mathbf{e} - \mathbf{e}'\cdot\mathbf{A}\mathbf{e}')\mathbf{e} + (\mathbf{e}\cdot\mathbf{A}\mathbf{e}')\mathbf{e}'. \tag{1.8}$$

Throughout, $\mathbf{e}$ is any given unit vector in the $\mathbf{n}$-plane and

$$\mathbf{e}' = \mathbf{n}\times\mathbf{e}. \tag{1.9}$$

Hence, the triplet $(\mathbf{n}, \mathbf{e}, \mathbf{e}')$ is an orthonormal system. It is evident that $(1.7)_{1,2}$ define two linear functions of $\mathbf{u}$ and $\mathbf{A}$ that are form-invariant under the maximal transverse isotropy group $D_{\infty h}(\mathbf{n})$.

Moreover, we denote

$$\begin{aligned} \mathbf{u}\wedge\mathbf{v} &= \mathbf{u}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{u}, \\ \mathbf{u}\vee\mathbf{v} &= \mathbf{u}\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}, \\ [\mathbf{u}, \mathbf{v}, \mathbf{r}] &= \mathbf{u}\cdot(\mathbf{v}\times\mathbf{r}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{r} = \mathbf{v}\cdot(\mathbf{r}\times\mathbf{u}), \end{aligned} \tag{1.10}$$

for vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{r}$. Throughout, $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\times\mathbf{v}$ are used to designate the inner product and the vector product of $\mathbf{u}$ and $\mathbf{v}$, respectively.

Let $<\mathbf{u}, \mathbf{v}>$ denote the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$. For any vector $\mathbf{z}$ in the $\mathbf{n}$-plane and for each positive integer r, the following notations are used:

$$\boldsymbol{\rho}_r(\mathbf{z}) = |\mathbf{z}|^r(\mathbf{e}\cos r <\mathbf{z}, \mathbf{e}> - \mathbf{e}'\sin r <\mathbf{z}, \mathbf{e}>); \tag{1.11}$$

$$\alpha_r(\mathbf{z}) = |\mathbf{z}|^r \cos r <\mathbf{z}, \mathbf{e}>, \qquad \beta_r(\mathbf{z}) = |\mathbf{z}|^r \sin r <\mathbf{z}, \mathbf{e}>. \tag{1.12}$$

When $\mathbf{z} = \mathbf{0}$, the angle $<\mathbf{z}, \mathbf{e}>$ is assumed to be zero. By means of the formulas

$$\begin{aligned} \cos <\mathbf{z}, \mathbf{e}> &= \frac{\mathbf{z}\cdot\mathbf{e}}{|\mathbf{z}|}, \\ \sin <\mathbf{z}, \mathbf{e}> &= \frac{\mathbf{z}\cdot\mathbf{e}'}{|\mathbf{z}|}, \end{aligned} \tag{1.13}$$

and Tschebysheff polynomials, it can easily be proved that each tensor function defined by (1.11) – (1.12) is a polynomial of the components $\mathbf{z}\cdot\mathbf{e}$ and $\mathbf{z}\cdot\mathbf{e}'$.

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2. A unified scheme of constructing functional bases and generating sets

Usually, quite different methods are used to derive functional bases and generating sets separately, which are generally cumbersome. In this section we shall describe a simple, unified scheme for constructing both functional bases and generating sets. Such a scheme enables us to derive generating sets for general vector-valued and second order tensor-valued anisotropic functions only from those for the same types of vector-valued and second order tensor-valued anisotropic functions with not more than three variables (see XIAO [33]) (two variables for the anisotropic functions considered here; see XIAO [34]) and, especially, at the same time it enables us to obtain functional bases for scalar-valued anisotropic functions directly using the generating sets obtained for vector-valued and second order tensor-valued functions. For the sake of definiteness, in the subsequent account of such a unified scheme we shall consider only the anisotropy groups of interest in this paper.

Let g be any subgroup of $C_{\infty h}$. Henceforth, we denote the domain $V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$ by $\mathcal{D}$. For any given set of vectors and second order tensors, $X = (\mathbf{u}_1, \cdots, \mathbf{u}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{W}_c) \in \mathcal{D}$, by means of the following facts

$$(2.1) \qquad g(\mathbf{W}) \cap C_{\infty h}(\mathbf{n}) = \begin{cases} C_{1h}(\mathbf{n}), & \mathbf{Wn} = \mathbf{0}\,, \\ C_1, & \mathbf{Wn} \neq \mathbf{0}\,, \end{cases}$$

$$(2.2) \qquad g(\mathbf{A}) \cap C_{\infty h}(\mathbf{n}) = \begin{cases} C_{\infty h}(\mathbf{n}), & \mathbf{A} = x\mathbf{I} + y\mathbf{n} \otimes \mathbf{n}\,, \\ C_{2h}(\mathbf{n}), & \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0}, \quad \mathbf{q}(\mathbf{A}) \neq \mathbf{0}\,, \\ S_2, & \overset{\circ}{\mathbf{A}}\mathbf{n} \neq \mathbf{0}\,, \end{cases}$$

$$(2.3) \qquad g(\mathbf{u}) \cap C_{\infty h}(\mathbf{n}) = \begin{cases} C_{\infty h}(\mathbf{n}), & \mathbf{u} = \mathbf{0}\,, \\ C_{\infty}(\mathbf{n}), & \mathbf{u} = x\mathbf{n}, \quad x \neq 0\,, \\ C_{1h}(\mathbf{n}), & \mathbf{u}\cdot\mathbf{n} = 0, \quad \mathbf{n} \times \mathbf{u} \neq \mathbf{0}\,, \\ C_1, & (\mathbf{u}\cdot\mathbf{n})(\mathbf{n} \times \mathbf{u}) \neq \mathbf{0}\,, \end{cases}$$

it can be proved (see §2.2 in XIAO [34]) that there are $X_0 \subset X$, where $X_0 \in \{(\mathbf{u}, \mathbf{v}), (\mathbf{u}, \mathbf{W}), (\mathbf{u}, \mathbf{A})\}$ is a subset of X with two elments, such that

$$(2.4) \qquad g \cap g(X) = g \cap g(X_0)\,,$$

and accordingly, that irreducible generating sets for general anisotropic vector-valued and second order tensor-valued functions of the variables X relative to the group $g \subset C_{\infty h}$ can be formed by union of irreducible generating sets for the same types of anisotropic functions of not more than two variables (see Theorem 2.4 in XIAO [34]). Generally, the just-stated fact implies that if $G(\mathbf{u}, \mathbf{v})$, $G(\mathbf{u}, \mathbf{W})$ and $G(\mathbf{u}, \mathbf{A})$ are irreducible generating sets for vector-valued or skewsymmetric

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tensor-valued or symmetric tensor-valued anisotropic functions of the two variables $(\mathbf{u}, \mathbf{v})$, $(\mathbf{u}, \mathbf{W})$ and $(\mathbf{u}, \mathbf{A})$ relative to the subgroup $g \subset C_{\infty h}$ respectively, then the union

$$\bigcup_{i,j=1}^{a} \bigcup_{\alpha=1}^{b} \bigcup_{\sigma=1}^{c} (G(\mathbf{u}_i, \mathbf{u}_j) \cup G(\mathbf{u}_i, \mathbf{W}_\alpha) \cup G(\mathbf{u}_i, \mathbf{A}_\sigma))$$

furnishes an irreducible generating set for vector-valued or skewsymmetric tensor-valued or symmetric tensor-valued anisotropic functions of the vectors $\mathbf{u}_1, \cdots, \mathbf{u}_a$, the skewsymmetric tensors $\mathbf{W}_1, \cdots, \mathbf{W}_b$ and the symmetric tensors $\mathbf{A}_1, \cdots, \mathbf{A}_c$ relative to the subgroup $g \subset C_{\infty h}$. In the above, each generating set of two variables may be constructed by applying the related results for isotropic extension of anisotropic functions given in XIAO [35, 39]. Moreover, in constructing generating sets for one and two variables we can, as mentioned before, derive functional bases for general scalar-valued anisotropic functions of the variables X, directly utilizing the results for the generating sets obtained. To realize this goal the following fact is essential.

Let $X \in \mathcal{D}$ be a set of vectors and second order tensors with a proper subset X_0 satisfying (2.4). Moreover, let $I(X_0)$ and

$$V(X_0) = \{\mathbf{h}_1(X_0), \cdots, \mathbf{h}_r(X_0)\},$$
$$\mathrm{Skw}(X_0) = \{\boldsymbol{\Omega}_1(X_0), \cdots, \boldsymbol{\Omega}_s(X_0)\},$$
$$\mathrm{Sym}(X_0) = \{\boldsymbol{\Psi}_1(X_0), \cdots, \boldsymbol{\Psi}_t(X_0)\},$$

be a functional basis and generating sets for vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables X_0 relative to the group g, respectively. For a generic vector $\mathbf{r} \in X/X_0$, a generic skewsymmetric tensor $\mathbf{H} \in X/X_0$ and a generic symmetric tensor $\mathbf{B} \in X/X_0$, construct the following invariants under g:

$$\{\mathbf{h}_1(X_0)\cdot\mathbf{r}, \cdots, \mathbf{h}_r(X_0)\cdot\mathbf{r}\},$$
$$\{\mathrm{tr}\,\boldsymbol{\Omega}_1(X_0)\mathbf{H}, \cdots, \mathrm{tr}\,\boldsymbol{\Omega}_s(X_0)\mathbf{H}\},$$
$$\{\mathrm{tr}\,\boldsymbol{\Psi}_1(X_0)\mathbf{B}, \cdots, \mathrm{tr}\,\boldsymbol{\Psi}_t(X_0)\mathbf{B}\}.$$

The union of the above invariants for all $\mathbf{r}, \mathbf{H}, \mathbf{B} \in X/X_0$ is denoted by $I^0(X)$. Then X can be determined to within an orthogonal tensor pertaining to the group g by $I(X_0)$ and $I^0(X)$.

The proof is as follows. First, from (2.4) we deduce

$$g \cap g(X_0) \subset g(\mathbf{z}), \ \forall\, \mathbf{z} \in X\,. \tag{2.5}$$

By virtue of this and the obvious fact:

$$g_1 \subset g_2 \implies M(g_2) \subset M(g_1)$$

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for any two subgroups $g_1, g_2 \subset \text{Orth}$ and $M \in \{V, \text{Skw}, \text{Sym}\}$, we infer

$$\mathbf{r} \in V(g(\mathbf{r})) \subset V(g \cap g(X_0)) = \operatorname{span} V(X_0),$$

$$\mathbf{H} \in \text{Skw}(g(\mathbf{H})) \subset \text{Skw}(g \cap g(X_0)) = \operatorname{span} \text{Skw}(X_0),$$

$$\mathbf{B} \in \text{Sym}(g(\mathbf{B})) \subset \text{Sym}(g \cap g(X_0)) = \operatorname{span} \text{Sym}(X_0),$$

for any vector $\mathbf{r} \in X/X_0$, any skewsymmetric tensor $\mathbf{H} \in X/X_0$ and any symmetric tensor $\mathbf{B} \in X/X_0$, where the last equality in each of the above three expressions can be derived from (2.13), (2.16) and (2.15) in XIAO [33]. From the above three expressions we know that X/X_0 is determined by the union $I^0(X)$ indicated before if the three generating sets $V(X_0)$, $\text{Skw}(X_0)$ and $\text{Sym}(X_0)$ are known. On the other hand, X_0 is determined to within an orthogonal tensor $\mathbf{Q} \in g$ by a functional basis $I(X_0)$ of X_0 under g. Since each generator is form-invariant under g, we infer that the three generating sets just mentioned can be determined to within an orthogonal tensor $\mathbf{Q} \in g$ by the functional basis $I(X_0)$.

Combining the above facts we conclude that the aforementioned fact is true.

Owing to the fact proved above, in the process of constructing generating sets for vector-valued and second order tensor-valued functions of the two variables $X_0 \subset X$, we can obtain general functional bases of the variables X merely by forming the corresponding inner product between each generic variable $\mathbf{x} \in X/X_0$ and each presented generator and moreover, by constructing functional bases of the two variables $X_0 \subset X$.

The above process of constructing generating sets and functional bases may be simplified due to the fact that some or even all the three lists of two variables, $(\mathbf{u}, \mathbf{v})$, $(\mathbf{u}, \mathbf{W})$ and $(\mathbf{u}, \mathbf{A})$, may be further reduced to some lists of a single variable. Generally, simplification concerning each list X_0 of two variables is possible (see (2.6) below). Specifically, we design the following scheme.

1. For each variable $\mathbf{x} \in \{\mathbf{u}, \mathbf{W}, \mathbf{A}\}$, construct irreducible generating sets $V^0(\mathbf{x})$, $\text{Skw}^0(\mathbf{x})$ and $\text{Sym}^0(\mathbf{x})$ for vector-valued and skewsymmetric and symmetric tensor-valued functions, respectively. Then form the corresponding inner product between each generic variable $\mathbf{y} \in \{\mathbf{r}, \mathbf{H}, \mathbf{B}\}$ and each presented vector, skewsymmetric tensor, and symmetric tensor generator and moreover, construct an irreducible functional basis of the single variable $\mathbf{x}$.

2. For each list $X_0 = (\mathbf{u}, \mathbf{x})$ of two variables, where $\mathbf{x} \in \{\mathbf{v}, \mathbf{W}, \mathbf{A}\}$, consider the union $G(\mathbf{u}) \cup G(\mathbf{x})$, where $G(\mathbf{u})$ and $G(\mathbf{x})$ are the generating sets for vector-, skewsymmetric tensor-, and symmetric tensor-valued functions of the single variables $\mathbf{u}$ and $\mathbf{x}$ respectively, constructed in the first step. This union obeys the criterion (1.1) for the cases

$$g(\mathbf{z}) \cap g = g(X_0) \cap g, \qquad \mathbf{z} \in \{\mathbf{u}, \mathbf{x}\}.$$

Thus, it suffices to treat the case other than the above cases, which is specified

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by the conditions

$$g(\mathbf{z}) \cap g \neq g(\mathbf{u}, \mathbf{x}) \cap g, \qquad \mathbf{z} = \mathbf{u}, \mathbf{x}. \tag{2.6}$$

Then, analyze the latter case and judge whether or not the aforementioned union also obeys the criterion (1.1) for the latter case. If not, then add some generators with two variables $\mathbf{u}$ and $\mathbf{x}$ into this union so that an irreducible generating set for the two variables $(\mathbf{u}, \mathbf{x})$ is formed. If yes, then the aforementioned union is already the desired result. Moreover, form the corresponding inner product between each variable $\mathbf{y} \in \{\mathbf{r}, \mathbf{H}, \mathbf{B}\}$ and each presented vector, skewsymmetric tensor, and symmetric tensor generator with two variables (the generators with a single variable have been covered in the first step) and moreover, construct an irreducible functional basis of the two variables $(\mathbf{u}, \mathbf{x})$ specified by (2.6).

3. Combine all vector generators, skewsymmetric tensor generators, symmetric tensor generators and invariants obtained, respectively, and let each generic variable concerned run over the set X. Then the desired general irreducible representations are available (see Theorem 2.4 in Xiao [34]).

In the above procedures, the related results for isotropic extension of anisotropic functions given in Xiao [35, 39] may be applied to construct irreducible generating sets for one or two variables. Furthermore, according to Theorem 3.7 in Xiao [34], for skewsymmetric and symmetric tensor-valued functions we need only to consider two vector variables $(\mathbf{u}, \mathbf{v})$, a single skewsymmetric tensor variable $\mathbf{W}$, and a single symmetric tensor variable $\mathbf{A}$, respectively. As to the scalar-valued and vector-valued functions of two variables X_0, the condition (2.6) usually leads to such X_0 that construction of functional bases and generating sets for X_0 can be considerably simplified.

Following the above scheme, in the succeeding sections we shall construct general irreducible representations for anisotropic functions of the variables $X \in \mathcal{D}$ under all subgroups $g \in C_{\infty h}$. In view of the fact stated above, in the second step above we shall omit generating sets for skewsymmetric and symmetric tensor-valued functions as well as the invariants obtained by forming the corresponding inner product between each variable $\mathbf{x} \in \{\mathbf{r}, \mathbf{H}, \mathbf{B}\}$ and each generator with one variable.

Finally, it should be pointed out that the above construction scheme in itself does not mean the irreducibility of each invariant obtained by means of the inner product. As a result, additiontial proof for the latter is needed. To this end, one may contruct the pair (X, X') fulfilling (1.6).

3. The triclinic and monoclinic crystal classes C_1, S_2 and C_{1h}

3.1. The triclinic group C_1

Henceforth, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal basis of V. Trivially,

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V	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
Skw	$\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1$
Sym	$\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_2 \otimes \mathbf{e}_2, \mathbf{e}_3 \otimes \mathbf{e}_3, \mathbf{e}_1 \vee \mathbf{e}_2, \mathbf{e}_2 \vee \mathbf{e}_3, \mathbf{e}_3 \vee \mathbf{e}_1$
R	$\mathbf{r}\cdot\mathbf{e}_i;\ \mathbf{e}_i\cdot\mathbf{H}\mathbf{e}_j;\ \mathbf{e}_i\cdot\mathbf{B}\mathbf{e}_j;\ i,j = 1,2,3,$

where $\mathbf{r} = \mathbf{u}_1, \cdots, \mathbf{u}_a$, $\mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$ and $\mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$, provide irreducible representations for scalar-. vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables $X \in \mathcal{D}$ relative to the triclinic group C_1.

3.2. The triclinic group S_2

It is evident that for any $X \in \mathcal{D}$ there is a vector $\mathbf{u} \in X$ such that

$$g(X) \cap S_2 = g(\mathbf{u}) \cap S_2\,.$$

Accordingly we construct the following table.

V	$\mathbf{u}, \mathbf{u}\times\mathbf{e}_1, \mathbf{u}\times\mathbf{e}_2, \mathbf{u}\times\mathbf{e}_3$
Skw	$\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1$
Sym	$\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_2 \otimes \mathbf{e}_2, \mathbf{e}_3 \otimes \mathbf{e}_3, \mathbf{e}_1 \vee \mathbf{e}_2, \mathbf{e}_2 \vee \mathbf{e}_3, \mathbf{e}_3 \vee \mathbf{e}_1$
R	$\mathbf{u}\cdot\mathbf{r}, [\mathbf{u},\mathbf{r},\mathbf{e}_i];\ \mathbf{e}_i\cdot\mathbf{H}\mathbf{e}_j;\ \mathbf{e}_i\cdot\mathbf{B}\mathbf{e}_j;\ <(\mathbf{u}\cdot\mathbf{e}_i)(\mathbf{u}\cdot\mathbf{e}_j)>;\ i,j = 1,2,3$

Here and hereafter the invariants in the angle brackets supply an irreducible functional basis for one variable or two variables under consideration. The other invarants are obtained by forming the corresponding inner product between each generator given and a variable $\mathbf{x} \in \{\mathbf{r}, \mathbf{H}, \mathbf{B}\}$.

Thus, we obtain the main result of this subsection as follows.

THEOREM 3.1. *The table given above, together with* $\mathbf{u}, \mathbf{r} = \mathbf{u}_1, \cdots, \mathbf{u}_a, \mathbf{u} \neq \mathbf{r}$; $\mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$; $\mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the triclinic group* S_2.

REMARK. An irreducible functional basis of two vector variables under S_2 was derived by LIU [11], in which 18 invariants are employed. Here only 16 invariants are used.

3.3. The monoclinic crystal class C_{1h}

Since C_{1h} has only two subgroups, i.e. C_1 and C_{1h}, we infer that for any $X \in \mathcal{D}$ there exists a single vector or second order tensor $\mathbf{x} \in X$ such that

$$g(X) \cap C_{1h}(\mathbf{n}) = g(\mathbf{x}) \cap C_{1h}(\mathbf{n}).$$

Consequently, it suffices to treat the cases for a single variable. As mentioned before, $\mathbf{e}$ and $\mathbf{e}'$ are two orthonormal vectors in the $\mathbf{n}$-plane. Henceforth we denote

$$\mathbf{N} = \mathbf{e} \wedge \mathbf{e}' = \mathbf{E}\mathbf{n}\,. \tag{3.1}$$

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Throughout, $\mathbf{E}$ is used to denote the third-order Eddington tensor, i.e. the permutation tensor. It is evident that $\mathbf{N}$ is invariant under $C_{\infty h}(\mathbf{n})$ and independent of the choice of the orthonornormal vectors $\mathbf{e}$ and $\mathbf{e}'$ in the $\mathbf{n}$-plane.

CASE 1. A single vector variable $\mathbf{u}$

V $\quad \mathbf{e}, \mathbf{e}', (\mathbf{u}\cdot\mathbf{n})\mathbf{n}$

Skw $\quad \mathbf{N}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\mathbf{e}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\mathbf{e}'$

Sym $\quad \mathbf{n}\otimes\mathbf{n}, \mathbf{e}\otimes\mathbf{e}, \mathbf{e}'\otimes\mathbf{e}', \mathbf{e}\vee\mathbf{e}', (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\mathbf{e}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\mathbf{e}'$

R $\quad \mathbf{r}\cdot\mathbf{e}, \mathbf{r}\cdot\mathbf{e}', (\mathbf{r}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{n});$
$\quad \operatorname{tr}\mathbf{HN}, (\mathbf{u}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{He}), (\mathbf{u}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{He}');$
$\quad \mathbf{n}\cdot\mathbf{Bn}, \mathbf{e}\cdot\mathbf{Be}, \mathbf{e}'\cdot\mathbf{Be}', \mathbf{e}\cdot\mathbf{Be}', (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{Be}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{Be}';$
$\quad <\mathbf{u}\cdot\mathbf{e}, \mathbf{u}\cdot\mathbf{e}', (\mathbf{u}\cdot\mathbf{n})^2> .$

CASE 2. A single skewsymmetric tensor variable $\mathbf{W}$

V $\quad \mathbf{e}, \mathbf{e}', (\mathbf{n}\cdot\mathbf{We})\mathbf{n}, (\mathbf{n}\cdot\mathbf{We}')\mathbf{n}$

Skw $\quad \mathbf{N}, \mathbf{n}\wedge\mathbf{Wn}, \mathbf{n}\wedge(\mathbf{n}\times\mathbf{Wn})$

Sym $\quad \{\mathbf{n}\otimes\mathbf{n}, \mathbf{e}\otimes\mathbf{e}, \mathbf{e}'\otimes\mathbf{e}', \mathbf{e}\vee\mathbf{e}', \mathbf{n}\vee\mathbf{Wn}, \mathbf{n}\vee(\mathbf{n}\times\mathbf{Wn})\}\ (=\mathrm{Sym}_0(\mathbf{W}))$

R $\quad \mathbf{r}\cdot\mathbf{e}, \mathbf{r}\cdot\mathbf{e}', (\mathbf{r}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{We}, (\mathbf{r}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{We}';$
$\quad \operatorname{tr}\mathbf{HN}, \mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Hn}];$
$\quad \mathbf{n}\cdot\mathbf{Bn}, \mathbf{e}\cdot\mathbf{Be}, \mathbf{e}'\cdot\mathbf{Be}', \mathbf{e}\cdot\mathbf{Be}', \mathbf{n}\cdot\mathbf{WBn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Bn}];$
$\quad <\operatorname{tr}\mathbf{WN}, (\mathbf{n}\cdot\mathbf{We})^2, (\mathbf{n}\cdot\mathbf{We}')^2, (\mathbf{n}\cdot\mathbf{We})(\mathbf{n}\cdot\mathbf{We}')>\ \ (=I_0(\mathbf{W})).$

It can be readily verified that the first three sets given above are irreducible generating sets for vector-valued, skewsymmetric and symmetric tensor-valued anisotropic functions of the skewsymmetric tensor variable $\mathbf{W}\in\mathrm{Skw}$ relative to $C_{1h}(\mathbf{n})$, respectively, by means of (1.2) – (1.4) and the fact

$$g(\mathbf{W})\cap C_{1h}(\mathbf{n}) = \begin{cases} C_{1h}(\mathbf{n}), & \mathbf{Wn}=\mathbf{0}, \\ C_1, & \mathbf{Wn}\neq\mathbf{0}. \end{cases}$$

Moreover, by means of criterion (1.5) it can be proved easily that the four invariants of $\mathbf{W}$ listed in the angle brackets form an irreducible functional basis of the variable $\mathbf{W}\in\mathrm{Skw}$ under $C_{1h}(\mathbf{n})$.

CASE 3. A single symmetric tensor variable $\mathbf{A}$

V $\quad \mathbf{e}, \mathbf{e}', (\mathbf{n}\cdot\mathbf{Ae})\mathbf{n}, (\mathbf{n}\cdot\mathbf{Ae}')\mathbf{n}$

Skw $\quad \mathbf{N}, \mathbf{n}\wedge\mathbf{An}, \mathbf{n}\wedge(\mathbf{n}\times\mathbf{An})$

Sym $\quad \{\mathbf{n}\otimes\mathbf{n}, \mathbf{e}\otimes\mathbf{e}, \mathbf{e}'\otimes\mathbf{e}', \mathbf{e}\vee\mathbf{e}', \mathbf{n}\vee\mathbf{An}, \mathbf{n}\vee(\mathbf{n}\times\mathbf{An})\}\ (=\mathrm{Sym}_0(\mathbf{A}))$

R $\quad \mathbf{r}\cdot\mathbf{e}, \mathbf{r}\cdot\mathbf{e}', (\mathbf{r}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{Ae}, (\mathbf{r}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{Ae}';$
$\quad \operatorname{tr}\mathbf{HN}, \mathbf{n}\cdot\mathbf{HAn}, [\mathbf{n}, \mathbf{An}, \mathbf{Hn}];$
$\quad \mathbf{n}\cdot\mathbf{Bn}, \mathbf{e}\cdot\mathbf{Be}, \mathbf{e}'\cdot\mathbf{Be}', \mathbf{e}\cdot\mathbf{Be}', \mathbf{n}\cdot\mathbf{ABn}, [\mathbf{n}, \mathbf{An}, \mathbf{Bn}];$
$\quad <\mathbf{n}\cdot\mathbf{An}, \mathbf{e}\cdot\mathbf{Ae}, \mathbf{e}'\cdot\mathbf{Ae}', \mathbf{e}\cdot\mathbf{Ae}', (\mathbf{n}\cdot\mathbf{Ae})^2, (\mathbf{n}\cdot\mathbf{Ae}')^2,$
$\quad (\mathbf{n}\cdot\mathbf{Ae})(\mathbf{n}\cdot\mathbf{Ae}')>\ \ (=I_0(\mathbf{A})).$

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It can be readily proved that the first three sets given above are irreducible generating sets for vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the symmetric tensor variable $\mathbf{A} \in \mathrm{Sym}$ relative to $C_{1h}(\mathbf{n})$, respectively, by means of the criterion (1.1), (1.2) – (1.4) and the fact

$$g(\mathbf{A}) \cap C_{1h}(\mathbf{n}) = \begin{cases} C_{1h}(\mathbf{n}), & \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0}\,, \\ C_1, & \overset{\circ}{\mathbf{A}}\mathbf{n} \neq \mathbf{0}\,. \end{cases}$$

Moreover, the given irreducible functional basis of the symmetric tensor variable $\mathbf{A} \in \mathrm{Sym}$ under $C_{1h}(\mathbf{n})$ can be found in XIAO [36].

Henceforth, we denote the irreducible functional bases and generating sets for scalar-valued and symmetric tensor-valued functions of the single variables $\mathbf{W}$ and $\mathbf{A}$, given in the two tables for Case 2 and Case 3 respectively, by $I_0(\mathbf{W})$, $I_0(\mathbf{A})$, $\mathrm{Sym}_0(\mathbf{W})$, and $\mathrm{Sym}_0(\mathbf{A})$, respectively.

Combining the above three cases, we arrive at the main result of this subsection as follows.

THEOREM 3.2. *The four sets given by*

$I_0(\mathbf{W}); I_0(\mathbf{A}); \mathbf{u}\cdot\mathbf{e}, \mathbf{u}\cdot\mathbf{e}', (\mathbf{u}\cdot\mathbf{n})^2;$
$(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n});$
$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{W}\mathbf{e}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{W}\mathbf{e}';$
$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{A}\mathbf{e}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{A}\mathbf{e}';$
$\mathbf{n}\cdot\mathbf{W}\mathbf{H}\mathbf{n}, [\mathbf{n}, \mathbf{W}\mathbf{n}, \mathbf{H}\mathbf{n}];$
$\mathbf{n}\cdot\mathbf{A}\mathbf{B}\mathbf{n}, [\mathbf{n}, \mathbf{A}\mathbf{n}, \mathbf{B}\mathbf{n}];$
$\mathbf{n}\cdot\mathbf{A}\mathbf{W}\mathbf{n}, [\mathbf{n}, \mathbf{W}\mathbf{n}, \mathbf{A}\mathbf{n}];$

and

$\mathbf{e}, \mathbf{e}', (\mathbf{u}\cdot\mathbf{n})\mathbf{n};$
$(\mathbf{n}\cdot\mathbf{W}\mathbf{e})\mathbf{n}, (\mathbf{n}\cdot\mathbf{W}\mathbf{e}')\mathbf{n};$
$(\mathbf{n}\cdot\mathbf{A}\mathbf{e})\mathbf{n}, (\mathbf{n}\cdot\mathbf{A}\mathbf{e}')\mathbf{n};$

and

$\mathbf{N}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \wedge \mathbf{e}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \wedge \mathbf{e}';$
$\mathbf{n} \wedge \mathbf{W}\mathbf{n}, \mathbf{n} \wedge (\mathbf{n} \times \mathbf{W}\mathbf{n});$
$\mathbf{n} \wedge \mathbf{A}\mathbf{n}, \mathbf{n} \wedge (\mathbf{n} \times \mathbf{A}\mathbf{n});$

and

$\mathrm{Sym}_0(\mathbf{W}), \mathrm{Sym}_0(\mathbf{A}), (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \vee \mathbf{e}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \vee \mathbf{e}';$

where $\mathbf{u}, \mathbf{v} = \mathbf{u}_1, \cdots, \mathbf{u}_a$; $\mathbf{W}, \mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$; $\mathbf{A}, \mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$; $\mathbf{u} \neq \mathbf{v}$, $\mathbf{W} \neq \mathbf{H}$, $\mathbf{A} \neq \mathbf{B}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the monoclinic group* $C_{1h}(\mathbf{n})$, *respectively.*

4. The classes C_{2mh} and C_{2m}

The classes C_{2mh} and C_{2m} include the monoclinic crystal classes C_{2h} and C_2, the tetrahedral crystal classes C_{4h} and C_4, and the hexagonal crystal classes

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C_{6h} and C_6 as the particular cases when $m = 1, 2, 3$. Moreover, the transverse isotropy groups $C_{\infty h}$ and C_∞ can be treated as the particular case of the classes at issue when $m = \infty$. It should be pointed out that the former can not be regarded as the particular cases of the classes $S_{4m+2}(\mathbf{n})$ and $C_{2m+1}(\mathbf{n})$ when $m = \infty$, since the element $\mathbf{R}_{\mathbf{n}}^{\pi}$ is not contained in either of the latter.

4.1. The classes C_{2mh}

Six cases will be discussed.

CASE 1. A single vector variable $\mathbf{u}$

From the criterion (1.1) and the formula (1.2) – (1.4) as well as

$$g(\mathbf{u}) \cap C_{2mh}(\mathbf{n}) = \begin{cases} C_{2mh}(\mathbf{n}), & \mathbf{u} = \mathbf{0}, \\ C_{2m}(\mathbf{n}), & \overset{\circ}{\mathbf{u}} = \mathbf{0}, \quad \mathbf{u}\cdot\mathbf{n} \neq 0, \\ C_{1h}(\mathbf{n}), & \mathbf{u}\cdot\mathbf{n} = 0, \quad \overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \\ C_1, & (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \end{cases} \tag{4.1}$$

$$g(\mathbf{u}) \cap g(\mathbf{N}) = \begin{cases} C_{\infty h}(\mathbf{n}), & \mathbf{u} = \mathbf{0}, \\ C_{\infty}(\mathbf{n}), & \overset{\circ}{\mathbf{u}} = \mathbf{0}, \quad \mathbf{u}\cdot\mathbf{n} \neq 0, \\ C_{1h}(\mathbf{n}), & \mathbf{u}\cdot\mathbf{n} = 0, \quad \overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \\ C_1, & (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \end{cases} \tag{4.2}$$

we deduce that the following fact holds: generating sets for vector-valued (for $m \geq 1$), skewsymmetric tensor-valued (for $m \geq 1$) and symmetric tensor-valued (for $m \geq 2$) anisotropic functions of the vector variable $\mathbf{u}$ under $C_{2mh}(\mathbf{n})$ can be derived from those for vector-, skewsymmetric and symmetric tensor-valued isotropic functions of the extended variables $(\mathbf{u}, \mathbf{N})$, respectively. Moreover, with the aid of the fact that $(-\mathbf{I})\mathbf{u} = -\mathbf{u}$, $-\mathbf{I} \in C_{2mh}(\mathbf{n})$ and the invariance condition stated at the start of the introduction, it may be readily understood that functional bases for scalar-valued anisotropic functions of the vector variable $\mathbf{u}$ under $C_{2mh}(\mathbf{n})$ can be obtained from those for scalar-valued anisotropic functions of the symmetric tensor variable $\mathbf{u} \otimes \mathbf{u} \in \mathrm{Sym}$ under $C_{2mh}(\mathbf{n})$. The latter can be found in XIAO [36] for $m = 1$ and in Case 3 below for $m \geq 2$.

Applying the above facts, we construct the following table:

V	$\{\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}} \times \mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\}$ $(= V^0(\mathbf{u}))$
Skw	$\{\mathbf{N}, (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}} \wedge \mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}} \wedge (\overset{\circ}{\mathbf{u}} \times \mathbf{n})\}$ $(= \mathrm{Skw}^0(\mathbf{u}))$
Sym	$\{\mathbf{n} \otimes \mathbf{n}, \delta_{1m}\mathbf{e} \otimes \mathbf{e}, \delta_{1m}\mathbf{e}' \otimes \mathbf{e}', \delta_{1m}\mathbf{e} \vee \mathbf{e}', (1 - \delta_{1m})\mathbf{I}, (1 - \delta_{1m})\overset{\circ}{\mathbf{u}} \otimes \overset{\circ}{\mathbf{u}},$ $(1 - \delta_{1m})\overset{\circ}{\mathbf{u}} \vee (\overset{\circ}{\mathbf{u}} \times \mathbf{n}); (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}} \vee \mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \vee (\overset{\circ}{\mathbf{u}} \times \mathbf{n})\}$ $(= \mathrm{Sym}_m^0(\mathbf{u}))$

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$$
\begin{array}{ll}
R & \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{r}}, [\mathbf{n}, \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{r}}], (\mathbf{u}\cdot\mathbf{n})(\mathbf{r}\cdot\mathbf{n}); \\
 & \mathrm{tr}\,\mathbf{HN}, (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}}\cdot\mathbf{Hn}, (\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \overset{\circ}{\mathbf{u}}, \mathbf{Hn}]; \\
 & \mathbf{n}\cdot\mathbf{Bn}, \delta_{1m}\mathbf{e}\cdot\mathbf{Be}, \delta_{1m}\mathbf{e}'\cdot\mathbf{Be}', \delta_{1m}\mathbf{e}\cdot\mathbf{Be}', (1-\delta_{1m})\mathrm{tr}\,\mathbf{B}, \\
 & (1-\delta_{1m})\overset{\circ}{\mathbf{u}}\cdot\mathbf{B}\overset{\circ}{\mathbf{u}}, (1-\delta_{1m})[\mathbf{n}, \overset{\circ}{\mathbf{u}}, \mathbf{B}\overset{\circ}{\mathbf{u}}], (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}}\cdot\mathbf{Bn}, (\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathbf{Bn}, \overset{\circ}{\mathbf{u}}]; \\
 & < (\mathbf{u}\cdot\mathbf{n})^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e})^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e}')^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e})(\mathbf{u}\cdot\mathbf{e}'), \\
 & (1-\delta_{1m})\alpha_{2m}(\overset{\circ}{\mathbf{u}}), (1-\delta_{1m})\beta_{2m}(\overset{\circ}{\mathbf{u}}) > \ (= I^0_m(\mathbf{u})).
\end{array}
$$

Henceforth, the first three sets consisting of vector generators, skewsymmetric tensor generators and symmetric tensor generators, respectively, are denoted by $V^0(\mathbf{u})$, $\mathrm{Skw}^0(\mathbf{u})$, and $\mathrm{Sym}^0_m(\mathbf{u})$ respectively. We have $\mathrm{Sym}^0_1 = \mathrm{Sym}_0(\mathbf{u})$ for $m = 1$ and

$$
(4.3) \qquad \mathrm{Sym}^0_m(\mathbf{u}) = \{\mathbf{I},\ \mathbf{n}\otimes\mathbf{n},\ \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}},\ \overset{\circ}{\mathbf{u}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}}),\ (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}},\ (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}})\} \equiv \mathrm{Sym}^0(\mathbf{u})
$$

for $m \geq 2$. Moreover, the set of invariants given in the angle brackets is signified by $I^0_m(\mathbf{u})$.

It should be pointed out that in the above table, each element with the coefficients δ_{1m} or $(1-\delta_{1m})$ comes into play only when $m = 1$ or $m \geq 2$. Here and hereafter δ_{rs} is used to represent the Kronecker delta. Such a difference between C_{2h} and C_{2mh} for $m \geq 2$, which will also appear in the next five cases, arises from the fact

$$
\dim\mathrm{Sym}(C_{2mh}) = \begin{cases} 4, & m = 1\,, \\ 2, & \dim\mathrm{Sym}(C_{2mh}) = 2, \quad m \geq 2\,. \end{cases}
$$

CASE 2. A single skewsymmetric tensor **W**

Every vector-valued function of the variable $\mathbf{W} \in \mathrm{Skw}$ that is form-invariant under C_{2mh} vanishes. Moreover, from the criterion (1.1) and the formula (1.3)–(1.4) as well as

$$
(4.4) \qquad g(\mathbf{W}) \cap C_{2mh}(\mathbf{n}) = \begin{cases} C_{2mh}(\mathbf{n}), & \mathbf{Wn} = \mathbf{0}\,, \\ S_2, & \mathbf{Wn} \neq \mathbf{0}\,, \end{cases}
$$

$$
(4.5) \qquad g(\mathbf{W}) \cap g(\mathbf{N}) = \begin{cases} C_{\infty h}(\mathbf{n}), & \mathbf{Wn} = \mathbf{0}\,, \\ S_2, & \mathbf{Wn} \neq \mathbf{0}\,, \end{cases}
$$

we infer that the following fact holds: generating sets for skewsymmetric tensor-valued (for $m \geq 1$) and symmetric tensor-valued (for $m \geq 2$) anisotropic functions of the variable $\mathbf{W} \in \mathrm{Skw}$ under $C_{2mh}(\mathbf{n})$ can be derived from those for skewsymmetric and symmetric tensor-valued isotropic functions of the extended variables $(\mathbf{W}, \mathbf{N})$. Thus, we construct the following table.

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Skw	$\{\mathbf{N}, \mathbf{n}\wedge\mathbf{Wn}, \mathbf{n}\wedge(\mathbf{n}\times\mathbf{Wn})\}$ $(=\mathrm{Skw}^0(\mathbf{W}))$
Sym	$\{\mathbf{n}\otimes\mathbf{n}, \delta_{1m}\mathbf{e}\otimes\mathbf{e}, \delta_{1m}\mathbf{e}'\otimes\mathbf{e}', \delta_{1m}\mathbf{e}\vee\mathbf{e}', (1-\delta_{1m})\mathbf{I}, (1-\delta_{1m})\mathbf{Wn}\otimes\mathbf{Wn},$ $(1-\delta_{1m})\mathbf{Wn}\vee(\mathbf{n}\times\mathbf{Wn}), \mathbf{n}\vee\mathbf{Wn}, \mathbf{n}\vee(\mathbf{n}\times\mathbf{Wn})\}$ $(=\mathrm{Sym}^0_m(\mathbf{W}))$
R	$\mathrm{tr}\,\mathbf{HN}, \mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Hn}]$; $\mathbf{n}\cdot\mathbf{Bn}, \delta_{1m}\mathbf{e}\cdot\mathbf{Be}, \delta_{1m}\mathbf{e}'\cdot\mathbf{Be}', \delta_{1m}\mathbf{e}\cdot\mathbf{Be}'$, $(1-\delta_{1m})\mathbf{n}\cdot\mathbf{WBWn}, (1-\delta_{1m})[\mathbf{n}, \mathbf{Wn}, \mathbf{WBn}], \mathbf{n}\cdot\mathbf{WBn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Bn}]$; $<\mathrm{tr}\,\mathbf{WN}, \delta_{1m}(\mathbf{n}\cdot\mathbf{We})^2, \delta_{1m}(\mathbf{n}\cdot\mathbf{We}')^2, \delta_{1m}(\mathbf{n}\cdot\mathbf{We})(\mathbf{n}\cdot\mathbf{We}')$, $(1-\delta_{1m})\alpha_{2m}(\mathbf{Wn}), (1-\delta_{1m})\beta_{2m}(\mathbf{Wn})>$ $(=I^0_m(\mathbf{W}))$.

Henceforth, the two generating sets in the above table are denoted by $\mathrm{Skw}^0(\mathbf{W})$ and $\mathrm{Sym}^0_m(\mathbf{W})$ and the functional basis given in the angle brackets is denoted by $I^0_m(\mathbf{W})$.

We have $\mathrm{Sym}^0_1(\mathbf{W}) = \mathrm{Sym}_0(\mathbf{W})$ and

$$\mathrm{Sym}^0_m(\mathbf{W}) = \{\mathbf{I},\ \mathbf{n}\otimes\mathbf{n},\ \mathbf{Wn}\otimes\mathbf{Wn},\ \mathbf{Wn}\vee(\mathbf{n}\times\mathbf{Wn}),\ \mathbf{n}\vee\mathbf{Wn},\ \mathbf{n}\vee(\mathbf{n}\times\mathbf{Wn})\} \equiv \mathrm{Sym}^0(\mathbf{W}), \tag{4.6}$$

for each $m \geq 2$.

We need only to show that the set $I^0_m(\mathbf{W})$ is an irreducible functional basis of $\mathbf{W}$ under $C_{2mh}(\mathbf{n})$. To this end, we prove that $I^0_m(\mathbf{W})$ obeys the criterion (1.5). The proof for $m = 1$ is easy. Let $m \geq 2$. Observing the fact that the last three invariants in the above table form a functional basis of the vector $\mathbf{Wn}$ in the $\mathbf{n}$-plane (see Case 1), we infer that for $\overline{\mathbf{W}}, \mathbf{W} \in \mathrm{Skw}$,

$$I^0_m(\overline{\mathbf{W}}) = I^0_m(\mathbf{W}) \implies \exists\, \mathbf{Q}\in C_{2mh}(\mathbf{n}):\ \overline{\mathbf{W}}\mathbf{n} = \mathbf{Q}(\mathbf{Wn}),\ \mathrm{tr}\,\overline{\mathbf{W}}\mathbf{N} = \mathrm{tr}\,\mathbf{WN}.$$

In the above, we can assume $\mathbf{Q} \in C_{2m}(\mathbf{n})$, since $\mathbf{Wn}$ lies on the $\mathbf{n}$-plane. Thus, by means of the identity

$$\mathbf{W} = \frac{1}{2}(\mathrm{tr}\,\mathbf{WN})\mathbf{N} + (\mathbf{Wn})\wedge\mathbf{n} \tag{4.7}$$

as well as the facts: $\mathbf{QNQ}^{\mathrm{T}} = \mathbf{N}$, $\mathbf{Qn} = \mathbf{n}$ for every $\mathbf{Q}\in C_{2m}(\mathbf{n})$, we deduce

$$I^0_m(\overline{\mathbf{W}}) = I^0_m(\mathbf{W}) \implies \exists\, \mathbf{Q}\in C_{2m}(\mathbf{n}):\ \overline{\mathbf{W}} = \mathbf{QWQ}^{\mathrm{T}}.$$

Thus, we conclude that $I^0_m(\mathbf{W})$ obeys the criterion (1.5). The irreducibility of this basis is evident.

CASE 3. A single symmetric tensor variable $\mathbf{A}$

Every vector-valued function of the variable $\mathbf{A}\in\mathrm{Sym}$ that is form-invariant under $C_{2mh}(\mathbf{n})$ vanishes. Moreover, from the facts

$$g(\mathbf{A})\cap C_{2mh}(\mathbf{n}) = \begin{cases} C_{2mh}(\mathbf{n}), & \mathbf{A} = x\mathbf{I} + y\mathbf{n}\otimes\mathbf{n}, \\ C_{2h}(\mathbf{n}), & \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0}, \quad \mathbf{q}(\mathbf{A}) \neq \mathbf{0}, \\ S_2, & \overset{\circ}{\mathbf{A}}\mathbf{n} \neq \mathbf{0}, \end{cases} \tag{4.8}$$

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$$(4.9)\qquad g(\mathbf{A})\cap g(\mathbf{N}) = \begin{cases} C_{\infty h}(\mathbf{n}), & \mathbf{A} = x\mathbf{I} + y\mathbf{n}\otimes\mathbf{n}\,,\\ C_{2h}(\mathbf{n}), & \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0}, \quad \mathbf{q}(\mathbf{A}) \neq \mathbf{0}\,,\\ S_2, & \overset{\circ}{\mathbf{A}}\mathbf{n} \neq \mathbf{0}\,, \end{cases}$$

and the criterion (1.1) as well as the formula (1.3) – (1.4), we infer that the following fact holds: generating sets for skewsymmetric tensor-valued (for $m \geq 1$) and symmetric tensor-valued (for $m \geq 2$) anisotropic functions of the variable $\mathbf{A} \in \mathrm{Sym}$ under $C_{2mh}(\mathbf{n})$ can be derived from those for skewsymmetric and symmetric tensor-valued isotropic functions of the extended variables $(\mathbf{A}, \mathbf{N})$. Using these facts, we construct the following table:

Skw	$\{\mathbf{N}, \mathbf{n}\wedge\mathbf{An}, \mathbf{n}\wedge(\mathbf{n}\times\mathbf{An})\}$ $(= \mathrm{Skw}^0(\mathbf{A}))$
Sym	$\{\mathbf{n}\otimes\mathbf{n}, \delta_{1m}\mathbf{e}\otimes\mathbf{e}, \delta_{1m}\mathbf{e}'\otimes\mathbf{e}', \delta_{1m}\mathbf{e}\vee\mathbf{e}', (1-\delta_{1m})\mathbf{I}, (1-\delta_{1m})\overset{\circ}{\mathbf{A}}\mathbf{n}\otimes\overset{\circ}{\mathbf{A}}\mathbf{n},$ $(1-\delta_{1m})\overset{\circ}{\mathbf{A}}\mathbf{n}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{A}}\mathbf{n}), \mathbf{A}, \mathbf{AN}-\mathbf{NA}\}$ $(= \mathrm{Sym}^0_m(\mathbf{A}))$
R	$\mathrm{tr}\,\mathbf{HN}, \mathbf{n}\cdot\mathbf{AHn}, [\mathbf{n}, \mathbf{An}, \mathbf{Hn}]$; $\mathbf{n}\cdot\mathbf{Bn}, \mathrm{tr}\,\mathbf{AB}, \mathrm{tr}\,\mathbf{ABN}, \delta_{1m}\mathbf{e}\cdot\mathbf{Be}, \delta_{1m}\mathbf{e}'\cdot\mathbf{Be}', \delta_{1m}\mathbf{e}\cdot\mathbf{Be}',$ $(1-\delta_{1m})\mathrm{tr}\,\mathbf{B}, (1-\delta_{1m})\mathbf{n}\cdot\overset{\circ}{\mathbf{A}}\mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}, (1-\delta_{1m})[\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}]$; $<I_0(\mathbf{A})>$ (for $m=1$); $<I^0_m(\mathbf{A})>$ (for $m \geq 2$).

It can be readily shown that the presented set for skewsymmetric tensor-valued functions, denoted by $\mathrm{Skw}^0(\mathbf{A})$ henceforth, obeys the criterion (1.1), and it is evident that this set is irreducible. The presented set for symmetric tensor-valued functions is denoted by $\mathrm{Sym}^0_m(\mathbf{A})$ henceforth. $\mathrm{Sym}^0_1(\mathbf{A})$ can be found in Xiao [32] and $\mathrm{Sym}^0_m(\mathbf{A})$ for $m \geq 2$ is an equivalent form of the minimal generating set given in Xiao [36]. We have $\mathrm{Sym}^0_1(\mathbf{A}) = \mathrm{Sym}_0(\mathbf{A})$ (cf. Sec. 3.3) and

$$(4.10)\qquad \mathrm{Sym}^0_m(\mathbf{A}) = \{\mathbf{I},\ \mathbf{n}\otimes\mathbf{n},\ \mathbf{A},\ \mathbf{AN}-\mathbf{NA},\ \overset{\circ}{\mathbf{A}}\mathbf{n}\otimes\overset{\circ}{\mathbf{A}}\mathbf{n},\ \overset{\circ}{\mathbf{A}}\mathbf{n}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{A}}\mathbf{n})\} \equiv \mathrm{Sym}^0(\mathbf{A})\,,$$

for each $m \geq 2$.

Moreover, in the above table, the basis $I_0(\mathbf{A})$ for $m = 1$ is given in Sec. 3.3 and the basis $I^0_m(\mathbf{A})$ for $m \geq 2$ is given by (cf. Xiao [38])

$$(4.11)\qquad I^0_m(\mathbf{A}) = \{\mathbf{n}\cdot\mathbf{An}, \mathrm{tr}\,\mathbf{A}, \mathrm{tr}\,\mathbf{A}^3, [\mathbf{n}, \mathbf{An}, \mathbf{A}^2\mathbf{n}], \alpha_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), \beta_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), \alpha_m(\mathbf{q}(\mathbf{A})), \beta_m(\mathbf{q}(\mathbf{A}))\}.$$

Case 4. Two vector variables $(\mathbf{u}, \mathbf{v})$

The condition (2.6) yields

$$(4.12)\qquad g(\mathbf{z})\cap C_{2mh}(\mathbf{n}) \neq g(\mathbf{u}, \mathbf{v})\cap C_{2mh}(\mathbf{n}), \qquad \mathbf{z} = \mathbf{u}, \mathbf{v}\,.$$

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Hence,

$$g(\mathbf{z}) \cap C_{2mh}(\mathbf{n}) \neq C_1\,, \qquad \mathbf{z} = \mathbf{u}, \mathbf{v}.$$

By using (4.1) and the latter we infer

$$(\mathbf{z}\cdot\mathbf{n})\mathbf{z} \times \mathbf{n} = \mathbf{0}, \qquad \mathbf{z} = \mathbf{u}, \mathbf{v}.$$

The above result and (4.12) yield

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad \mathbf{v} \times \mathbf{n} = \mathbf{0} \qquad \text{or} \qquad \mathbf{v}\cdot\mathbf{n} = 0, \quad \mathbf{u} \times \mathbf{n} = \mathbf{0}\,, \tag{4.13}$$

where $(\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) \neq 0$, and hence

$$g(\mathbf{u}, \mathbf{v}) \cap C_{2mh}(\mathbf{n}) = C_1\,.$$

Accordingly, we construct the following table (the first case in (4.13) is considered).

V	$\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}} \times \mathbf{n}, (\mathbf{v}\cdot\mathbf{n})\mathbf{n}$
Skw	$\mathbf{N}, \mathbf{u} \wedge \mathbf{v}, (\mathbf{u} \times \mathbf{n}) \wedge \mathbf{v} + (\mathbf{v} \times \mathbf{n}) \wedge \mathbf{u}$
Sym	$\mathbf{n} \otimes \mathbf{n}, \delta_{1m}\mathbf{e} \otimes \mathbf{e}, \delta_{1m}\mathbf{e}' \otimes \mathbf{e}', \delta_{1m}\mathbf{e} \vee \mathbf{e}', (1-\delta_{1m})\overset{\circ}{\mathbf{u}} \otimes \overset{\circ}{\mathbf{u}},$ $(1-\delta_{1m})\overset{\circ}{\mathbf{u}} \vee (\overset{\circ}{\mathbf{u}} \times \mathbf{n}), \mathbf{u} \vee \mathbf{v}, (\mathbf{n} \times \mathbf{u}) \vee \mathbf{v} + (\mathbf{n} \times \mathbf{v}) \vee \mathbf{u}$
R	$\mathbf{u}\cdot\mathbf{H}\mathbf{v}, [\mathbf{n}, \mathbf{u}, \mathbf{H}\mathbf{v}] + [\mathbf{n}, \mathbf{v}, \mathbf{H}\mathbf{u}];$ $\mathbf{u}\cdot\mathbf{B}\mathbf{v}, [\mathbf{n}, \mathbf{u}, \mathbf{B}\mathbf{v}] + [\mathbf{n}, \mathbf{v}, \mathbf{B}\mathbf{u}];$ $< (\mathbf{v}\cdot\mathbf{n})^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e})^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e}')^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e})(\mathbf{u}\cdot\mathbf{e}'),$ $(1-\delta_{1m})\alpha_{2m}(\overset{\circ}{\mathbf{u}}), (1-\delta_{1m})\beta_{2m}(\overset{\circ}{\mathbf{u}}) >$

Owing to (4.13), the above table can easily be constructed.

CASE 5. A vector variable **u** and a skewsymmetric tensor variable **W**
The condition (2.6) yields

$$g(\mathbf{z}) \cap C_{2mh}(\mathbf{n}) \neq g(\mathbf{u}, \mathbf{W}) \cap C_{2mh}(\mathbf{n}), \qquad \mathbf{z} = \mathbf{u}, \mathbf{W}\,. \tag{4.14}$$

Hence,

$$g(\mathbf{u}) \cap C_{2mh}(\mathbf{n}) \neq C_1, \qquad \mathbf{u} \neq \mathbf{0}\,,$$
$$g(\mathbf{W}) \cap C_{2mh}(\mathbf{n}) \neq C_{2mh}(\mathbf{n})\,.$$

From the latter and (4.1) and (4.4) we derive

$$\begin{aligned} &(\mathbf{u}\cdot\mathbf{n})\mathbf{u} \times \mathbf{n} = \mathbf{0}, \qquad \mathbf{u} \neq \mathbf{0}\,, \\ &g(\mathbf{W}) \cap C_{2mh}(\mathbf{n}) = S_2, \qquad \text{i.e.} \quad \mathbf{W}\mathbf{n} \neq \mathbf{0}\,, \end{aligned} \tag{4.15}$$

and hence

$$g(\mathbf{u}, \mathbf{W}) \cap C_{2mh}(\mathbf{n}) = C_1\,.$$

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Thus, we construct the following table for vector generators and invariants (second order tensor generators and related invariants have been covered by Case 2 due to $(4.15)_2$).

V $\quad \overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{u}}\times\mathbf{n}, \mathbf{Wu}, \mathbf{W}(\mathbf{u}\times\mathbf{n})+\mathbf{n}\times\mathbf{Wu}$

R $\quad \mathbf{r}\cdot\mathbf{Wu}, [\mathbf{n},\mathbf{u},\mathbf{Wr}]+[\mathbf{n},\mathbf{r},\mathbf{Wu}];$

$\quad < I^0_m(\mathbf{u}), I^0_m(\mathbf{W}), (1-\delta_{1m})(\overset{\circ}{\mathbf{u}}\cdot\mathbf{Wn})^2, (1-\delta_{1m})(\overset{\circ}{\mathbf{u}}\cdot\mathbf{Wn})[\overset{\circ}{\mathbf{u}}, \mathbf{Wn}, \mathbf{n}] > .$

We are in a position to prove that the above two sets, denoted by $V(\mathbf{u},\mathbf{W})$ and $I(\mathbf{u},\mathbf{W})$, are an irreducible generating set and an irreducible functional basis for scalar-valued and vector-valued anisotropic functions of the two variables $(\mathbf{u},\mathbf{W})$ specified by (4.15) relative to the group $C_{2mh}(\mathbf{n})$, respectively. For the case when $\mathbf{u}\times\mathbf{n}=\mathbf{0}$, the proof is easy. In the following, we treat the case when $\mathbf{u}\cdot\mathbf{n}=0$. We prove that the two sets in question obey the criteria (1.1) and (1.5), respectively. First, we have

$$\operatorname{rank} V(\mathbf{u},\mathbf{W}) = \begin{cases} \operatorname{rank}\{\mathbf{u},\mathbf{u}\times\mathbf{n},\mathbf{Wu}\}=3, & \mathbf{n}\cdot\mathbf{Wu}\neq 0\,, \\ \operatorname{rank}\{\mathbf{u},\mathbf{u}\times\mathbf{n},\mathbf{W}(\mathbf{u}\times\mathbf{n})\}=3, & \mathbf{n}\cdot\mathbf{Wu}=0\,, \end{cases}$$

where in the second equality, the fact

$$\mathbf{n}\cdot\mathbf{W}(\mathbf{u}\times\mathbf{n}) = (\mathbf{Wn})\cdot(\mathbf{u}\times\mathbf{n}) \neq 0$$

is used, which can be derived by using the facts: 1. The three vectors $\mathbf{u}, \mathbf{u}\times\mathbf{n}$ and $\mathbf{Wn}$ lie in the $\mathbf{n}$-plane and the first two are independent, and 2. $\mathbf{n}\cdot\mathbf{Wu} = (\mathbf{Wn})\cdot\mathbf{u}=0$. From the above, we know that the set $V(\mathbf{u},\mathbf{W})$ obeys (1.1) for the case $\mathbf{u}\cdot\mathbf{n}=0$.

Next, for $m=1$, it is readily verified that the set $I(\mathbf{u},\mathbf{W})$, i.e. $I^0_1(\mathbf{u})\cup I^0_1(\mathbf{W})$, obeys (1.5). For $m\geq 2$, let $I'(\mathbf{u},\mathbf{W}) = I^0_m(\mathbf{u})\cup I^0_m(\mathbf{W})$. Then, for the pair $(\overline{\mathbf{u}},\overline{\mathbf{W}})$ and $(\mathbf{u},\mathbf{W})$, where the two vectors lie on the $\mathbf{n}$-plane, we have

$$I'(\overline{\mathbf{u}},\overline{\mathbf{W}}) = I'(\mathbf{u},\mathbf{W}) \implies \exists\, \mathbf{R},\mathbf{Q}\in C_{2mh}(\mathbf{n}):\ \overline{\mathbf{u}}=\mathbf{Ru},\ \ \overline{\mathbf{W}}=\mathbf{QWQ}^{\mathrm{T}}.$$

Denoting $\mathbf{Q}_0=\mathbf{Q}^{\mathrm{T}}\mathbf{R}=\epsilon\mathbf{R}^{\psi}_{\mathbf{n}}$, $\epsilon^2=1$, and noting that both $\overline{\mathbf{u}}$ and $\overline{\mathbf{W}}\mathbf{n}$ are on the $\mathbf{n}$-plane, we infer

$$\begin{aligned} (\overset{\circ}{\overline{\mathbf{u}}}\cdot\overline{\mathbf{W}}\mathbf{n})^2 &= ((\mathbf{Q}_0\overset{\circ}{\mathbf{u}})\cdot\mathbf{Wn})^2 \\ &= |\overset{\circ}{\mathbf{u}}|^2\cdot|\mathbf{Wn}|^2\cos^2(\theta+\psi)\,, \\ (\overset{\circ}{\overline{\mathbf{u}}}\cdot\overline{\mathbf{W}}\mathbf{n})[\overset{\circ}{\overline{\mathbf{u}}},\overline{\mathbf{W}}\mathbf{n},\mathbf{n}] &= \frac{1}{2}|\overset{\circ}{\mathbf{u}}|^2|\mathbf{Wn}|^2\sin^2(\theta+\psi), \end{aligned}$$

where θ is the angle between $\overset{\circ}{\mathbf{u}}$ and $\mathbf{Wn}$, and $(\theta+\psi)$ the angle between $\mathbf{R}^{\psi}_{\mathbf{n}}\mathbf{u}$ and $\mathbf{Wn}$. When $\psi=0$, the above two equalities yield $(\overset{\circ}{\mathbf{u}}\cdot\mathbf{Wn})^2$ and $(\overset{\circ}{\mathbf{u}}\cdot\mathbf{Wn})[\overset{\circ}{\mathbf{u}},\mathbf{Wn},\mathbf{n}]$.

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Hence, we deduce

$$\begin{cases} (\overline{\overset{\circ}{\mathbf{u}}}\cdot\overline{\mathbf{W}}\mathbf{n})^2 = (\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n})^2\,, \\ (\overline{\overset{\circ}{\mathbf{u}}}\cdot\overline{\mathbf{W}}\mathbf{n})[\overline{\overset{\circ}{\mathbf{u}}}, \overline{\mathbf{W}}\mathbf{n}, \mathbf{n}] = (\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n})[\overset{\circ}{\mathbf{u}}, \mathbf{W}\mathbf{n}, \mathbf{n}]\,, \end{cases} \Longrightarrow \begin{cases} \cos 2(\theta+\psi) = \cos 2\theta\,, \\ \sin 2(\theta+\psi) = \sin 2\theta\,, \end{cases}$$

$$\Longrightarrow \psi = k\pi, \quad k = 0, \pm 1, \pm 2, \cdots .$$

Therefore,

$$\mathbf{R} = \mathbf{Q}\mathbf{Q}_0 = \epsilon\mathbf{Q}\mathbf{R}_{\mathbf{n}}^{k\pi}, \qquad \epsilon^2 = 1.$$

Let $\mathbf{R}_0 = (-1)^k\epsilon\mathbf{Q}$. Then, using the facts

$$\mathbf{u}\cdot\mathbf{n} = 0 \implies \mathbf{R}_{\mathbf{n}}^{k\pi}\mathbf{u} = (-1)^k\mathbf{u},$$

we deduce

$$\begin{aligned} \mathbf{R}_0\mathbf{u} &= (-1)^k\epsilon\mathbf{Q}\mathbf{u} = \epsilon\mathbf{Q}\mathbf{R}_{\mathbf{n}}^{k\pi}\mathbf{u} = \mathbf{R}\mathbf{u} = \overline{\mathbf{u}}, \\ \mathbf{R}_0\mathbf{W}\mathbf{R}_0^{\mathrm{T}} &= \mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}} = \overline{\mathbf{W}}, \end{aligned}$$

where $\mathbf{R}_0 \in C_{2mh}(\mathbf{n})$.

Thus, we conclude that the set $I(\mathbf{u}, \mathbf{W})$ obeys the criterion (1.5) for $(\mathbf{u}, \mathbf{W})$ specified by (4.15).

Case 6. A vector variable $\mathbf{u}$ and a symmetric tensor variable $\mathbf{A}$

The condition (2.6) yields

$$g(\mathbf{z}) \cap C_{2mh}(\mathbf{n}) \neq g(\mathbf{u}, \mathbf{A}) \cap C_{2mh}(\mathbf{n}), \qquad \mathbf{z} = \mathbf{u}, \mathbf{A}\,. \tag{4.16}$$

From this we derive

$$\begin{aligned} (\mathbf{u}\cdot\mathbf{n})\mathbf{u}\times\mathbf{n} &= \mathbf{0}, \qquad \mathbf{u} \neq \mathbf{0}\,, \\ g(\mathbf{A}) \cap C_{2mh}(\mathbf{n}) &= S_2\,, \qquad \text{i.e.}\ \overset{\circ}{\mathbf{A}}\mathbf{n} \neq \mathbf{0}\,, \end{aligned} \tag{4.17}$$

and

$$g(\mathbf{u}, \mathbf{A}) \cap C_{2mh}(\mathbf{n}) = C_1\,.$$

Thus, we construct the following table for vector generators and invariants (second order tensor generators and related invariants have been covered by Case 3 due to $(4.17)_2$).

V $\quad \overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{u}}\times\mathbf{n}, \mathbf{A}\mathbf{u}, \mathbf{A}(\mathbf{u}\times\mathbf{n}) + \mathbf{n}\times\mathbf{A}\mathbf{u}$

R $\quad \mathbf{r}\cdot\mathbf{A}\mathbf{u}, [\mathbf{n}, \mathbf{u}, \mathbf{A}\mathbf{r}] + [\mathbf{n}, \mathbf{r}, \mathbf{A}\mathbf{u}];$
$\quad < I_m^0(\mathbf{u}), I_m^0(\mathbf{A}), (1-\delta_{1m})(\overset{\circ}{\mathbf{u}}\cdot\mathbf{A}\mathbf{n})^2, (1-\delta_{1m})(\overset{\circ}{\mathbf{u}}\cdot\mathbf{A}\mathbf{n})[\overset{\circ}{\mathbf{u}}, \mathbf{A}\mathbf{n}, \mathbf{n}] >$

It is easy to treat the case when $\mathbf{u}\times\mathbf{n} = \mathbf{0}$. Let $\mathbf{u}\cdot\mathbf{n} = 0$. Following the same procedures used in the last case, we can prove the following facts: 1. The set of

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vector generators given in the above table is a generating set for vector-valued anisotropic functions of the two variables $(\mathbf{u}, \mathbf{A})$ specified by (4.17) under C_{2mh}, and 2. The set of invariants listed in the angle brackets is a functional basis of the two variables $(\mathbf{u}, \mathbf{A})$ specified by (4.17) under C_{2mh}, $m \geq 2$. Besides, it can be readily verified that the union $I_1^0(\mathbf{u}) \cup I_1^0(\mathbf{A})$ is a functional basis of the two variables $(\mathbf{u}, \mathbf{A})$ at issue under $C_{2h}(\mathbf{n})$. Hence, for each $m \geq 1$, the set listed in the angle brackets gives a functional basis of the two variables $(\mathbf{u}, \mathbf{A})$ specified by (4.17) under C_{2mh}.

Combining the above six cases, we arrive at the desired representations under each subgroup C_{2mh}. Moreover, it can be easily seen that the analysis given also applies to the transverse isotropy group $C_{\infty h}$ if the functional bases $I_m^0(\mathbf{u})$, $I_m^0(\mathbf{W})$ and $I_m^0(\mathbf{A})$ are respectively replaced by three bases of the single variables $\mathbf{u}$, $\mathbf{W}$ and $\mathbf{A}$ under the group $C_{\infty h}$. These facts are summerized as follows.

THEOREM 4.1. *The four sets given by*

$I_m^0(\mathbf{u}); I_m^0(\mathbf{W}); I_m^0(\mathbf{A});\ (\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}); \mathring{\mathbf{u}}\cdot\mathring{\mathbf{v}}, [\mathbf{n}, \mathring{\mathbf{u}}, \mathring{\mathbf{v}}];$

$(\mathbf{u}\cdot\mathbf{n})\mathring{\mathbf{u}}\cdot\mathbf{Wn}, (\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathbf{Wn}, \mathring{\mathbf{u}}], (1-\delta_{1m})(\mathring{\mathbf{u}}\cdot\mathbf{Wn})^2, (1-\delta_{1m})(\mathring{\mathbf{u}}\cdot\mathbf{Wn})[\mathring{\mathbf{u}}, \mathbf{Wn}, \mathbf{n}];$

$(\mathbf{u}\cdot\mathbf{n})\mathring{\mathbf{u}}\cdot\mathbf{An}, (\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathbf{An}, \mathring{\mathbf{u}}], (1-\delta_{1m})\mathring{\mathbf{u}}\cdot\mathbf{A}\mathring{\mathbf{u}}, (1-\delta_{1m})[\mathbf{n}, \mathring{\mathbf{u}}, \mathbf{A}\mathring{\mathbf{u}}],$

$(1-\delta_{1m})(\mathring{\mathbf{u}}\cdot\mathbf{An})^2, (1-\delta_{1m})(\mathring{\mathbf{u}}\cdot\mathbf{An})[\mathring{\mathbf{u}}, \mathbf{An}, \mathbf{n}];$

$\mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Hn}];$

$\mathbf{n}\cdot\mathbf{WAn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{An}], (1-\delta_{1m})\mathbf{n}\cdot\mathbf{WAWn}, (1-\delta_{1m})[\mathbf{n}, \mathbf{Wn}, \mathbf{AWn}];$

$\mathrm{tr}\,\mathbf{AB}, \mathrm{tr}\,\mathbf{ABN}, (1-\delta_{1m})\mathbf{n}\cdot\mathring{\mathbf{A}}\mathbf{B}\mathring{\mathbf{A}}\mathbf{n}, (1-\delta_{1m})[\mathbf{n}, \mathring{\mathbf{A}}\mathbf{n}, \mathbf{B}\mathring{\mathbf{A}}\mathbf{n}];$

$\mathbf{u}\cdot\mathbf{Wv}, [\mathbf{n}, \mathbf{u}, \mathbf{Wv}] + [\mathbf{n}, \mathbf{v}, \mathbf{Wu}];$

$\mathbf{u}\cdot\mathbf{Av}, [\mathbf{n}, \mathbf{u}, \mathbf{Av}] + [\mathbf{n}, \mathbf{v}, \mathbf{Au}];$

and

$\mathring{\mathbf{u}}, \mathring{\mathbf{u}} \times \mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n};$

$\mathbf{Wu}, \mathbf{W}(\mathbf{u}\times\mathbf{n}) + \mathbf{n}\times\mathbf{Wu};$

$\mathbf{Au}, \mathbf{A}(\mathbf{u}\times\mathbf{n}) + \mathbf{n}\times\mathbf{Au};$

and

$\mathrm{Skw}^0(\mathbf{u}); \mathrm{Skw}^0(\mathbf{W}); \mathrm{Skw}^0(\mathbf{A}); \mathbf{u}\wedge\mathbf{v}, (\mathbf{u}\times\mathbf{n})\wedge\mathbf{v} + (\mathbf{v}\times\mathbf{n})\wedge\mathbf{u};$

and

$\mathrm{Sym}_m^0(\mathbf{u}); \mathrm{Sym}_m^0(\mathbf{W}); \mathrm{Sym}_m^0(\mathbf{A}); \mathbf{u}\vee\mathbf{v}, (\mathbf{u}\times\mathbf{n})\vee\mathbf{v} + (\mathbf{v}\times\mathbf{n})\vee\mathbf{u};$

where $\mathbf{u}, \mathbf{v} = \mathbf{u}_1, \cdots, \mathbf{u}_a$; $\mathbf{W}, \mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$; $\mathbf{A}, \mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$; $\mathbf{u} \neq \mathbf{v}$, $\mathbf{W} \neq \mathbf{H}$, $\mathbf{A} \neq \mathbf{B}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the group* $C_{2mh}(\mathbf{n})$ *for each* $m = 1, 2, \cdots, \infty$, *respectively. For* $m = \infty$, *i.e. for the transverse isotropy group* $C_{\infty h}(\mathbf{n})$, *it is assumed that* $\delta_{1\infty} = 0$ *and moreover that*

$$I_\infty^0(\mathbf{u}) = \{(\mathbf{u}\cdot\mathbf{n})^2, |\mathbf{u}|^2\}, \qquad I_\infty^0(\mathbf{W}) = \{\mathrm{tr}\,\mathbf{WN}, \mathrm{tr}\,\mathbf{W}^2\},$$
$$I_\infty^0(\mathbf{A}) = \{\mathrm{tr}\,\mathbf{A}, \mathrm{tr}\,\mathbf{A}^2, \mathrm{tr}\,\mathbf{A}^3, \mathbf{n}\cdot\mathbf{An}, \mathbf{n}\cdot\mathbf{A}^2\mathbf{n}, [\mathbf{n}, \mathbf{An}, \mathbf{A}^2\mathbf{n}]\}.$$

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REMARK. Irreducible nonpolynomial representations for transversely isotropic functions of the variables $X \in \mathcal{D}$ relative to $C_{\infty h}$ were first derived by ZHENG [41], which belongs to the case when $m = \infty$ in the above theorem. It can be seen that the new results presented in the above theorem are more compact than those given in ZHENG [41], e.g. the presented functional basis of two symmetric tensor variables and the presented generating set for a single symmetric tensor variable, respectively, consist of eighteen invariants and six generators, respectively, while the corresponding results given in ZHENG [41] include nineteen invariants and eight generators, respectively.

4.2. The classes C_{2m}

Let $I^0(\mathbf{H}_1, \cdots, \mathbf{H}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c)$, $\mathrm{Skw}^0(\mathbf{H}_1, \cdots, \mathbf{H}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c)$ and $\mathrm{Sym}^0(\mathbf{H}_1, \cdots, \mathbf{H}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c)$ be, respectively, an irreducible functional basis and irreducible generating sets for scalar-valued, skewsymmetric and symmetric tensor-valued anisotropic functions of $(a + b)$ skewsymmetric tensor variables and c symmetric tensor variables under an orthogonal subgroup g containing the central inversion $-\mathbf{I}$. Then, according to Theorem 2.1 and 2.2 in XIAO [35], the four sets

$$\begin{aligned}
&I^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),\\
\mathbf{E} : &\mathrm{Skw}^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),\\
&\mathrm{Skw}^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),\\
&\mathrm{Sym}^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),
\end{aligned}$$

supply, respectively, an irreducible functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of a vector variables, b skewsymmetric tensor variables and c symmetric tensor variables under the rotation subgroup of g, i.e. $g \cap \mathrm{Orth}^+$. Here, the second set above is obtained by forming the double dot product between each skewsymmetric tensor generator and the third order Eddington tensor $\mathbf{E}$. From this fact and Theorem 4.1, we derive the following result.

THEOREM 4.2. *The four sets given by*

$I^0_m(\mathbf{W}), I^0_m(\mathbf{A})$;
$\mathbf{u}\cdot\mathbf{n}, \delta_{m\infty}|\mathbf{u}|^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e})^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e}')^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e})(\mathbf{u}\cdot\mathbf{e}')$,
$(1-\delta_{1m})\alpha_{2m}(\mathring{\mathbf{u}}), (1-\delta_{1m})\beta_{2m}(\mathring{\mathbf{u}})$;
$\mathring{\mathbf{u}}\cdot\mathring{\mathbf{v}}, [\mathbf{n}, \mathring{\mathbf{u}}, \mathring{\mathbf{v}}]$;
$[\mathbf{n}, \mathbf{u}, \mathbf{Wn}], \mathbf{u}\cdot\mathbf{Wn}$;
$[\mathbf{n}, \mathbf{u}, \mathbf{An}], \mathbf{u}\cdot\mathbf{An}, (1-\delta_{1m})\mathring{\mathbf{u}}\cdot\mathbf{A}\mathring{\mathbf{u}}, (1-\delta_{1m})[\mathbf{n}, \mathbf{u}, \mathbf{Au}]$;
$\mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Hn}]$;
$\mathbf{n}\cdot\mathbf{WAn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{An}], (1-\delta_{1m})\mathbf{n}\cdot\mathbf{WAWn}, (1-\delta_{1m})[\mathbf{n}, \mathbf{Wn}, \mathbf{AWn}]$;
$\mathrm{tr}\,\mathbf{AB}, \mathrm{tr}\,\mathbf{ABN}, (1-\delta_{1m})\mathbf{n}\cdot\mathring{\mathbf{A}}\mathbf{B}\mathring{\mathbf{A}}\mathbf{n}, (1-\delta_{1m})[\mathbf{n}, \mathring{\mathbf{A}}\mathbf{n}, \mathbf{B}\mathring{\mathbf{A}}\mathbf{n}]$;

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and

$\mathbf{n}, \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}} \times \mathbf{n}$;

$\mathbf{W}\mathbf{n}, \mathbf{n} \times \mathbf{W}\mathbf{n}$;

$\mathbf{A}\mathbf{n}, \mathbf{n} \times \mathbf{A}\mathbf{n}$;

and

$\mathbf{N}, \mathbf{n} \wedge \overset{\circ}{\mathbf{u}}, \mathbf{n} \wedge (\mathbf{n} \times \overset{\circ}{\mathbf{u}}); \mathrm{Skw}^0(\mathbf{W}); \mathrm{Skw}^0(\mathbf{A})$;

and

$\mathrm{Sym}^0_m(\mathbf{E}\mathbf{u}), \mathrm{Sym}^0_m(\mathbf{W}), \mathrm{Sym}^0_m(\mathbf{A})$;

where $\mathbf{u}, \mathbf{v} = \mathbf{u}_1, \cdots, \mathbf{u}_a$; $\mathbf{W}, \mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$; $\mathbf{A}, \mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$; $\mathbf{u} \neq \mathbf{v}$, $\mathbf{W} \neq \mathbf{H}$, $\mathbf{A} \neq \mathbf{B}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the group* $C_{2m}(\mathbf{n})$ *for each* $m = 1, 2, \cdots, \infty$, *respectively. When* $m = \infty$, *the bases* $I^0_\infty(\mathbf{W})$ *and* $I^0_\infty(\mathbf{A})$ *are given in Theorem 4.1 and moreover*

$$\delta_{m\infty} = \begin{cases} 1, & m = \infty, \\ 0, & m = 1, 2, \cdots. \end{cases}$$

5. The classes S_{4m}

The classes S_{4m} include the tetrahedral crystal class S_4 as the particular case when $m = 1$.

As pointed out in Sec. 2, for any $X \in \mathcal{D}$ there is $X_0 \subset X$ such that (2.4) holds, where X_0 consists of two vectors or a vector and a second order tensor, i.e. $X_0 = (\mathbf{u}, \mathbf{x})$, $\mathbf{x} \in V \cup \mathrm{Skw} \cup \mathrm{Sym}$. Furthermore, from the facts

$$g(\mathbf{r}) \cap S_{4m}(\mathbf{n}) = \begin{cases} S_{4m}(\mathbf{n}), & \mathbf{u} = \mathbf{0}, \\ C_{2m}(\mathbf{n}), & \overset{\circ}{\mathbf{u}} = \mathbf{0}, \quad \mathbf{u}\cdot\mathbf{n} \neq 0, \\ C_1, & \overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \end{cases} \tag{5.1}$$

$$g(\mathbf{W}) \cap S_{4m}(\mathbf{n}) = \begin{cases} S_{4m}(\mathbf{n}), & \mathbf{W}\mathbf{n} = \mathbf{0}, \\ C_1, & \mathbf{W}\mathbf{n} \neq \mathbf{0}, \end{cases} \tag{5.2}$$

$$g(\mathbf{A}) \cap S_{4m}(\mathbf{n}) = \begin{cases} S_{4m}(\mathbf{n}), & \overset{\circ}{\mathbf{A}} = x\mathbf{I} + y\mathbf{n} \otimes \mathbf{n}, \\ C_2(\mathbf{n}), & \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0}, \quad \mathbf{q}(\mathbf{A}) \neq \mathbf{0}, \\ C_1, & \overset{\circ}{\mathbf{A}}\mathbf{n} \neq \mathbf{0}, \end{cases} \tag{5.3}$$

we infer that for any $X \in \mathcal{D}$ there is a single vector or a single second order tensor $\mathbf{z} \in V \cup \mathrm{Skw} \cup \mathrm{Sym}$ such that

$$g(\mathbf{z}) \cap S_{4m}(\mathbf{n}) = g(X) \cap S_{4m}(\mathbf{n}).$$

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The above fact indicates that the cases for two variables can further be reduced to the cases for single variables, which are discussed as follows.

Case 1. A single vector variable $\mathbf{u}$

According to the related results in Xiao [35, 39], representations for scalar-, vector-, and second order tensor-valued anisotropic functions of the vector variable $\mathbf{u}$ under the group $S_{4m}(\mathbf{n})$ may be obtained from those for scalar-, vector-, and second order tensor-valued isotropic functions of the extended variables $(\mathbf{u}, \boldsymbol{\phi}_m(\mathbf{u}), \mathbf{N})$ respectively, where

$$\boldsymbol{\phi}_m(\mathbf{u}) = \mathbf{n} \vee \boldsymbol{\rho}_{2m-1}(\mathring{\mathbf{u}}) + \delta_{1m}(\mathbf{u}\cdot\mathbf{n})(\mathbf{e}\otimes\mathbf{e} - \mathbf{e}'\otimes\mathbf{e}'), \tag{5.4}$$

and where $\boldsymbol{\rho}_{2m-1}(\mathring{\mathbf{u}})$ is a vector depending on $\mathring{\mathbf{u}}$, given by (1.11).

We mention that the second term at the right-hand side of (5.4) comes into play only when the group $S_4(\mathbf{n})$ is concerned. This fact implies the particular property of the group $S_4(\mathbf{n})$, which will be seen below.

Applying the above fact and the related results for isotropic functions and then removing some redundant elements, we construct the following table.

V	$\{\mathring{\mathbf{u}}, \mathring{\mathbf{u}}\times\mathbf{n}, \alpha_{2m}(\mathring{\mathbf{u}})\mathbf{n}, \beta_{2m}(\mathring{\mathbf{u}})\mathbf{n}\}$ $(= V_m^1(\mathbf{u}))$
Skw	$\{\mathbf{N}, \mathbf{n}\wedge\boldsymbol{\rho}_{2m-1}(\mathring{\mathbf{u}}), \mathbf{n}\wedge(\boldsymbol{\rho}_{2m-1}(\mathring{\mathbf{u}})\times\mathbf{n})\}$ $(= \mathrm{Skw}_m^1(\mathbf{u}))$
Sym	$\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathring{\mathbf{u}}\otimes\mathring{\mathbf{u}}, \mathring{\mathbf{u}}\vee(\mathring{\mathbf{u}}\times\mathbf{n}), \boldsymbol{\phi}_m(\mathbf{u}), \boldsymbol{\phi}_m(\mathbf{u})\mathbf{N} - \mathbf{N}\boldsymbol{\phi}_m(\mathbf{u})\}$ $(= \mathrm{Sym}_m^1(\mathbf{u}))$
R	$\mathring{\mathbf{u}}\cdot\mathring{\mathbf{r}}, [\mathbf{n}, \mathring{\mathbf{u}}, \mathring{\mathbf{r}}], (\mathbf{r}\cdot\mathbf{n})\alpha_{2m}(\mathring{\mathbf{u}}), (\mathbf{r}\cdot\mathbf{n})\beta_{2m}(\mathring{\mathbf{u}})$; $\mathrm{tr}\,\mathbf{HN}, \boldsymbol{\rho}_{2m-1}(\mathring{\mathbf{u}})\cdot\mathbf{Hn}, [\mathbf{n}, \mathbf{Hn}, \boldsymbol{\rho}_{2m-1}(\mathring{\mathbf{u}})]$; $\mathrm{tr}\,\mathbf{B}, \mathbf{n}\cdot\mathbf{Bn}, \mathring{\mathbf{u}}\cdot\mathbf{B}\mathring{\mathbf{u}}, [\mathbf{n}, \mathring{\mathbf{u}}, \mathbf{B}\mathring{\mathbf{u}}], \mathrm{tr}\,\boldsymbol{\phi}_m(\mathbf{u})\mathbf{B}, \mathrm{tr}\,\boldsymbol{\phi}_m(\mathbf{u})\mathbf{BN}$; $< (\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\mathring{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\mathring{\mathbf{u}}), \alpha_{4m}(\mathring{\mathbf{u}}), \beta_{4m}(\mathring{\mathbf{u}}) >$ $(= I_m^1(\mathbf{u}))$

In the following, we prove that the first three sets given in the above table, denoted by $V_m^1(\mathbf{u})$ and $\mathrm{Skw}_m^1(\mathbf{u})$ and $\mathrm{Sym}_m^1(\mathbf{u})$ henceforth, are irreducible generating sets for vector-valued and skewsymmetric and symmetric tensor-valued anisotropic functions of the vector variable $\mathbf{u}$ under $S_{4m}(\mathbf{n})$, respectively. To this end, we prove that each of these sets obeys the criterion (1.1). In fact, we have

$$\alpha_{2m}(\mathring{\mathbf{u}}) = \beta_{2m}(\mathring{\mathbf{u}}) = 0 \iff \boldsymbol{\rho}_{2m-1}(\mathring{\mathbf{u}}) = \mathbf{0} \iff \mathring{\mathbf{u}} = \mathbf{0}. \tag{5.5}$$

Then, by using (4.1) and the latter, as well as (1.2) and (1.3), we deduce that $V_m^1(\mathbf{u})$ and $\mathrm{Skw}_m^1(\mathbf{W})$ obey (1.1) respectively. For $\mathrm{Sym}_m^1(\mathbf{u})$, the group S_4 and the groups S_{4m} for $m \geq 2$ should be considered separately due to the following particular property of the former:

$$\dim\mathrm{Sym}(g(\mathbf{u})\cap S_4(\mathbf{n})) = \begin{cases} \dim\mathrm{Sym}(S_4(\mathbf{n})) = 2, & \mathbf{u} = \mathbf{0}, \\ \dim\mathrm{Sym}(C_2(\mathbf{n})) = 4, & \mathring{\mathbf{u}} = \mathbf{0}, \quad \mathbf{u}\cdot\mathbf{n} \neq 0, \\ \mathrm{rank}\,\mathrm{Sym}(C_1) = 6, & \mathring{\mathbf{u}} \neq \mathbf{0}, \end{cases} \tag{5.6}$$

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while for $m \geq 2$,

$$\dim \mathrm{Sym}(g(\mathbf{u}) \cap S_{4m}(\mathbf{n})) = \begin{cases} 2, & \mathring{\mathbf{u}} = \mathbf{0}, \\ 6, & \mathring{\mathbf{u}} \neq \mathbf{0}. \end{cases} \tag{5.7}$$

For $m = 1$, we have

$$\mathrm{rank}\,\mathrm{Sym}_1^1(\mathbf{u}) = \begin{cases} \mathrm{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}\} = 2, & \mathbf{u} = \mathbf{0}, \\ \{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{e}\otimes\mathbf{e} - \mathbf{e}'\otimes\mathbf{e}', \mathbf{e}\vee\mathbf{e}'\} = 4, & \mathring{\mathbf{u}} = \mathbf{0}, \quad \mathbf{u} \neq \mathbf{0}, \\ \{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathring{\mathbf{u}}\otimes\mathring{\mathbf{u}}, \mathring{\mathbf{u}}\otimes(\mathring{\mathbf{u}}\times\mathbf{n}), \mathbf{n}\vee\boldsymbol{\rho}_1(\mathring{\mathbf{u}}), & \\ \qquad \mathbf{n}\vee(\mathbf{n}\times\boldsymbol{\rho}_1(\mathring{\mathbf{u}})\} = 6, & \mathring{\mathbf{u}} \neq \mathbf{0}, \end{cases}$$

and for $m \geq 2$ we have

$$\mathrm{rank}\,\mathrm{Sym}_m^1(\mathbf{u}) = \begin{cases} \mathrm{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}\} = 2, & \mathring{\mathbf{u}} = \mathbf{0}, \\ 6, & \mathring{\mathbf{u}} \neq \mathbf{0}, \end{cases}$$

Thus, we deduce that the set $\mathrm{Sym}_m(\mathbf{u})$ obeys (1.1) for each $m \geq 1$.

It is evident that both $\mathrm{Skw}_m^1(\mathbf{u})$ and $\mathrm{Sym}_m^1(\mathbf{u})$ are irreducible. Moreover, by considering the facts

$$\beta_{2m}(\mathring{\mathbf{u}}) = 0 \qquad \text{for} \quad \mathbf{u} = \mathbf{e},$$
$$\alpha_{2m}(\mathring{\mathbf{u}}) = 0 \qquad \text{for} \quad \mathbf{u} = \mathbf{R}_{\mathbf{n}}^{\pi/4m}\mathbf{e},$$

we know that $V_m^1(\mathbf{u})$ is also irreducible.

Next, we prove that the set given in the above table in the angle brackets, denoted by $I_m^1(\mathbf{u})$ henceforth, is an irreducible functional basis of the vector variable $\mathbf{u}$ under $S_{4m}(\mathbf{n})$. First, suppose $\mathbf{u}\cdot\mathbf{n} \neq 0$. Then we infer

$$\begin{aligned} I_m^1(\overline{\mathbf{u}}) = I_m^1(\mathbf{u}) \Longrightarrow |\mathring{\overline{\mathbf{u}}}| = |\mathring{\mathbf{u}}|, \quad \overline{\mathbf{u}}\cdot\mathbf{n} = \delta\mathbf{u}\cdot\mathbf{n}, \quad \cos 2m\overline{\theta} = \delta\cos 2m\theta, \\ \sin 2m\overline{\theta} = \delta\sin 2m\theta \\ \Longrightarrow \overline{\theta} = \frac{4p+1-\delta}{4m}\pi + \theta, \quad p = 0, \pm1, \pm2, \cdots; \quad \overline{\mathbf{u}}\cdot\mathbf{n} = \delta\mathbf{u}\cdot\mathbf{n}, \\ |\mathring{\overline{\mathbf{u}}}| = |\mathring{\mathbf{u}}|, \end{aligned}$$

where $\delta^2 = 1$, $\overline{\theta} =< \mathring{\overline{\mathbf{u}}}, \mathbf{e} >$ and $\theta =< \mathring{\mathbf{u}}, \mathbf{e} >$. Thus, we have $\overline{\mathbf{u}} = \mathbf{Q}\mathbf{u}$, where $\mathbf{Q} \in S_{4m}(\mathbf{n})$ depends on δ, given by

$$\mathbf{Q} = \begin{cases} \mathbf{R}_{\mathbf{n}}^{2p\pi/2m}, & \delta = 1, \\ -\mathbf{R}_{\mathbf{n}}^{(2m+2p+1)\pi/2m}, & \delta = -1. \end{cases}$$

Hence the set $I_m^1(\mathbf{u})$ obeys (1.5) when $\mathbf{u}\cdot\mathbf{n} \neq 0$.

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Second, suppose $\mathbf{u}\cdot\mathbf{n}=0$, i.e. $\overset{\circ}{\mathbf{u}}=\mathbf{u}$. Since $\mathbf{R}_{\mathbf{n}}^{\pi}\in S_{4m}(\mathbf{n})$, we have

$$f(\mathbf{u})=f(\mathbf{R}_{\mathbf{n}}^{\pi}\mathbf{u})=f(-\mathbf{u})=f((-\mathbf{I})\mathbf{u})$$

for each invariant $f(\mathbf{u})$ under $S_{4m}(\mathbf{n})$. Since the group $S_{4m}(\mathbf{n})$ and the central inversion $-\mathbf{I}$ generate the group $C_{4mh}(\mathbf{n})$, the above fact implies that for the vector variable $\mathbf{u}=\overset{\circ}{\mathbf{u}}$ each invariant of $\mathbf{u}$ under $S_{4m}(\mathbf{n})$ turns out to be an invariant of $\mathbf{u}$ under the larger group $C_{4mh}(\mathbf{n})\supset S_{4m}(\mathbf{n})$. Thus, for the variable $\mathbf{u}$ obeying $\mathbf{u}\cdot\mathbf{n}=0$, a functional basis of $\mathbf{u}$ under S_{4m} is provided by a functional basis of $\mathbf{u}$ under C_{4mh}, the latter being formed by the last two invariants listed in the presented table.

From the above, we know that $I_m^1(\mathbf{u})$ is a functional basis of the vector $\mathbf{u}$ under $S_{4m}(\mathbf{n})$. Furthermore, from the following four pairs $(X,X')=(\mathbf{u},\mathbf{u}')$ fulfilling the condition (1.6) we infer that each of its elements is irreducible.

$$\beta_{4m}(\overset{\circ}{\mathbf{u}}):\mathbf{u}=\mathbf{R}_{\mathbf{n}}^{\pi/8m}\mathbf{e},\ \mathbf{u}'=2\mathbf{u};\ \alpha_{4m}(\overset{\circ}{\mathbf{u}}):\ \mathbf{u}=\mathbf{e},\ \mathbf{u}'=2\mathbf{e};$$
$$(\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\overset{\circ}{\mathbf{u}}):\mathbf{u}=\mathbf{n}+\mathbf{e},\ \mathbf{u}'=2\mathbf{u};\ \beta_{2m}(\overset{\circ}{\mathbf{u}}):\ \mathbf{u}=\mathbf{n}+\mathbf{R}_{\mathbf{n}}^{\pi/4m}\mathbf{e},\ \mathbf{u}'=2\mathbf{u}.$$

Case 2. A single skewsymmetric tensor variable **W**

Every scalar-valued (resp. second order tensor-valued) anisotropic function of $\mathbf{W}$ under $S_{4m}(\mathbf{n})$ is equivalent to a scalar-valued (resp. second order tensor-valued) anisotropic function of $\mathbf{W}$ under $C_{4mh}(\mathbf{n})$. Irreducible representations for the latter can be obtained by merely replacing m with $2m$ in the table for Case 2 of Sec. 4.1. As a result, we need only to consider vector-valued functions.

Representations for vector-valued anisotropic functions of the skewsymmetric tensor variable $\mathbf{W}$ under $S_{4m}(\mathbf{n})$ can be derived from those for isotropic functions of the extended variables $(\mathbf{W},\boldsymbol{\rho}_{2m-1}(\mathbf{Wn}),\mathbf{N})$ (see Xiao [35, 39]), where $\boldsymbol{\rho}_{2m-1}(\mathbf{Wn})$ is defined by (1.11). Applying this fact, we derive an irreducible generating set for vector-valued anisotropic functions of the variable $\mathbf{W}\in\mathrm{Skw}$ and the related invariants formed by the inner product between the generic vector variable $\mathbf{r}$ and each vector generator as follows.

V	$\{\boldsymbol{\rho}_{2m-1}(\mathbf{Wn}),\ \mathbf{n}\times\boldsymbol{\rho}_{2m-1}(\mathbf{Wn}),\ (\alpha_{2m}(\mathbf{Wn})\mathbf{n},\ \beta_{2m}(\mathbf{Wn})\mathbf{n}\}\ (=V_m^1(\mathbf{W}))$,
R	$(\mathbf{r}\cdot\mathbf{n})\alpha_{2m}(\mathbf{Wn}),\ (\mathbf{r}\cdot\mathbf{n})\beta_{2m}(\mathbf{Wn})$.

In the above table, the generic vector variable $\mathbf{r}$ is treated as being subject to the condition that $\mathbf{u}\times\mathbf{n}=\mathbf{0}$, since the case when there is an $\mathbf{r}$ such that $\mathbf{r}\times\mathbf{n}\neq\mathbf{0}$, i.e. $g(\mathbf{r})\cap S_{4m}(\mathbf{n})=C_1$, has been covered by Case 1. The same is true for the next case.

Case 3. A single symmetric tensor variable **A**

Every scalar-valued (resp. second order tensor-valued) anisotropic function of $\mathbf{A}$ under $S_{4m}(\mathbf{n})$ is equivalent to a scalar-valued (resp. second order tensor-valued)

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anisotropic function of $\mathbf{W}$ under $C_{4mh}(\mathbf{n})$. Irreducible representations for the latter can be obtained by merely replacing m with $2m$ in the Table for Case 3 of Sec. 4.1. As a result, we only need to consider vector-valued functions and the related invariants. According to XIAO [35, 39], generating sets for vector-valued anisotropic functions of the symmetric tensor variable $\mathbf{A}$ under $S_{4m}(\mathbf{n})$ can be derived from those for isotropic functions of the four extended variables $(\mathbf{N}, \mathbf{A}, \beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n}, \boldsymbol{\rho}'_m(\mathbf{A}))$, where

$$\boldsymbol{\rho}'_m(\mathbf{A}) = \boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n}) + |\mathbf{q}(\mathbf{A})|^{m-1}\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n}. \tag{5.8}$$

Basing on the above facts, we construct the following table (refer to the remark at the end of the last case).

V $\{\boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n}) + |\mathbf{q}(\mathbf{A})|^{m-1}\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n}, \mathbf{n}\times\boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n}) + |\mathbf{q}(\mathbf{A})|^{m-1}\beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n},$
$\alpha_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n})\mathbf{n}, \beta_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n})\mathbf{n}\}(= V^1_m(\mathbf{A})),$

R $(\mathbf{r}\cdot\mathbf{n})\alpha_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), (\mathbf{r}\cdot\mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), (\mathbf{r}\cdot\mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A})), (\mathbf{r}\cdot\mathbf{n})\beta_m(\mathbf{q}(\mathbf{A}))$.

We proceed to prove that the set $I^1_m(\mathbf{A})$ given above is an irreducible generating set for the vector-valued anisotropic functions of a symmetric tensor under the group $S_{4m}(\mathbf{n})$. By using (1.2) and the fact (see (5.8) and Case 1)

$$\boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n})\cdot\overset{\circ}{\mathbf{A}}\mathbf{n} = \left[\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}, \boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n})\right] = 0 \iff \boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n}) = \mathbf{0} \iff \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0}$$

as well as $g(\mathbf{A})\cap S_{4m}(\mathbf{n}) = C_1$ for $\overset{\circ}{\mathbf{A}}\mathbf{n} \neq \mathbf{0}$, we infer that the criterion (1.1) can be satisfied when $\overset{\circ}{\mathbf{A}}\mathbf{n} \neq \mathbf{0}$ and each of the presented generators is irreducible. On the other hand, let $\overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0}$. Then by using (1.2) and the facts

$$g(\mathbf{A})\cap S_{4m}(\mathbf{n}) = \begin{cases} C_2(\mathbf{n}), & \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0},\ \mathbf{q}(\mathbf{A}) \neq \mathbf{0}, \\ S_{4m}(\mathbf{n}), & \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{q}(\mathbf{A}) = \mathbf{0}, \end{cases}$$

$$\overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0},\ \mu_m(\overset{\circ}{\mathbf{A}}) = \nu_m(\overset{\circ}{\mathbf{A}}) = 0 \iff \overset{\circ}{\mathbf{A}} = \mathbf{O},$$

we deduce that the criterion (1.1) can also be satisfied when $\overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{O}$, i.e. $\mathbf{n}\times\mathbf{A}\mathbf{n} = \mathbf{0}$.

Combining the above three cases, we arrive at the main result of this section.

THEOREM 5.1. *The four sets given by*
$I^1_m(\mathbf{u}), I^0_{2m}(\mathbf{W}), I^0_{2m}(\mathbf{A})$;
$\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{v}}, [\mathbf{n}, \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{v}}], (\mathbf{v}\cdot\mathbf{n})\alpha_{2m}(\overset{\circ}{\mathbf{u}}), (\mathbf{v}\cdot\mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{u}})$;

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$(\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\mathbf{Wn}), (\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\mathbf{Wn}), \boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{Wn}, [\mathbf{n}, \mathbf{Wn}, \boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{u}})]$;

$\overset{\circ}{\mathbf{u}}\cdot\mathbf{A}\overset{\circ}{\mathbf{u}}, [\mathbf{n}, \overset{\circ}{\mathbf{u}}, \mathbf{A}\overset{\circ}{\mathbf{u}}], \operatorname{tr}\boldsymbol{\phi}_m(\overset{\circ}{\mathbf{u}})\mathbf{A}, \operatorname{tr}\boldsymbol{\phi}_m(\overset{\circ}{\mathbf{u}})\mathbf{AN}, (\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), (\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n})$,
$(\mathbf{u}\cdot\mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})\beta_m(\mathbf{q}(\mathbf{A}))$;

$\mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Hn}]$;

$\mathbf{n}\cdot\mathbf{WAn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{An}], \mathbf{n}\cdot\mathbf{WAWn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{WAn}]$;

$\operatorname{tr}\mathbf{AB}, \operatorname{tr}\mathbf{ABN}, \mathbf{n}\cdot\overset{\circ}{\mathbf{A}}\mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}, [\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}]$;

and

$V_m^1(\mathbf{u}), V_m^1(\mathbf{W}), V_m^1(\mathbf{A})$;

and

$\operatorname{Skw}_m^1(\mathbf{u}), \operatorname{Skw}^0(\mathbf{W}), \operatorname{Skw}^0(\mathbf{A})$;

and

$\operatorname{Sym}_m^1(\mathbf{u}), \operatorname{Sym}^0(\mathbf{W}), \operatorname{Sym}^0(\mathbf{A})$;

where $\mathbf{u}, \mathbf{v} = \mathbf{u}_1, \cdots, \mathbf{u}_a$; $\mathbf{W}, \mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$; $\mathbf{A}, \mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$; $\mathbf{u} \neq \mathbf{v}$, $\mathbf{W} \neq \mathbf{H}$, $\mathbf{A} \neq \mathbf{B}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the group* $S_{4m}(\mathbf{n})$ *for each integer* $m \geq 1$, *respectively.*

6. The classes C_{2m+1h}

The classes C_{2m+1h} include the hexagonal crystal class C_{3h} as the particular case when $m = 1$.

Following the scheme designed in Sec. 2, in what follows we discuss various cases for a single variable and two variables.

CASE 1. A single vector variable $\mathbf{u}$

By means of the criterion (1.1) and (1.2)-(1.4) as well as the facts

$$g(\mathbf{u}) \cap C_{2m+1h}(\mathbf{n}) = \begin{cases} C_{2m+1h}(\mathbf{n}), & \mathbf{u} = \mathbf{0}, \\ C_{2m+1}(\mathbf{n}), & \overset{\circ}{\mathbf{u}} = \mathbf{0}, \quad \mathbf{u}\cdot\mathbf{n} \neq 0, \\ C_{1h}(\mathbf{n}), & \mathbf{u}\cdot\mathbf{n} = 0, \quad \overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \\ C_1, & (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \end{cases} \tag{6.1}$$

we infer that the sets $V^0(\mathbf{u})$, $\operatorname{Skw}^0(\mathbf{u})$ and $\operatorname{Sym}^0(\mathbf{u})$ listed in the table for Case 1 in Sec. 4.1 are also generating sets for vector-valued and skewsymmetric and symmetric tensor-valued anisotropic functions of the vector variable $\mathbf{u}$ under $C_{2m+1h}(\mathbf{n})$. In the following, we show that the set given below is an irreducible functional basis of the vector variable $\mathbf{u}$ under $C_{2m+1h}(\mathbf{n})$.

$$I_m^2(\mathbf{u}) = \{(\mathbf{u}\cdot\mathbf{n})^2, \alpha_{2m+1}(\overset{\circ}{\mathbf{u}}), \beta_{2m+1}(\overset{\circ}{\mathbf{u}})\}. \tag{6.2}$$

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In fact, we have

$$I_m^2(\bar{\mathbf{u}}) = I_m^2(\mathbf{u}) \Longrightarrow \begin{cases} \bar{\mathbf{u}}\cdot\mathbf{n} = \delta\mathbf{u}\cdot\mathbf{n}, \ |\overset{\circ}{\bar{\mathbf{u}}}| = |\overset{\circ}{\mathbf{u}}|, \\ \cos(2m+1)\bar{\theta} = \cos(2m+1)\theta, \ \sin(2m+1)\bar{\theta} = \sin(2m+1)\theta, \end{cases}$$

$$\Longrightarrow \bar{\mathbf{u}}\cdot\mathbf{n} = \delta\mathbf{u}\cdot\mathbf{n}, |\overset{\circ}{\bar{\mathbf{u}}}| = |\overset{\circ}{\mathbf{u}}|, \bar{\theta} = \frac{2k\pi}{2m+1} + \theta, k = 0, \pm 1, \cdots$$

$$\Longrightarrow \bar{\mathbf{u}} = \mathbf{Q}\mathbf{u},$$

where $\delta^2 = 1$, $\bar{\theta} =< \overset{\circ}{\bar{\mathbf{u}}}, \mathbf{e} >$, $\theta =< \overset{\circ}{\mathbf{u}}, \mathbf{e} >$, and $\mathbf{Q} \in C_{2m+1h}(\mathbf{n})$ depends on δ, given by

$$\mathbf{Q} = \begin{cases} \mathbf{R}_{\mathbf{n}}^{2k\pi/2m+1}, & \delta = 1, \\ -\mathbf{R}_{\mathbf{n}}^{\pi}\mathbf{R}_{\mathbf{n}}^{2k\pi/2m+1}, & \delta = -1. \end{cases}$$

Thus, we infer that the set $I_m^2(\mathbf{u})$ obeys the criterion (1.5) and hence that it is a functional basis of $\mathbf{u}$ under $C_{2m+1h}(\mathbf{n})$. It is readily shown that this basis is irreducible.

From the above, we conclude that for the case at issue, the derived results are obtained by taking $\delta_{1m} = 0$ and replacing the basis $I_m^0(\mathbf{u})$ with the basis $I_m^2(\mathbf{u})$ in the Table for Case 1 in Sec. 4.1.

CASE 2. A single second order tensor variable

Since a scalar-valued (resp. second order tensor-valued) anisotropic function of a second order tensor variable $\mathbf{x}$ under $C_{2m+1h}(\mathbf{n})$ is equivalent to a scalar-valued (resp. second order tensor-valued) anisotropic functions of $\mathbf{x}$ under $C_{4m+2h}(\mathbf{n})$, we know that irreducible functional basis or generating sets for scalar-valued and skewsymmetric and symmetric tensor-valued anisotropic functions of the variable $\mathbf{x} = \mathbf{W} \in$ Skw (resp. $\mathbf{x} = \mathbf{A} \in$ Sym) under $C_{2m+1h}(\mathbf{n})$, as well as the invariants formed by means of the inner product between each presented second order tensor generator and a generic second order tensor variable, are given by the corresponding ones listed in the Tables for Cases 2—3 in Sec. 4.1 with m replaced by $2m+1$.

In view of the above facts, in what follows we need only to derive a generating set for the vector-valued anisotropic functions of the variable $\mathbf{W} \in$ Skw or $\mathbf{A} \in$ Sym under $C_{2m+1h}(\mathbf{n})$, and meanwhile provide the invariants formed by the inner product between each presented vector generator and the generic vector variable $\mathbf{r}$. The desired generating sets are obtainable from those for vector-valued isotropic functions of the extended variables $(\mathbf{W}, \boldsymbol{\rho}_{2m}(\mathbf{Wn}), \mathbf{N})$ or $(\mathbf{A}, \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{A}\mathbf{n}}), \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), \mathbf{N})$ (see XIAO [35, 39]).

Applying the above facts, we construct the following tables for the single variables $\mathbf{W}$ and $\mathbf{A}$, respectively.

V	$\{\boldsymbol{\rho}_{2m}(\mathbf{Wn}), \mathbf{n}\times\boldsymbol{\rho}_{2m}(\mathbf{Wn}), \alpha_{2m+1}(\mathbf{Wn})\mathbf{n}, \beta_{2m+1}(\mathbf{Wn})\mathbf{n}\}$ $(= V_m^2(\mathbf{W}))$,
R	$\mathbf{r}\cdot\boldsymbol{\rho}_{2m}(\mathbf{Wn}), [\mathbf{n}, \mathbf{r}, \boldsymbol{\rho}_{2m}(\mathbf{Wn})], (\mathbf{r}\cdot\mathbf{n})\alpha_{2m+1}(\mathbf{Wn}), (\mathbf{r}\cdot\mathbf{n})\beta_{2m+1}(\mathbf{Wn})$.

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and

$$\begin{array}{ll} V & \{\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), \mathbf{n}\times\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), |\mathbf{q}(\mathbf{A})|^{m+1}\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})) + \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n})\mathbf{n}, \\ & |\mathbf{q}(\mathbf{A})|^{m+1}\mathbf{n}\times\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})) + \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n})\mathbf{n}\} \ (= V_m^2(\mathbf{A})), \\ R & \mathbf{r}\cdot\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), [\mathbf{n},\mathbf{r},\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n})], \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n})(\mathbf{r}\cdot\mathbf{n}), \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n})(\mathbf{r}\cdot\mathbf{n}). \end{array}$$

We need to prove that the two sets $V_m^2(\mathbf{W})$ and $V_m^2(\mathbf{A})$ given in the above tables obey the criterion (1.1). First, let $\mathbf{z}$ be a vector in the $\mathbf{n}$-plane. Then we have

$$\boldsymbol{\rho}_m(\mathbf{z}) = \mathbf{0} \iff \mathbf{z} = \mathbf{0}. \tag{6.3}$$

From this and

$$g(\mathbf{W}) \cap C_{2m+1h}(\mathbf{n}) = \begin{cases} C_{2m+1h}(\mathbf{n}), & \mathbf{W}\mathbf{n} = \mathbf{0}, \\ C_1, & \mathbf{W}\mathbf{n} \neq \mathbf{0}, \end{cases} \tag{6.4}$$

$$g(\mathbf{A}) \cap C_{2m+1h}(\mathbf{n}) = \begin{cases} C_{2m+1h}(\mathbf{n}), & \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{q}(\mathbf{A}) = \mathbf{0}, \\ C_{1h}(\mathbf{n}), & \mathbf{q}(\mathbf{A}) \neq \mathbf{0}, \quad \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0}, \\ C_1, & \overset{\circ}{\mathbf{A}}\mathbf{n} \neq \mathbf{0}, \end{cases} \tag{6.5}$$

as well as (1.2), we conclude that $V_m^2(\mathbf{W})$ and $V_m^2(\mathbf{A})$ obey the criterion (1.1) separately.

Furthermore, let $\mathbf{W}_i = \mathbf{n}\wedge\mathbf{e}_i, \mathbf{A}_i = \mathbf{n}\vee\mathbf{e}_i, i = 1, 2$. Then for $i, j = 1, 2$,

$$\sin(2m+1) < \mathbf{W}_1\mathbf{n}, \mathbf{e} >= 0, \qquad \cos(2m+1) < \mathbf{W}_2\mathbf{n}, \mathbf{e} >= 0,$$
$$g(\mathbf{W}_i) \cap C_{2m+1h}(\mathbf{n}) = C_1;$$

$$\sin(2m+1) < \overset{\circ}{\mathbf{A}}_1\mathbf{n}, \mathbf{e} >= 0, \qquad \cos(2m+1) < \overset{\circ}{\mathbf{A}}_2\mathbf{n}, \mathbf{e} >= 0,$$
$$g(\mathbf{A}_i) \cap C_{2m+1h}(\mathbf{n}) = C_1.$$

Thus, we infer that either of the two generating sets $V_m^2(\mathbf{W})$ and $V_m^2(\mathbf{A})$ is irreducible.

Case 3. Two vector variables $(\mathbf{u}, \mathbf{v})$

From the condition (4.12), where $C_{2mh}(\mathbf{n})$ is replaced by $C_{2m+1h}(\mathbf{n})$, we infer that the vector variables $(\mathbf{u}, \mathbf{v})$ are specified by (4.13). Hence, we construct the following table (the first case in (4.13) is considered).

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V	$\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}} \times \mathbf{n}, (\mathbf{v}\cdot\mathbf{n})\mathbf{n}$
Skw	$\mathbf{N}, \mathbf{u} \wedge \mathbf{v}, (\mathbf{u} \times \mathbf{n}) \wedge \mathbf{v} + (\mathbf{v} \times \mathbf{n}) \wedge \mathbf{u}$
Sym	$\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \overset{\circ}{\mathbf{u}} \otimes \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}} \vee (\overset{\circ}{\mathbf{u}} \times \mathbf{n}); \mathbf{u} \vee \mathbf{v}, (\mathbf{u} \times \mathbf{n}) \vee \mathbf{v} + (\mathbf{v} \times \mathbf{n}) \vee \mathbf{u}$
R	$\mathbf{u}\cdot\mathbf{Hv}, [\mathbf{n}, \mathbf{u}, \mathbf{Hv}] + [\mathbf{n}, \mathbf{v}, \mathbf{Hu}];$ $\mathbf{u}\cdot\mathbf{Bv}, [\mathbf{n}, \mathbf{u}, \mathbf{Bv}] + [\mathbf{n}, \mathbf{v}, \mathbf{Bu}];$ $< (\mathbf{v}\cdot\mathbf{n})^2, \alpha_{2m+1}(\overset{\circ}{\mathbf{u}}), \beta_{2m+1}(\overset{\circ}{\mathbf{u}}) > .$

The results listed in the above table can easily be verified by means of the condition (4.13).

CASE 4. A vector variable $\mathbf{u}$ and a skewsymmetric tensor variable $\mathbf{W}$
From (6.1) and (6.4) we infer that there is $\mathbf{x} \in \{\mathbf{u}, \mathbf{W}\}$ such that

$$g(\mathbf{x}) \cap C_{2m+1h}(\mathbf{n}) = g(\mathbf{u}, \mathbf{W}) \cap C_{2m+1h}(\mathbf{n}).$$

Thus, the case at issue can be reduced to the cases for the single variables $\mathbf{u}$ and $\mathbf{W}$.

CASE 5. A vector variable $\mathbf{u}$ and a symmetric tensor variable $\mathbf{A}$
The condition (2.6) yields

$$g(\mathbf{u}, \mathbf{A}) \cap C_{2m+1h}(\mathbf{n}) \neq g(\mathbf{x}) \cap C_{2m+1h}(\mathbf{n}), \qquad \mathbf{x} = \mathbf{u}, \mathbf{A}.$$

From (6.1) and (6.5) and the above condition we derive (cf. §3.3 in XIAO [34] for detail)

$$\begin{aligned} &\mathbf{u} = a\mathbf{n}, \qquad a \neq 0, \\ &\mathbf{A} = x(\mathbf{e} \otimes \mathbf{e} - \mathbf{e}' \otimes \mathbf{e}') + y\mathbf{e} \vee \mathbf{e}' + c\mathbf{I} + d\mathbf{n} \otimes \mathbf{n}, \qquad x^2 + y^2 \neq 0. \end{aligned} \tag{6.6}$$

By means of the criterion (1.5), it can easily be verified that for the variables $(\mathbf{u}, \mathbf{A})$ specified above, the union $\{(\mathbf{u}\cdot\mathbf{n})^2\} \cup I_m^2(\mathbf{A})$ supplies an irreducible functonal basis of the variables $(\mathbf{u}, \mathbf{A})$ specified above under $C_{2m+1h}(\mathbf{n})$. On the other hand, generating sets for vector-valued and second order tensor-valued anisotropic functions of the variables $(\mathbf{u}, \mathbf{A})$ under $C_{2m+1h}(\mathbf{n})$ can be derived from vector-valued and second order tensor-valued isotropic functions of the extended variables $(\mathbf{u}, \mathbf{A}, \boldsymbol{\rho}_m(\mathbf{q}(A)), \mathbf{N})$ (see XIAO [35, 39]). Thus, we construct the following table for irreducible generating sets.

V	$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})) + \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n})\mathbf{n}, \mathbf{n} \times \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})) + \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n})\mathbf{n}$
Skw	$\mathbf{N}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \wedge \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \wedge (\mathbf{n} \times \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})))$
Sym	$\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathbf{A}, \mathbf{AN} - \mathbf{NA}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \vee \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \vee (\mathbf{n} \times \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))).$

The above results can easily be verified by means of the condition (6.6). We mentioned that in the above table, the generic variables $\mathbf{r}, \mathbf{H}, \mathbf{B} \in X \in \mathcal{D}$ are treated as being subject to the conditions: $\mathbf{r} \times \mathbf{n} = \mathbf{0}$, $\mathbf{Hn} = \mathbf{0}$ and $\overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0}$

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respectively (hence no new invariants appear), since any set $X \in \mathcal{D}$ violating the just-stated conditions has been covered by one of the preceding four cases.

Combining the above cases, we arrive at the main result of this section as follows.

THEOREM 6.1. *The four sets given by*

$I_m^2(\mathbf{u}), I_{2m+1}^0(\mathbf{W}), I_{2m+1}^0(\mathbf{A})$;

$\mathring{\mathbf{u}}\cdot\mathring{\mathbf{v}}, (\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}), [\mathbf{n}, \mathring{\mathbf{u}}, \mathring{\mathbf{v}}]$;

$\mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Hn}]$;

$\mathrm{tr}\,\mathbf{AB}, \mathrm{tr}\,\mathbf{ABN}, \mathbf{n}\cdot\mathring{\mathbf{A}}\mathbf{B}\mathring{\mathbf{A}}\mathbf{n}, [\mathbf{n}, \mathring{\mathbf{A}}\mathbf{n}, \mathbf{B}\mathring{\mathbf{A}}\mathbf{n}]$;

$\mathbf{n}\cdot\mathbf{WAn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{An}], \mathbf{n}\cdot\mathbf{WAWn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{WAn}]$;

$(\mathbf{u}\cdot\mathbf{n})\mathring{\mathbf{u}}\cdot\mathbf{Wn}, (\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathring{\mathbf{u}}, \mathbf{Wn}], \mathbf{u}\cdot\boldsymbol{\rho}_{2m}(\mathbf{Wn}), [\mathbf{n}, \mathbf{u}, \boldsymbol{\rho}_{2m}(\mathbf{Wn})]$,

$(\mathbf{u}\cdot\mathbf{n})\alpha_{2m+1}(\mathbf{Wn}), (\mathbf{u}\cdot\mathbf{n})\beta_{2m+1}(\mathbf{Wn})$;

$\mathring{\mathbf{u}}\cdot\mathbf{A}\mathring{\mathbf{u}}, [\mathbf{n}, \mathring{\mathbf{u}}, \mathbf{A}\mathring{\mathbf{u}}], (\mathbf{u}\cdot\mathbf{n})\mathring{\mathbf{u}}\cdot\mathbf{An}, (\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathring{\mathbf{u}}, \mathbf{An}]$,

$\mathbf{u}\cdot\boldsymbol{\rho}_{2m}(\mathring{\mathbf{A}}\mathbf{n}), [\mathbf{n}, \mathbf{u}, \boldsymbol{\rho}_{2m}(\mathring{\mathbf{A}}\mathbf{n})], \alpha_{2m+1}(\mathring{\mathbf{A}}\mathbf{n})(\mathbf{u}\cdot\mathbf{n}), \beta_{2m+1}(\mathring{\mathbf{A}}\mathbf{n})(\mathbf{u}\cdot\mathbf{n})$;

$\mathbf{u}\cdot\mathbf{Wv}, [\mathbf{n}, \mathbf{u}, \mathbf{Wv}] + [\mathbf{n}, \mathbf{v}, \mathbf{Wu}]$;

$\mathbf{u}\cdot\mathbf{Av}, [\mathbf{n}, \mathbf{u}, \mathbf{Av}] + [\mathbf{n}, \mathbf{v}, \mathbf{Au}]$;

and

$V^0(\mathbf{u}), V_m^2(\mathbf{W}), V_m^2(\mathbf{A})$;

and

$\mathrm{Skw}^0(\mathbf{u}), \mathrm{Skw}^0(\mathbf{W}), \mathrm{Skw}^0(\mathbf{A})$;

$\mathbf{u}\wedge\mathbf{v}, (\mathbf{u}\times\mathbf{n})\wedge\mathbf{v} + (\mathbf{v}\times\mathbf{n})\wedge\mathbf{u}$;

$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge(\mathbf{n}\times\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})))$;

and

$\mathrm{Sym}^0(\mathbf{u}), \mathrm{Sym}^0(\mathbf{W}), \mathrm{Sym}^0(\mathbf{A})$;

$\mathbf{u}\vee\mathbf{v}, (\mathbf{u}\times\mathbf{n})\vee\mathbf{v} + (\mathbf{v}\times\mathbf{n})\vee\mathbf{u}$;

$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee(\mathbf{n}\times\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})))$;

where $\mathbf{u}, \mathbf{v} = \mathbf{u}_1, \cdots, \mathbf{u}_a$; $\mathbf{W}, \mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$; $\mathbf{A}, \mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$; $\mathbf{u} \neq \mathbf{v}$, $\mathbf{W} \neq \mathbf{H}$, $\mathbf{A} \neq \mathbf{B}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the group* $C_{2m+1h}(\mathbf{n})$ *for each integer* $m \geq 1$, *respectively.*

7. The classes S_{4m+2} and C_{2m+1}

The classes at issue include the trigonal crystal classes S_6 and C_3 as the particular case when $m = 1$.

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7.1. The classes S_{4m+2}

With the aid of the fact

$$(7.1)\qquad g(\mathbf{r})\cap S_{4m+2}(\mathbf{n})=\begin{cases} S_{4m+2}(\mathbf{n}), & \mathbf{r}=\mathbf{0}\,,\\ C_{2m+1}(\mathbf{n}), & \overset{\circ}{\mathbf{r}}=\mathbf{0}, \quad \mathbf{r}\cdot\mathbf{n}\neq 0\,,\\ C_1, & \overset{\circ}{\mathbf{r}}\neq\mathbf{0}\,,\end{cases}$$

for any vector $\mathbf{r}\in V$, we infer that for any two vectors $(\mathbf{u},\mathbf{v})$, there exists $\mathbf{z}\in\{\mathbf{u},\mathbf{v}\}$ such that

$$g(\mathbf{u},\mathbf{v})\cap S_{4m+2}(\mathbf{n})=g(\mathbf{z})\cap S_{4m+2}\,.$$

This indicates that the case for two vector variables can be reduced to the case for a single vector variable. Accordingly, following the scheme outlined in Sec. 2, five classes are discussed as follows.

CASE 1. A single vector variable $\mathbf{u}$

A scalar-valued (resp. second order tensor-valued) anisotropic function of the vector variable $\mathbf{u}$ under $S_{4m+2}(\mathbf{n})$ is equivalent to a scalar-valued (resp. second order tensor-valued) anisotropic function of the symmetric tensor variable $\mathbf{u}\otimes\mathbf{u}$ under $S_{4m+2}(\mathbf{n})$. As a result, generating sets for the former can be derived by taking $\mathbf{A}=\mathbf{u}\otimes\mathbf{u}$ in the corresponding generating sets listed in the table for Case 3 below. Moreover, according to XIAO [35, 39], generating sets for vector-valued anisotropic functions of $\mathbf{u}$ under $S_{4m+2}(\mathbf{n})$ can be obtained from those for vector-valued isotropic functions of the extended variables $(\mathbf{u},\mathbf{E}\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}}),\mathbf{N})$, where $\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})$ is given by (1.11).

Basing on the above facts, we construct the following table.

V	$\{\mathbf{u},\overset{\circ}{\mathbf{u}}\times\mathbf{n},\alpha_{2m+1}(\overset{\circ}{\mathbf{u}})\mathbf{n},\beta_{2m+1}(\overset{\circ}{\mathbf{u}})\mathbf{n}\}\ (=V^3_m(\mathbf{u}))$
Skw	$\{\mathbf{N},\mathbf{n}\wedge\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}}),\mathbf{n}\wedge(\mathbf{n}\times\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}}))\}\ (=\mathrm{Skw}^3_m(\mathbf{u}))$
Sym	$\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}},\overset{\circ}{\mathbf{u}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}}),\mathbf{n}\vee\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}}),\mathbf{n}\vee(\mathbf{n}\times\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}}))\}\ (=\mathrm{Sym}^3_m(\mathbf{u}))$
R	$\mathbf{u}\cdot\mathbf{r},[\mathbf{n},\overset{\circ}{\mathbf{u}},\overset{\circ}{\mathbf{r}}],(\mathbf{r}\cdot\mathbf{n})\alpha_{2m+1}(\overset{\circ}{\mathbf{u}}),(\mathbf{r}\cdot\mathbf{n})\beta_{2m+1}(\overset{\circ}{\mathbf{u}})$; $\mathrm{tr}\,\mathbf{HN},\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{Hn},[\mathbf{n},\mathbf{Hn},\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})]$; $\mathrm{tr}\,\mathbf{B},\mathbf{n}\cdot\mathbf{Bn},\overset{\circ}{\mathbf{u}}\cdot\mathbf{B}\overset{\circ}{\mathbf{u}},[\mathbf{n},\overset{\circ}{\mathbf{u}},\mathbf{B}\overset{\circ}{\mathbf{u}}],\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{Bn},[\mathbf{n},\mathbf{Bn},\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})]$; $<(\mathbf{u}\cdot\mathbf{n})\alpha_{2m+1}(\overset{\circ}{\mathbf{u}}),(\mathbf{u}\cdot\mathbf{n})\beta_{2m+1}(\overset{\circ}{\mathbf{u}}),\alpha_{4m+2}(\overset{\circ}{\mathbf{u}}),\beta_{4m+2}(\overset{\circ}{\mathbf{u}})>\ (=I^3_m(\mathbf{u}))$.

With the aid of (7.1) and (1.2) – (1.4), it can be verified that the first three sets given above, denoted by $V^3_m(\mathbf{u})$, $\mathrm{Skw}^3_m(\mathbf{u})$ and $\mathrm{Sym}^3_m(\mathbf{u})$ henceforth, obey the criterion (1.1) and hence they are the desired generating sets.

CASE 2. A single skewsymmetric tensor variable $\mathbf{W}$

Every vector-valued anisotropic function of $\mathbf{W}$ under the group S_{4m+2} vanishes. Anisotropic functional bases of the variable $\mathbf{W}\in\mathrm{Skw}$ under $S_{4m+2}(\mathbf{n})$ can be obtained from isotropic functional bases of the extended variables

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$(\mathbf{W}, \mathbf{E}\boldsymbol{\rho}_{2m}(\mathbf{Wn}), \mathbf{N})$ (see XIAO [35, 39]). Moreover, the generating sets $\mathrm{Skw}^0(\mathbf{W})$ and $\mathrm{Sym}^0(\mathbf{W})$ under $C_{2mh}(\mathbf{n})$ for each $m \geq 2$ (cf. Case 2 in Sec. 4.1) also provide irreducible generating sets for skewsymmetric and symmetric tensor-valued anisotropic functions of the variable $\mathbf{W}$ under $S_{4m+2}(\mathbf{n})$.

An irreducible functional basis of $\mathbf{W}$ under $S_{4m+2}(\mathbf{n})$ is given by

$$I^3_m(\mathbf{W}) = \{\operatorname{tr}\mathbf{WN},\ \alpha_{2m+1}(\mathbf{Wn}),\ \beta_{2m+1}(\mathbf{Wn})\}\,. \tag{7.2}$$

This result can be verified by means of the procedure used at Case 1 in Sec. 6.

CASE 3. A single symmetric tensor variable $\mathbf{A}$

Every vector-valued anisotropic function of the variable $\mathbf{A} \in \mathrm{Sym}$ under $S_{4m+2}(\mathbf{n})$ vanishes. Irreducible representations for scalar-valued and second order tensor-valued anisotropic functions of the single variable $\mathbf{A} \in \mathrm{Sym}$ under $S_{4m+2}(\mathbf{n})$ can be obtained from those for scalar-valued and second order tensor-valued isotropic functions of the extended variables

$$(\mathbf{A}, \mathbf{E}\boldsymbol{\rho}_{2m}(\mathring{\mathbf{A}}\mathbf{n}), \mathbf{E}\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), \mathbf{N})$$

(see XIAO [35, 39]).

Based on this fact, we construct the following table.

Skw $\{\mathbf{N}, \mathbf{n}\wedge\mathring{\mathbf{A}}\mathbf{n}, \mathbf{n}\wedge(\mathbf{n}\times\mathring{\mathbf{A}}\mathbf{n}), \mathbf{n}\wedge\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), \mathbf{n}\wedge(\mathbf{n}\times\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})))\}$ $(=\mathrm{Skw}^3_m(\mathbf{A}))$

Sym $\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{A}, \mathbf{AN}-\mathbf{NA}, \mathring{\mathbf{A}}\mathbf{n}\otimes\mathring{\mathbf{A}}\mathbf{n}, \mathring{\mathbf{A}}\mathbf{n}\vee(\mathbf{n}\times\mathring{\mathbf{A}}\mathbf{n}),$
$\mathbf{n}\vee\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), \mathbf{n}\vee(\mathbf{n}\times\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})))\}$ $(=\mathrm{Sym}^3_m(\mathbf{A}))$

R $\operatorname{tr}\mathbf{HN}; \operatorname{tr}\mathbf{B}, \mathbf{n}\cdot\mathbf{Bn}, \operatorname{tr}\mathbf{BA}, \operatorname{tr}\mathbf{BAN},$
$\mathbf{n}\cdot\mathring{\mathbf{A}}\mathbf{B}\mathring{\mathbf{A}}\mathbf{n}, [\mathbf{n}, \mathring{\mathbf{A}}\mathbf{n}, \mathbf{B}\mathring{\mathbf{A}}\mathbf{n}], \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\cdot\mathbf{Bn}, [\mathbf{n}, \mathbf{Bn}, \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))];$
$< I^3_m(\mathbf{A}) >$

In the above, the set $I^3_m(\mathbf{A})$ is an irreducible functional basis of a symmetric tensor $\mathbf{A}$ under the group S_{4m+2}, given by (see XIAO [38])

$$\begin{aligned} I^3_m(\mathbf{A}) = \{&\mathbf{n}\cdot\mathbf{An}, \operatorname{tr}\mathbf{A}, \operatorname{tr}\mathbf{A}^3, [\mathbf{n}, \mathbf{An}, \mathbf{A}^2\mathbf{n}], \alpha_{2m+1}(\mathring{\mathbf{A}}\mathbf{n}), \beta_{2m+1}(\mathring{\mathbf{A}}\mathbf{n}), \\ &\alpha_{2m+1}(\mathbf{q}(\mathbf{A})), \beta_{2m+1}(\mathbf{q}(\mathbf{A}))\}. \end{aligned} \tag{7.3}$$

By means of the criterion (1.1) and the facts

$$g(\mathbf{A})\cap S_{4m+2}(\mathbf{n}) = \begin{cases} S_{4m+2}(\mathbf{n}), & \mathbf{A} = x\mathbf{I} + y\mathbf{n}\otimes\mathbf{n}\,, \\ S_2, & |\mathbf{q}(\mathbf{A})|^2 + |\mathring{\mathbf{A}}\mathbf{n}|^2 \neq 0\,, \end{cases} \tag{7.4}$$

it can be verified that the first two sets in the above table, denoted by $\mathrm{Skw}^3_m(\mathbf{A})$ and $\mathrm{Sym}^3_m(\mathbf{A})$ henceforth, are generating sets for skewsymmetric and symmetric

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tensor-valued anisotropic functions of the variable $\mathbf{A} \in \mathrm{Sym}$ under $S_{4m+2}(\mathbf{n})$ and moreover, that the two sets are irreducible.

It should be pointed out that in the above table, the generic skewsymmetric tensor variable $\mathbf{H}$ has been treated as being subject to the condition that $\mathbf{Hn} = \mathbf{0}$, since the case when $\mathbf{Wn} \neq \mathbf{0}$ has been covered by the last case. As a result, of the five invariants obtained by forming the inner product between $\mathbf{H}$ and each of the five skewsymmetric tensor generators presented, only one, i.e. $\mathrm{tr}\,\mathbf{HN}$, does not vanish.

REMARK. The generating set $\mathrm{Sym}^3_m(\mathbf{A})$ consists of eight generators. For $m = 1$, i.e. the trigonal crystal class S_6, a minimal generating set of six generators is available (see XIAO [37]). For the sake of consistency, here for all $m \geq 1$ we employ the set $\mathrm{Sym}^3_m(\mathbf{A})$, which supplies a unified form of all generating sets in question.

CASE 4. A vector variable $\mathbf{u}$ and a skewsymmetric tensor variable $\mathbf{W}$

The condition (2.6) yields

$$g(\mathbf{u}, \mathbf{W}) \cap S_{4m+2}(\mathbf{n}) \neq g(\mathbf{x}) \cap S_{4m+2}(\mathbf{n}), \qquad \mathbf{x} = \mathbf{u}, \mathbf{W}.$$

From (7.1) and the above condition we derive

$$\begin{aligned} &\mathbf{u} = a\mathbf{n}, \quad a \neq 0\,, \\ &\mathbf{Wn} \neq \mathbf{0}, \quad \text{i.e. } g(\mathbf{W}) \cap S_{4m+2}(\mathbf{n}) = S_2\,. \end{aligned} \tag{7.5}$$

As a result, $\mathrm{Skw}^0(\mathbf{W})$ and $\mathrm{Sym}^0(\mathbf{W})$ supply the desired irreducible generating sets for second order tensor-valued functions. It is evident that the union $I^3_m(\mathbf{W}) \cup \{(\mathbf{u}\cdot\mathbf{n})^2\}$ provides an irreducible functional basis of the variables $(\mathbf{u}, \mathbf{W})$ specified by (7.5) under $S_{4m+2}(\mathbf{n})$. Moreover, the generic vector variable $\mathbf{r}$ is subject to the condition that $\mathring{\mathbf{r}} = \mathbf{0}$ (the other case has been covered at Case 1). Thus, we need only to provide an irreducible generating set for vector-valued functions of the variables $(\mathbf{u}, \mathbf{W})$ specified by (7.5) under $S_{4m+2}(\mathbf{n})$, which is given as follows.

$$V^3_m(\mathbf{u}, \mathbf{W}) = \{\mathbf{u}, (\mathbf{u}\cdot\mathbf{n})\boldsymbol{\rho}_{2m}(\mathbf{Wn}), (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \times \boldsymbol{\rho}_{2m}(\mathbf{Wn})\}\,. \tag{7.6}$$

CASE 5. A vector variable $\mathbf{u}$ and a symmetric tensor variable $\mathbf{A}$

The condition (2.6) yields

$$g(\mathbf{u}, \mathbf{A}) \cap S_{4m+2}(\mathbf{n}) \neq g(\mathbf{x}) \cap S_{4m+2}(\mathbf{n}), \qquad \mathbf{x} = \mathbf{u}, \mathbf{A}.$$

From (7.1) and (7.4) and the above condition we derive

$$\begin{aligned} &\mathbf{u} = a\mathbf{n}, \quad a \neq 0\,, \\ &g(\mathbf{A}) \cap S_{4m+2}(\mathbf{n}) = S_2\,. \end{aligned} \tag{7.7}$$

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As a result, $\mathrm{Skw}^3_m(\mathbf{A})$ and $\mathrm{Sym}^3_m(\mathbf{A})$ supply the desired irreducible generating sets for second order tensor-valued functions. It is evident that the union $I^3_m(\mathbf{A}) \cup \{(\mathbf{u}\cdot\mathbf{n})^2\}$ provides an irreducible functional basis of the variables $(\mathbf{u}, \mathbf{A})$ specified by (7.7) under $S_{4m+2}(\mathbf{n})$. Moreover, the generic vector variable $\mathbf{r}$ is subject to the condition that $\overset{\circ}{\mathbf{r}} = \mathbf{0}$ (the other case has been covered at Case 1). Thus, we need only to provide an irreducible generating set for vector-valued functions of the variables $(\mathbf{u}, \mathbf{A})$ specified by (7.7) under $S_{4m+2}(\mathbf{n})$, which is given by

$$(7.8) \qquad V^3_m(\mathbf{u}, \mathbf{A}) = \{\mathbf{u}, (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{A}}\mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \times \overset{\circ}{\mathbf{A}}\mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \times \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\}.$$

Combining the above cases, we arrive at the main result of this subsection as follows.

THEOREM 7.1. *The four sets given by*

$I^3_m(\mathbf{u}), I^3_m(\mathbf{W}), I^3_m(\mathbf{A})$;

$\mathbf{u}\cdot\mathbf{v}, [\mathbf{n}, \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{v}}], (\mathbf{u}\cdot\mathbf{n})\alpha_{2m+1}(\overset{\circ}{\mathbf{v}}), (\mathbf{u}\cdot\mathbf{n})\beta_{2m+1}(\overset{\circ}{\mathbf{v}})$;

$\mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Hn}]$;

$\mathbf{n}\cdot\mathbf{WAn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{An}], \mathbf{n}\cdot\mathbf{WAWn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{AWn}]$;

$\operatorname{tr}\mathbf{AB}, \operatorname{tr}\mathbf{ABN}, \mathbf{n}\cdot\overset{\circ}{\mathbf{A}}\mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}, [\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}], \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{B}))\cdot\mathbf{An}, [\mathbf{n}, \mathbf{An}, \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{B}))]$;

$\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{Wn}, [\mathbf{n}, \mathbf{Wn}, \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})]$;

$\overset{\circ}{\mathbf{u}}\cdot\mathbf{A}\overset{\circ}{\mathbf{u}}, [\mathbf{n}, \overset{\circ}{\mathbf{u}}, \mathbf{A}\overset{\circ}{\mathbf{u}}], \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{An}, [\mathbf{n}, \mathbf{An}, \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})]$;

and

$V^3_m(\mathbf{u}),\ V^3_m(\mathbf{u}, \mathbf{W}),\ V^3_m(\mathbf{u}, \mathbf{A})$;

and

$\mathrm{Skw}^3_m(\mathbf{u}),\ \mathrm{Skw}^0(\mathbf{W}),\ \mathrm{Skw}^3_m(\mathbf{A})$;

and

$\mathrm{Sym}^3_m(\mathbf{u}),\ \mathrm{Sym}^0(\mathbf{W}),\ \mathrm{Sym}^3_m(\mathbf{A})$;

where $\mathbf{u}, \mathbf{v} = \mathbf{u}_1, \cdots, \mathbf{u}_a$; $\mathbf{W}, \mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$; $\mathbf{A}, \mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$; $\mathbf{u} \neq \mathbf{v}$, $\mathbf{W} \neq \mathbf{H}$, $\mathbf{A} \neq \mathbf{B}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the group* $S_{4m+2}(\mathbf{n})$ *for each integer* $m \geq 1$, *respectively.*

7.2. The classes C_{2m+1}

Applying the argument given in Sec. 4.2 and Theorem 7.1, we derive the following result.

THEOREM 7.2. *The four sets given by*

$I^3_m(\mathbf{W}), I^3_m(\mathbf{A}), \mathbf{u}\cdot\mathbf{n}, \alpha_{2m+1}(\overset{\circ}{\mathbf{u}}), \beta_{2m+1}(\overset{\circ}{\mathbf{u}}); \mathbf{u}\cdot\mathbf{v}, [\mathbf{u}, \mathbf{v}, \mathbf{n}]$;

$\mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Hn}]$;

$\mathbf{n}\cdot\mathbf{WAn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{An}], \mathbf{n}\cdot\mathbf{WAWn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{AWn}]$;

$\operatorname{tr}\mathbf{AB}, \operatorname{tr}\mathbf{ABN}, \mathbf{n}\cdot\overset{\circ}{\mathbf{A}}\mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}, [\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}], \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{B}))\cdot\mathbf{An}, [\mathbf{n}, \mathbf{An}, \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{B}))]$;

$\mathbf{u}\cdot\mathbf{Wn}, [\mathbf{n}, \mathbf{u}, \mathbf{Wn}]$;

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$\mathbf{u}\cdot\mathbf{An}, [\mathbf{n},\mathbf{u},\mathbf{An}], \overset{\circ}{\mathbf{u}}\cdot\mathbf{A}\overset{\circ}{\mathbf{u}}, [\mathbf{n},\mathbf{u},\mathbf{Au}]$;

and

$\mathbf{n}, \overset{\circ}{\mathbf{u}}, \mathbf{n}\times\overset{\circ}{\mathbf{u}}, \mathbf{Wn}, \mathbf{n}\times\mathbf{Wn}, \overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{n}\times\overset{\circ}{\mathbf{A}}\mathbf{n}, \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), \mathbf{n}\times\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))$;

and

$\mathrm{Skw}^0(\mathbf{W}), \mathrm{Skw}^3_m(\mathbf{A}), \mathbf{n}\wedge\overset{\circ}{\mathbf{u}}, \mathbf{n}\wedge(\mathbf{n}\times\overset{\circ}{\mathbf{u}})$;

and

$\mathrm{Sym}^0(\mathbf{W}), \mathrm{Sym}^3_m(\mathbf{A}), \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}}), \mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \mathbf{n}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}})$;

where $\mathbf{u},\mathbf{v}=\mathbf{u}_1,\cdots,\mathbf{u}_a$; $\mathbf{W},\mathbf{H}=\mathbf{W}_1,\cdots,\mathbf{W}_b$; $\mathbf{A},\mathbf{B}=\mathbf{A}_1,\cdots,\mathbf{A}_c$; $\mathbf{u}\neq\mathbf{v}$, $\mathbf{W}\neq\mathbf{H}$, $\mathbf{A}\neq\mathbf{B}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X\in\mathcal{D}$ *under the group* $C_{2m+1}(\mathbf{n})$ *for each integer* $m\geq 1$, *respectively.*

8. Remarks

In the previous sections, complete nonpolynomial representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of any finite number of vector and second order tensor variables under all kinds of subgroups of the transverse isotropy group $C_{\infty h}$ are derived, each of which consists of polynomial invariants or polynomial generators. The presented results offer unified forms of general representations for infinitely many subgroup classes concerned, respectively.

It can be seen that infinitely many different types of vector-valued or second order-tensor-valued anisotropic functions may have a common generating set. Indeed, the set $\mathrm{Skw}^0(\mathbf{W})$ is a common generating set which applies to all subgroups of $C_{\infty h}$ except the triclinic groups. Other examples are the sets $\mathrm{Skw}^0(\mathbf{u})$, $\mathrm{Skw}^0(\mathbf{A})$, $\mathrm{Sym}^0(\mathbf{W})$, $\mathrm{Sym}^0(\mathbf{A})$, etc.

The presented results for generating sets are irreducible. It has been shown that each presented invariant with a single variable is irreducible. Irreducibility of each presented invariant with two or three variables will be proved elsewhere.

The unified scheme described in Sec. 2 and the method for isotropic extension of anisotropic functions may be used to derive irreducible representations for other types of anisotropic functions. The results will be reported elsewhere.

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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Effective properties of physically nonlinear piezoelectric composites

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FOR PIEZOELECTRIC composites with periodic microstructure and subject to stronger electric fields, an effective model has been prepared. To this end the Γ-convergence theory has been applied. Detailed convergence proof has been given. Specific cases of the internal energy have been suggested. Comments on homogenization in the case of periodically nonuniform microstructure have also been provided.

1. Introduction

IN THE LAST DECADE various approaches were proposed to finding the effective properties of piezoelectric composites, cf. [1–7] and the references cited therein. In 1991 the first author published a paper [1] where he performed non-uniform homogenization of linear piezoelectric composites by using the Γ-convergence method. However, that paper did not contain the proof of convergence. The aim of the present contribution is to perform *nonlinear* homogenization of piezoelectric composite with periodic or non-uniformly periodic microstructure. As it was argued by TIERSTEN [8], for stronger electric fields one has to take into account higher order terms in the electric field $\mathbf{E}$. The form of the electric enthalpy $H(\mathbf{e}, \mathbf{E})$ proposed by this author, being of the third order in $\mathbf{E}$, cannot be *concave* in $\mathbf{E}$, where $\mathbf{e}$ denotes the strain tensor. It should be remembered that $H(\mathbf{e}, \mathbf{E})$ is here understood as a partial *concave* conjugate of the internal energy function $U(\mathbf{e}, \mathbf{D})$ with respect to $\mathbf{D}$, $\mathbf{D}$ being the electric displacement vector. In Sec. 2 we shall briefly discuss a plausible form of the internal energy which accounts for stronger electric fields.

The plan of the paper is as follows. Fundamental relations and nonlinear piezoelectric composites with εY-periodic microstructure are introduced in Sec. 2. In Sec. 3 we recall the basic notions of the Γ-convergence theory, which will next be of primary importance in Sec. 4. The heart of the paper constitutes Sec. 4, where we give the proof of the Γ-convergence of sequence of functionals J_ε defined by (4.2), to the limit functional J_h. Comments on non-uniform homogenization are provided in Sec. 5. The summation convention applies to repeated indices.

2. Basic relations

Let $V \subset \mathbb{R}^3$ be a bounded, sufficiently regular domain such that its closure $\overline{V}$ stands for a considered piezoelectric composite in its natural state. By $\gamma = \partial V$ we

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denote the boundary of V. If $\mathbf{u} = (u_i)$ is a displacement field, then $e_{ij}(\mathbf{u}) = u_{(i,j)}$ is the strain tensor; $i, j = 1, 2, 3$. By $\mathbf{D} = (D_i)$, $\mathbf{E} = (E_i)$ and $\boldsymbol{\sigma} = (\sigma_{ij})$ we denote the electric displacement vector, the electric field and the stress tensor, respectively. As usual, we set $E_i(\varphi) = -\varphi_{,i}$, where φ is the electric potential [9]. Let $\varepsilon > 0$ be a small parameter and $\varepsilon = l/L$. Here l, L are typical length scales associated with microinhomogeneities and the region V, respectively. The internal energy is $U = U(y, \mathbf{e}, \mathbf{D})$, $y \in Y$. Here Y is a so-called basic cell, cf. [10, 11, 12]. We set

$$U_\varepsilon(x, \mathbf{e}, \mathbf{D}) = U\left(\frac{x}{\varepsilon}, \mathbf{e}, \mathbf{D}\right), \tag{2.1}$$

where $x \in V$, $\mathbf{e} \in \mathbb{E}^3_s$ and $\mathbf{D} \in \mathbb{R}^3$; $\mathbb{E}^3_s$ stands for the space of symmetric 3×3 matrices. Consequently, the piezoelectric material occupying V exhibits the εY-periodic microstructure. We observe that the case of quadratic internal energy has been studied in [1]. In the general case the constitutive equations are given by

$$\boldsymbol{\sigma} = \frac{\partial U}{\partial \mathbf{e}}, \qquad \mathbf{E} = \frac{\partial U}{\partial \mathbf{D}}. \tag{2.2}$$

We make the following assumption:

(A) The function $U : (y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \in \mathbb{R}^3 \times \mathbb{E}^3_s \times \mathbb{R}^3 \to U(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \in \mathbb{R}$ is measurable and Y-periodic in y, convex in $(\mathbf{e}, \mathbf{D})$ and such that

$$\exists\, c_1 \geq c_0 > 0, \quad c_0(|\boldsymbol{\varepsilon}|^p + |\boldsymbol{\varrho}|^q) \leq U(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \leq c_1(|\boldsymbol{\varepsilon}|^p + |\boldsymbol{\varrho}|^q), \tag{2.3}$$

for each $(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho})$. Here $p > 1$ and $q > 1$. As usual, we set $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. The assumption (A) will significantly be weakened in the case of non-uniformly microperiodic composites, cf. Sec. 5 of our paper.

REMARK 1. As a particular case of the internal energy one can consider the following one:

$$U\left(\frac{x}{\varepsilon}, \mathbf{e}, \mathbf{D}\right) = U_1\left(\frac{x}{\varepsilon}, \mathbf{e}, \mathbf{D}\right) + U_2\left(\frac{x}{\varepsilon}, \mathbf{e}, \mathbf{D}\right), \tag{2.4}$$

where $U_1\left(\dfrac{x}{\varepsilon}, \mathbf{e}, \mathbf{D}\right)$ is a positive definite quadratic form in $\mathbf{e}$ and $\mathbf{D}$, typical for linear piezocomposites, cf. [1–3]. On the other hand, the function $U_2\left(\dfrac{x}{\varepsilon}, \mathbf{e}, \mathbf{D}\right)$ collects non-quadratic, higher order terms. Obviously, the function $U_2\left(\dfrac{x}{\varepsilon}, \mathbf{e}, \mathbf{D}\right)$ has still to be convex in $\mathbf{e}$ and $\mathbf{D}$.

A simple example is provided by

$$U_2(y, \mathbf{e}, \mathbf{D}) = \widetilde{U}_2(y, \mathbf{D}) = \frac{1}{4} b_{ijkl}(y) D_i D_j D_k D_l\,, \qquad y = \frac{x}{\varepsilon}, \tag{2.5}$$

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where $b_{ijkl} \in L^\infty(Y)$ is a completely symmetric tensor [13]. Further restrictions on the material functions are imposed by the requirement of $\widetilde{U}_2(y, \mathbf{D})$ being convex in $\mathbf{D}$. The present contribution is confined to small deformations and the internal energy $U(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho})$ is convex in $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varrho}$. Finite deformations are properly described by a nonconvex internal energy. Nonconvex homogenization is out of scope of the present contribution.

Let us pass to the formulation of the minimum principle. The following boundary conditions are assumed:

$$\mathbf{u} = 0 \quad \text{on } \gamma_0, \qquad \sigma_{ij}n_j = \Sigma_i \quad \text{on } \gamma_1, \tag{2.6}$$

$$\varphi = \varphi_0 \quad \text{on } \gamma_2, \qquad D_i n_i = 0 \quad \text{on } \gamma_3, \tag{2.7}$$

where Σ_i are the surface tractions, $\gamma = \overline{\gamma}_0 \cup \overline{\gamma}_1$, $\gamma_0 \cap \gamma_1 = \emptyset$; $\gamma = \overline{\gamma}_2 \cup \overline{\gamma}_3$, $\gamma_2 \cap \gamma_3 = \emptyset$, and $\mathbf{n} = (n_i)$ is the outward unit normal vector to γ; obviously $\emptyset$ denotes the empty set.

For fixed $\varepsilon > 0$ we set

$$F_\varepsilon(\mathbf{u}, \mathbf{D}) = \int_V U_\varepsilon(x, \mathbf{e}(\mathbf{u}), \mathbf{D})\, dx - L(\mathbf{u}, \mathbf{D}), \tag{2.8}$$

where

$$L(\mathbf{u}, \mathbf{D}) = \int_V b_i u_i \, dx + \int_{\gamma_1} \Sigma_i u_i d\gamma - \int_{\gamma_2} \varphi_0 D_i n_i d\gamma, \tag{2.9}$$

and

$$\mathbf{u} \in \mathbf{W}(V, \gamma_0) = \left\{ \mathbf{v} = (v_i) \mid v_i \in W^{1,p}(V), \;\; \mathbf{v} = \mathbf{0} \;\text{ on } \gamma_0 \right\},$$

$$\mathbf{D} \in \mathbf{W}(\mathrm{div}; V, \gamma_3) = \{ \mathbf{D} = (D_i) \mid D_i \in L^q(V), \; \mathrm{div}\, \mathbf{D} \in L^q(V), \; \mathbf{D}\cdot\mathbf{n} = 0 \text{ on } \gamma_3 \}.$$

For more details on the spaces just introduced the reader is refered to [14, 15].

The equlibrum problem of the physically nonlinear piezocomposites with the ε-Y periodic microstructure means evaluating

$$(\mathcal{P}_\varepsilon) \qquad F_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) = \inf \{ F_\varepsilon(\mathbf{u}, \mathbf{D}) \mid \mathbf{u} \in \mathbf{W}(V, \gamma_0), \; \mathbf{D} \in \mathbf{W}(\mathrm{div}; V, \gamma_3) \}.$$

The assumption (A) implies the existence of unique $(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \in \mathbf{W}(V, \gamma_0) \times \mathbf{W}(\mathrm{div}; V, \gamma_3)$ solving the problem $(\mathcal{P}_\varepsilon)$.

3. Γ-convergence

A detailed presentation of the theory of Γ-convergence is provided by ATTOUCH [10] and DAL MASO [11]. ATTOUCH [10] prefers to use the notion of epi-convergence, which in fact is a special case of Γ-convergence. In our specific case these notions coincide.

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DEFINITION 1. *Let* (X,τ) *be a metrisable topological space, and let* $\{G_\varepsilon\}_{\varepsilon>0}$ *be a sequence of functionals from* X *into* $\overline{\mathbb{R}}$ *– the extended reals.*

a. *The* $\Gamma(\tau)$*-limit inferior, denoted also by* G_i*, is the functional on* X *defined by*

$$G_i(u) = \Gamma(\tau)\text{-}\liminf_{\varepsilon\to 0} G_\varepsilon(u) = \min_{\{u_\varepsilon \xrightarrow{\tau} u\}} \liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon).$$

b. *The* $\Gamma(\tau)$*-limit superior, denoted also by* G_s*, is the functional on* X *defined by*

$$G_s(u) = \Gamma(\tau)\text{-}\limsup_{\varepsilon\to 0} G_\varepsilon(u) = \min_{\{u_\varepsilon \xrightarrow{\tau} u\}} \limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon).$$

c. *The sequence* $\{G_\varepsilon\}_{\varepsilon>0}$ *is said to be* $\Gamma(\tau)$*-convergent if* $G_i = G_s$*; we then write*

$$G = \Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon.$$

PROPERTIES. Let $G_\varepsilon : (X,\tau) \to \overline{\mathbb{R}}$ be a sequence of $\Gamma(\tau)$-convergent functionals and let $G = \Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon$. Then the following properties hold:

(i) The functionals G_i and G_s are τ-lower semicontinuous (τ-l.s.c.).

(ii) If the functionals G_ε are convex, then $G_s = \Gamma(\tau)\text{-}\limsup_{\varepsilon\to 0} G_\varepsilon$ is also a convex functional. Hence the $\Gamma(\tau)$-limit $G = \Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon$ is a τ-closed (τ-l.s.c) convex functional.

(iii) If $\Phi : X \to \mathbb{R}$ is a τ-continuous functional, called a perturbation functional, then

$$\Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} (G_\varepsilon + \Phi) = \Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon + \Phi = G + \Phi;$$

(iv)

$$G(u) = \Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon(u) \Leftrightarrow \begin{cases} \forall\, \{u_\varepsilon \xrightarrow{\tau} u\},\ G(u) \le \liminf\limits_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon),\ u \in X, \\ \forall\, u \in X \quad \exists\, u_\varepsilon \xrightarrow{\tau} u, \text{ such that} \\ G(u) \ge \limsup\limits_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon). \end{cases}$$

Further characterization is given by

THEOREM 1. *Let* $G = \Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon$*, and suppose that there exists a* τ*-relatively compact subset* $X_0 \subset X$ *such that* $\inf_{X_0} G_\varepsilon = \inf_X G_\varepsilon$ $(\forall \varepsilon > 0)$*. Then* $\inf_X G = \lim_{\varepsilon\to 0}(\inf_X G_\varepsilon)$*. Moreover, if* $\{u_\varepsilon\}_{\varepsilon>0}$ *is such that* $G_\varepsilon(u_\varepsilon) - \inf_X G_\varepsilon \underset{\varepsilon\to 0}{\to} 0$*, then every* τ*-cluster point of the sequence* $\{u_\varepsilon : \varepsilon \to 0\}$ *minimizes* G *on* X*.*

REMARK 2. From a practical point of view the following sufficient condition of existence of compact set X_0 is very useful. If $(X, \|\,.\,\|)$ is a Banach space with

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τ-relatively compact balls, then a sufficient condition of existence of compact set X_0 is that the sequence $\{G_\varepsilon\}_{\varepsilon>0}$ satisfies the condition of equi-coercivity

$$\limsup_{\varepsilon} G_\varepsilon(u_\varepsilon) < +\infty \Longrightarrow \limsup_{\varepsilon} \|u_\varepsilon\| < +\infty. \tag{3.1}$$

4. Γ-convergence of the sequence of functionals $\{F_\varepsilon\}_{\varepsilon>0}$

We proceed to find the limit functional

$$\Gamma\left[(s - L^p(V)^3) \times (w - L^q(V)^3)\right] - \lim_{\varepsilon\to 0} F_\varepsilon = F_h\,, \tag{4.1}$$

where $L^p(V)^3 = [L^p(V)]^3$ and $s - L^p(V)^3(w - L^q(V)^3)$ stands for the strong topology of $L^p(V)^3$ (the weak topology of $L^q(V)^3$). The loading functional L may be assumed to be continuous in the topology $\tau = (s - L^p(V)^3) \times (w - L^q(V)^3)$. To this end it is sufficient to assume that $\mathbf{b} \in L^{p'}(V)^3$, $\mathbf{\Sigma} \in L^{p'}(\gamma_1)^3$ and $\varphi_0 \in W^{1-\frac{1}{q'},q'}(\gamma_2)$. As a particular case, one can impose φ_0 being continuous on γ_2 and vanishing on $\partial\gamma_2$. According to the property (iii), the functional L plays the role of a perturbation functional. Consequently it suffices to study the $\Gamma(\tau)$-limit of the following sequence of functionals $\{J_\varepsilon\}_{\varepsilon>0}$ given by

$$J_\varepsilon(\mathbf{u},\mathbf{D}) = \int_V U_\varepsilon(x, \mathbf{e}(\mathbf{u}), \mathbf{D})dx. \tag{4.2}$$

The main result of this paper is formulated as:

THEOREM 2. *Let the assumption (A) be satisfied. The sequence of functionals* $\{J_\varepsilon\}_{\varepsilon>0}$ *is* $\Gamma(\tau)$*-convergent to the functional*

$$J_h(\mathbf{u},\mathbf{D}) = \int_V U_h(\mathbf{e}(\mathbf{u}), \mathbf{D})dx, \tag{4.3}$$

where $\mathbf{u} \in W^{1,p}(V)^3$, $\mathbf{D} \in L^q(V)^3$ *and*

$$U_h(\boldsymbol{\varepsilon}, \boldsymbol{\varrho}) = \inf\left\{ \frac{1}{|Y|}\int_Y U(y, \mathbf{e}^y(\mathbf{v}(y)) + \boldsymbol{\varepsilon}, \mathbf{d}(y) + \boldsymbol{\varrho})dy \;\middle|\; \mathbf{v} \in W^{1,p}_{\text{per}}(Y)^3,\ \mathbf{d} \in \Delta_{\text{per}}(Y) \right\}. \tag{4.4}$$

Here $\boldsymbol{\varepsilon} \in \mathbb{E}^3_s$, $\boldsymbol{\varrho} \in \mathbb{R}^3$, $e^y_{ij}(\mathbf{v}) = \dfrac{1}{2}\left(\dfrac{\partial v_i}{\partial y_j} + \dfrac{\partial v_j}{\partial y_i}\right)$ *and*

$$W^{1,p}_{\text{per}}(Y)^3 = \{\mathbf{v} \in W^{1,p}(Y)^3 | \mathbf{v} \text{ is } Y\text{−periodic}\}, \tag{4.5}$$

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(4.6) $$\Delta_{\mathrm{per}}(Y) = \left\{\mathbf{d} \in L^q(Y)^3 \mid \operatorname{div}_y \mathbf{d} = 0 \text{ in } Y,\ \langle \mathbf{d} \rangle = 0,\ \mathbf{d} \text{ is anti-periodic}\right\},$$

$$\langle \mathbf{d} \rangle = \frac{1}{|Y|} \int_Y \mathbf{d}(y)dy. \qquad \square$$

REMARK 3. A function $\mathbf{v} \in W^{1,p}_{\mathrm{per}}(Y)^3$ is Y-periodic if the traces of $\mathbf{v}$ on the opposite faces of Y are equal. It means that on these faces, the values of v_i are equal almost everywhere, at least. Similarly, if $\mathbf{d} \in \Delta_{\mathrm{per}}(Y)$, then the traces $\mathbf{d}\cdot\mathbf{N}$ are opposite on the opposite faces of Y. Here $\mathbf{N}$ stands for the outward unit normal vector to ∂Y.

PROPERTIES OF U_h

(i) The function U_h is convex.

Proof. This evident property follows immediately from the convexity of the function $U(y,\cdot,\cdot)$ and the linearity of the operator $\mathbf{e}^y(\cdot)$, cf. [16].

(ii) $\exists\, c_1 \geq c_0' > 0$ such that

$$c_0'(|\boldsymbol{\varepsilon}|^p + |\boldsymbol{\varrho}|^q) \leq U_h(\boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \leq c_1(|\boldsymbol{\varepsilon}|^p + |\boldsymbol{\varrho}|^q),$$

for each $\boldsymbol{\varepsilon} \in \mathbb{E}^3_s$, $\boldsymbol{\varrho} \in \mathbb{R}^3$. The constant c_1 is the same as in (2.3).

Proof. Indeed, from (2.3) and (4.4) we obtain

$$U_h(\boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \leq \langle U(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}\rangle \leq c_1(|\boldsymbol{\varepsilon}|^p + |\boldsymbol{\varrho}|^q).$$

Similarly, let $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{d}}) \in W^{1,p}_{\mathrm{per}}(Y)^3 \times \Delta_{\mathrm{per}}(Y)$ be a minimizer of the minimization problem occurring on the r.h.s. of (4.4). This minimizer is unique, provided that $\langle \widetilde{\mathbf{v}} \rangle = 0$. Taking into account (2.3) we have

(4.7) $$\begin{aligned} U_h(\boldsymbol{\varepsilon}, \boldsymbol{\varrho}) &= \langle U(y, \mathbf{e}^y(\widetilde{\mathbf{v}}) + \boldsymbol{\varepsilon}, \widetilde{\mathbf{d}}(y) + \boldsymbol{\varrho}\rangle \\ &\geq \langle |\mathbf{e}^y(\widetilde{\mathbf{v}}) + \boldsymbol{\varepsilon}|^p + |\widetilde{\mathbf{d}}(y) + \boldsymbol{\varrho}|^q \rangle \geq c_0'(|\boldsymbol{\varepsilon}|^p + |\boldsymbol{\varrho}|^q). \end{aligned}$$

Indeed, from (2.3) we conclude

(4.8) $$c_2(|\boldsymbol{\varepsilon}^*|^{p'} + |\boldsymbol{\varrho}^*|^{q'}) \leq U^*(y, \boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*) \leq c_3(|\boldsymbol{\varepsilon}^*|^{p'} + |\boldsymbol{\varrho}^*|^{q'}),$$

where $c_3 \geq c_2 > 0$ are constants and $\boldsymbol{\varepsilon}^* \in \mathbb{E}^3_s$, $\boldsymbol{\varrho}^* \in \mathbb{R}^3$. Here U^* denotes Fenchel's conjugate of U.

We recall that if $f \leq g$ then $g^* \leq f^*$ [16, 17]. Since $U^*(y, \boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*) \geq 0$ and $U^*(y, \mathbf{0}, \mathbf{0}) = 0$, we may apply Remark 4.3, Chap. I, of EKELAND and TEMAM [17] and (4.8) immediately follows.

By using the formula for the dual effective potential (4.9) below we conclude

$$U_h^*(\boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*) \leq \langle U^*(y, \boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*)\rangle \leq c_3(|\boldsymbol{\varepsilon}^*|^{p'} + |\boldsymbol{\varrho}^*|^{q'}).$$

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Hence

$$U_h(\boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \geq c_0'(|\boldsymbol{\varepsilon}|^p + |\boldsymbol{\varrho}|^q).$$

REMARK 4. For $p = q = 2$ another proof of (4.7) is more straightforward. In this case we have

$$\begin{aligned} U_h(\boldsymbol{\varepsilon}, \boldsymbol{\varrho}) &\geq c_0 \langle |\mathbf{e}^y(\widetilde{\mathbf{v}}) + \boldsymbol{\varepsilon}|^2 + |\widetilde{\mathbf{d}}(y) + \boldsymbol{\varrho}|^2 \rangle \\ &= \frac{c_0}{|Y|} \int\limits_Y \Big[|\mathbf{e}^y(\widetilde{\mathbf{v}}(y))|^2 + 2\boldsymbol{\varepsilon} : \mathbf{e}^y(\widetilde{\mathbf{v}}) + |\boldsymbol{\varepsilon}|^2 + |\widetilde{\mathbf{d}}(y)|^2 + 2\widetilde{\mathbf{d}}(y) \cdot \boldsymbol{\varrho} + |\boldsymbol{\varrho}|^2 \Big] \, dy \\ &\geq \frac{c_0}{|Y|} \int\limits_Y (|\boldsymbol{\varepsilon}|^2 + |\boldsymbol{\varrho}|^2) dy = c_0'(|\boldsymbol{\varepsilon}|^2 + |\boldsymbol{\varrho}|^2), \end{aligned}$$

since $\langle \widetilde{\mathbf{d}}(y) \rangle = 0$ and $\int\limits_Y \mathbf{e}^y(\widetilde{\mathbf{v}}(y)) dy = 0$; here $c_0' = c_0/|Y|$. □

Prior to passing to the proof of Theorem 2, we shall formulate two lemmas.

LEMMA 1. The dual macroscopic potential U_h^* is given by

$$(4.9) \qquad U_h^*(\boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*) = \inf \Big\{ \langle U^*(y, \mathbf{t}(y) + \boldsymbol{\varepsilon}^*, \, \mathbf{E}^y(\xi) + \boldsymbol{\varrho}^*) \rangle \ | \ \mathbf{t} \in \mathcal{S}_{\text{per}}(Y), \ \ \xi \in W^{1,q'}_{\text{per}}(Y) \Big\}$$

where U^* is the Fenchel coniugate of $U(y, \cdot, \cdot)$, $\boldsymbol{\varepsilon}^* \in \mathbb{E}^3_s$, $\boldsymbol{\varrho}^* \in \mathbb{R}^3$ and

$$(4.10) \qquad \mathcal{S}_{\text{per}}(Y) = \Big\{ \mathbf{t} \in L^{p'}(Y, \mathbb{E}^3_s) | \ \ \operatorname{div}_y \mathbf{t} = 0 \ \text{ in } Y, \ \ \langle \mathbf{t} \rangle = 0, \ \ \mathbf{t} \cdot \mathbf{N} \text{ is antiperiodic} \Big\}.$$

Proof. We have

$$\begin{aligned} (4.11) \qquad U_h^*(\boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*) &= \sup \Big\{ \boldsymbol{\varepsilon}^* : \boldsymbol{\varepsilon} + \boldsymbol{\varrho}^* \cdot \boldsymbol{\varrho} - U_h(\boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \ | \ \ \boldsymbol{\varepsilon} \in \mathbb{E}^3_s, \ \ \boldsymbol{\varrho} \in \mathbb{R}^3 \Big\} \\ &= \sup_{\boldsymbol{\varepsilon} \in \mathbb{E}^3_s, \ \boldsymbol{\varrho} \in \mathbb{R}^3} \Bigg\{ \frac{1}{|Y|} \int\limits_Y (\boldsymbol{\varepsilon}^* : \boldsymbol{\varepsilon} + \boldsymbol{\varrho}^* \cdot \boldsymbol{\varrho}) \, dy \\ &\quad - \inf_{\mathbf{v} \in W^{1,p}(Y)^3, \, \mathbf{d} \in \Delta_{\text{per}}(Y)} \int\limits_Y U(y, \mathbf{e}^y(\mathbf{v}(y)) + \boldsymbol{\varepsilon}, \ \mathbf{d}(y) + \boldsymbol{\varrho}) \, dy \Bigg\} \\ &= \sup \frac{1}{|Y|} \Bigg\{ \int\limits_Y \Big[(\boldsymbol{\varepsilon}^* : (\mathbf{e}^y(\mathbf{v}(y)) + \boldsymbol{\varepsilon}) + \boldsymbol{\varrho}^* \cdot (\mathbf{d}(y) + \boldsymbol{\varrho}) \\ &\quad - U(y, \mathbf{e}^y(\mathbf{v}(y)) + \boldsymbol{\varepsilon}, \ \mathbf{d}(y) + \boldsymbol{\varrho}) \Big] \, dy | \ \mathbf{v} \in W^{1,p}(Y)^3, \\ &\quad \mathbf{d} \in \Delta_{\text{per}}(Y), \ \ \boldsymbol{\varepsilon} \in \mathbb{E}^3_s, \ \ \boldsymbol{\varrho} \in \mathbb{R}^3 \Bigg\}, \end{aligned}$$

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since $\int_Y \boldsymbol{\varepsilon}^* : \mathbf{e}^y(\mathbf{v})dy = 0$ and $\langle \mathbf{d}(y)\rangle = 0$. The last relation is written as follows

$$U_h^*(\boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*) = \frac{1}{|Y|}(j + I_{\mathcal{H}})^*(\boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*), \tag{4.12}$$

where

$$j(\mathbf{t}, \boldsymbol{\gamma}) = \int_Y U(y, \mathbf{t}(y), \boldsymbol{\gamma}(y))\, dy, \tag{4.13}$$

$$\mathcal{H} = \left(\mathbf{e}^y(W^{1,p}_{\mathrm{per}}(Y)^3) \oplus \mathbb{E}^3_s\right) \times \left(\Delta_{\mathrm{per}}(Y) \oplus \mathbb{R}^3\right). \tag{4.14}$$

Identifying $\boldsymbol{\varepsilon}^*$ with a constant element of $L^{p'}(Y, \mathbb{E}^3_s)$ and $\boldsymbol{\varrho}^*$ with a constant element of $L^{q'}(Y)^3$ we have, cf. [18]

$$(j + I_{\mathcal{H}})^* = (j^* \square I_{\mathcal{H}^\perp}), \tag{4.15}$$

where $\square$ denotes the inf-convolution and

$$j^*(\mathbf{t}^*, \boldsymbol{\gamma}^*) = \int_Y U^*(y, \mathbf{t}^*(y), \boldsymbol{\gamma}^*(y))dy,$$

$$\mathcal{H}^\perp = \left[\left(\mathbf{e}^y(W^{1,p}_{\mathrm{per}}(Y)^3)\right)^\perp \cap (\mathbb{E}^3_s)^\perp\right] \times \left[(\Delta_{\mathrm{per}}(Y))^\perp \cap (\mathbb{R}^3)^\perp\right]. \tag{4.16}$$

We find

$$(\mathbb{E}^3_s)^\perp = \left\{\boldsymbol{\tau} \in L^{p'}(Y, \mathbb{E}^3_s) \mid \langle \boldsymbol{\tau}\rangle = 0\right\},$$

$$\left(\mathbf{e}(W^{1,p}_{\mathrm{per}}(Y)^3)\right)^\perp = \left\{\boldsymbol{\tau} \in L^{p'}(Y, \mathbb{E}^3_s) \mid \int_Y \boldsymbol{\tau}(y) : \mathbf{e}^y(\mathbf{v})dy = 0 \;\; \forall \mathbf{v} \in W^{1,p}_{\mathrm{per}}(Y)^3\right\}$$

$$= \Big\{\boldsymbol{\tau} \in L^{p'}(Y, \mathbb{E}^3_s) \mid \mathrm{div}_y \boldsymbol{\tau} \in L^{p'}(Y)^3,\ \mathrm{div}_y \boldsymbol{\tau} = 0 \text{ in } Y;$$

$$\boldsymbol{\tau}\cdot\mathbf{N} \text{ is anti-periodic}\Big\},$$

$$(\Delta_{\mathrm{per}}(Y))^\perp = \left\{\boldsymbol{\phi} \in L^{q'}(Y)^3 \mid \int_Y \boldsymbol{\phi}\cdot\mathbf{d}(y)dy = 0\ \forall \mathbf{d} \in \Delta_{\mathrm{per}}(Y)\right\}$$

$$= \left\{\boldsymbol{\phi} \in L^{q'}(Y)^3 \mid \boldsymbol{\phi} = \mathbf{E}^y(\varphi),\ \varphi \in W^{1,q'}_{\mathrm{per}}(Y)\right\},$$

since

$$\int_Y \mathbf{d}(y)\cdot \mathbf{E}^y(\varphi)dy = \int_Y \varphi(\mathrm{div}_y \mathbf{d})dy - \int_{\partial Y} \varphi d_i N_i ds = 0 \quad \forall \mathbf{d} \in \Delta_{\mathrm{per}}(Y),$$

provided that $\varphi \in W^{1,q'}_{\mathrm{per}}(Y)$. Here $\mathbf{E}^y(\varphi) = -\partial\varphi/\partial y_i$.

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Taking into account (4.15) and (4.16) in (4.12) we obtain

$$U_h^*(\boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*) = \frac{1}{|Y|} (j^* \Box I_{\mathcal{H}^\perp}) (\boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*)$$

$$= \inf \frac{1}{|Y|} \Big\{ j^*(\mathbf{t}_1, \boldsymbol{\gamma}_1) + I_{\mathcal{H}^\perp}(\mathbf{t}_2, \boldsymbol{\gamma}_2) \mid \boldsymbol{\varepsilon}^* = \mathbf{t}_1 + \mathbf{t}_2,$$

$$\boldsymbol{\varrho}^* = \boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2, \ \ \mathbf{t}_\alpha \in \mathcal{S}_{\text{per}}(Y), \ \ \boldsymbol{\gamma} = \mathbf{E}^y(\varphi_\alpha), \ \varphi_\alpha \in W^{1,q'}_{\text{per}}(Y), \ \ \alpha = 1, 2 \Big\}$$

$$= \inf \frac{1}{|Y|} \left\{ j^*(\boldsymbol{\varepsilon}^* - \mathbf{t}, \ \ \boldsymbol{\varrho}^* - \boldsymbol{\gamma}) \mid \ \mathbf{t} \in \mathcal{S}_{\text{per}}(Y), \ \ \boldsymbol{\gamma} = \mathbf{E}^y(\varphi), \ \varphi \in W^{1,q'}_{\text{per}}(Y) \right\}$$

$$= \inf \left\{ \langle U^*(y, \mathbf{t} + \boldsymbol{\varepsilon}^*, \ \mathbf{E}^y(\varphi) + \boldsymbol{\varrho}^* \rangle | \ \ \mathbf{t} \in \mathcal{S}_{\text{per}}(Y), \varphi \in W^{1,q'}_{\text{per}}(Y) \right\},$$

because $\mathcal{S}_{\text{per}}(Y)$ and $W^{1,q'}_{\text{per}}(Y)$ are linear spaces. This establishes the formula (4.9). □

COROLLARY 1. The macroscopic electric enthalpy $H_h(\mathbf{e}, \mathbf{E})$ can be calculated as the partial concave conjugate of U_h, cf. [1]

$$H_h(\mathbf{e}, \mathbf{E}) = \inf \left\{ -\mathbf{E} \cdot \mathbf{D} + U_h(\mathbf{e}, \mathbf{E}) \mid \mathbf{D} \in \mathbb{R}^3 \right\}. \tag{4.17}$$

Proceeding similarly to the proof of (4.9) we finally obtain

$$H_h(\boldsymbol{\varepsilon}, \boldsymbol{\varrho}^*) = \inf_{\mathbf{v} \in W^{1,p}_{\text{per}}(Y)^3} \ \sup_{\xi \in W^{1,q'}_{\text{per}}(Y)} \langle H(y, \mathbf{e}^y(\mathbf{v}) + \boldsymbol{\varepsilon}, \mathbf{E}^y(\xi) + \boldsymbol{\varrho}^*) \rangle, \tag{4.18}$$

where $\boldsymbol{\varepsilon} \in \mathbb{E}^3_s$, $\boldsymbol{\varrho}^* \in \mathbb{R}^3$ and

$$H(y, \mathbf{e}, \mathbf{E}) = \inf \left\{ -\mathbf{E} \cdot \mathbf{D} + U(y, \mathbf{e}, \mathbf{E}) \mid \mathbf{D} \in \mathbb{R}^3 \right\}, \tag{4.19}$$

is the microscopic electric enthalpy. As we have already mentioned, the homogenization of linear piezocomposites was performed in [1], cf. also [2, 3, 6, 7, 21, 22]. □

LEMMA 2. Let $\mathbf{t} \in \mathcal{S}_{\text{per}}(Y)$, $\xi \in W^{1,q'}_{\text{per}}(Y)$, $\psi \in \mathcal{D}(V)$. Let the bounded sequences $\{\mathbf{u}^\varepsilon\}_{\varepsilon > 0} \subset W^{1,p}(V)^3$, $\{\mathbf{D}^\varepsilon\}_{\varepsilon > 0} \subset \mathbf{W}(\text{div}, V)$ be such that

$$\mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{strongly in } L^p(V)^3,$$

$$\mathbf{D}^\varepsilon \rightharpoonup \mathbf{D} \quad \text{weakly in } L^q(V)^3,$$

when $\varepsilon \to 0$. Then

$$\lim_{\varepsilon \to 0} \int_V \psi(x) t_{ij} \left(\frac{x}{\varepsilon} \right) e_{ij}(\mathbf{u}^\varepsilon(x)) dx = 0, \tag{4.20}$$

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(4.21) $$D_i^\varepsilon(x)E_i\left(\xi\left(\frac{x}{\varepsilon}\right)\right) \rightharpoonup D_i(x)E_i(\langle\xi\rangle) = 0 \quad \text{in } \mathcal{D}'(V) \text{ when } \varepsilon \to 0.$$

Here $\mathcal{D}'(V)$ is the space of distributions or the dual of the space $\mathcal{D}(V)$.

P r o o f. To prove (4.20) we set

$$R_\varepsilon = \int\limits_V \psi(x)t_{ij}\left(\frac{x}{\varepsilon}\right)e_{ij}(\mathbf{u}^\varepsilon(x))dx.$$

Using integration by parts we obtain

$$R_\varepsilon = -\int\limits_V \psi_{,j}(x)t_{ij}\left(\frac{x}{\varepsilon}\right)u_i^\varepsilon(x)\,dx - \int\limits_V \psi(x)t_{ij,j}\left(\frac{x}{\varepsilon}\right)u_i^\varepsilon(x)\,dx.$$

After the rescaling $y \to x/\varepsilon$ we have $\dfrac{\partial}{\partial y_i} = \varepsilon\dfrac{\partial}{\partial x_i}$ and consequently, $\mathrm{div}_y\mathbf{t} = 0$ in Y implies $\varepsilon(\mathrm{div}\mathbf{t})(x/\varepsilon) = 0$ or $(\mathrm{div}\mathbf{t})(x/\varepsilon) = 0$ in V. Since $t_{ij,j}(x/\varepsilon) \rightharpoonup \langle t_{ij,j}(y)\rangle = 0$ in $L^{p'}(V)$ weakly as $\varepsilon \to 0$, therefore $\lim\limits_{\varepsilon\to 0} R_\varepsilon = 0$ as claimed. □

To prove (4.21) we shall exploit the following result due to MURAT. [19].

PROPOSITION 1. Let q and q' be such that

$$1 < q,\ q' < \infty, \qquad \frac{1}{q} + \frac{1}{q'} = 1,$$

and let V be a bounded or unbounded domain of $\mathbb{R}^N$. We define the following spaces:

$$\mathbf{W}(\mathrm{div}, V) = \left\{\mathbf{u} \in L^q(V)^N \mid \mathrm{div}\mathbf{u} \in L^q(V)\right\},$$
$$\mathbf{W}(\mathrm{rot}, V) = \left\{\mathbf{v} \in L^{q'}(V)^N \mid \mathrm{rot}\mathbf{v} \in L^{q'}(V, \mathbb{E}^N)\right\},$$

where

$$(\mathrm{rot}\,\mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}, \qquad 1 \le i, j \le N,$$

and $\mathbb{E}^N$ is the space of $N \times N$ matrices. The spaces $\mathbf{W}(\mathrm{div}, V)$ and $\mathbf{W}(\mathrm{rot}, V)$ are equipped with the norms:

$$\|\mathbf{u}\|_{\mathbf{W}(\mathrm{div},V)} = \|\mathbf{u}\|_{L^q(V)^N} + \|\mathrm{div}\mathbf{u}\|_{L^q(V)},$$
$$\|\mathbf{v}\|_{\mathbf{W}(\mathrm{rot},V)} = \|\mathbf{v}\|_{L^{q'}(V)^N} + \|\mathrm{rot}\mathbf{v}\|_{L^{q'}(V,\mathbb{E}^N)}.$$

If two sequences $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subset \mathbf{W}(\mathrm{div}, V)$, $\{\mathbf{v}_n\}_{n\in\mathbb{N}} \subset \mathbf{W}(\mathrm{rot}, V)$ satisfy the conditions

$\mathbf{u}_n$ is bounded in $\mathbf{W}(\mathrm{div}, V)$, $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in $L^q(V)^N$,

$\mathbf{v}_n$ is bounded *in* $\mathbf{W}(\mathrm{rot}, V)$, $\mathbf{v}_n \rightharpoonup \mathbf{u}$ weakly in $L^{q'}(V)^N$,

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then

$$\mathbf{u}_n \cdot \mathbf{v}_n \rightharpoonup \mathbf{u}\cdot\mathbf{v} \quad \text{in } \mathcal{D}'(V) \text{ when } n \to \infty. \qquad \square$$

Proof of (4.21). We observe that $\operatorname{rot} \mathbf{E}^y(\xi) = 0$ and consequently $\mathbf{E}(\xi(\cdot/\varepsilon) \in \mathbf{W}(\operatorname{rot}, V)$. By using Proposition 1 we conclude that

$$D_i^\varepsilon(x) E_i\left(\xi\left(\frac{x}{\varepsilon}\right)\right) \rightharpoonup D_i(x) E_i(\langle\xi\rangle) = 0 \quad \text{in } \mathcal{D}'(V) \text{ when } \varepsilon \to 0,$$

because $\xi(x/\varepsilon) \rightharpoonup \langle\xi(y)\rangle$ weakly in $L^{q'}(V)$, cf. [12, 20]. The proof is complete. $\square$

Now we are in a position to prove Th. 2.

Proof of Th. 2. It falls naturally into two parts.

I. Let us first show that

$$J_i = \Gamma(\tau)\text{-} \liminf_{\varepsilon\to 0} J_\varepsilon \geq J_h.$$

We recall that $\tau = w\text{-}(W^{1,p}(\Omega)^3 \times L^q(\Omega)^3)$. Let $\{\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon\}_{\varepsilon>0} \subset W^{1,p}(\Omega)^3 \times L^q(\Omega)^3$ be a bounded sequence such that

$$\begin{aligned} \mathbf{u}^\varepsilon \to \mathbf{u} \quad &\text{strongly in} \quad L^p(\Omega)^3,\\ \mathbf{D}^\varepsilon \rightharpoonup \mathbf{D} \quad &\text{weakly in} \quad L^q(\Omega)^3, \end{aligned}$$

when $\varepsilon \to 0$.

We have to show that

$$\begin{aligned} \liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \geq J_h(\mathbf{u}, \mathbf{D}) &= \int_V U_h(\mathbf{e}(\mathbf{u}), \mathbf{D})\, dx \\ &= \sup\left\{ \int_V [\boldsymbol{\sigma} : \mathbf{e}(\mathbf{u}) + \mathbf{D}^* \cdot \mathbf{D} - U_h^*(\boldsymbol{\sigma}, \mathbf{D}^*)]\, dx \;\middle|\; \boldsymbol{\sigma} \in L^{p'}(V, \mathbb{E}_s^3),\ \ \mathbf{D}^* \in L^{q'}(V)^3 \right\}. \end{aligned} \tag{4.22}$$

STEP 1. First we take $\boldsymbol{\sigma}$ and $\mathbf{D}^*$ in the following form:

$$\begin{aligned} \boldsymbol{\sigma}(x) &= \sum_{K\in\mathcal{K}} \chi_{V_K}(x)\, \boldsymbol{\sigma}^K, \qquad \boldsymbol{\sigma}^K \in \mathbb{E}_s^3, \\ \mathbf{D}^*(x) &= \sum_{K\in\mathcal{K}} \chi_{V_K}(x) \mathbf{D}^{*K}, \qquad \mathbf{D}^{*K} \in \mathbb{R}^3, \end{aligned} \tag{4.23}$$

where

$$\chi_{V_K}(x) = \begin{cases} 1 & \text{if } x \in V_K\,, \\ 0 & \text{if } x \notin V_K\,. \end{cases}$$

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Here $\{V_K\}_{K\in\mathcal{K}}$ is a family of open disjoint sets such that $\overline{V} = \bigcup_{K\in\mathcal{K}} \overline{V}_K$.

For $\delta > 0$ we set

$$V_K^\delta = \{x \in V_K \mid \operatorname{dist}(x, \partial V_K) > \delta\}. \tag{4.24}$$

Let $\psi_K^\delta \in \mathcal{D}(V_K)$ be such that $0 \le \psi_K^\delta \le 1$ and $\psi_K^\delta = 1$ for $x \in V_K^\delta$.

Let $(\mathbf{t}^K, \varphi^K) \in \mathcal{S}_{\text{per}}(Y) \times W^{1,q'}_{\text{per}}(Y)$, $K \in \mathcal{K}$. By using Lemma 2 and recalling that $U \ge 0$ we obtain

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \ge \liminf_{\varepsilon\to 0} \Bigg\{ \int_V U\left(\frac{x}{\varepsilon}, \mathbf{e}(\mathbf{u}^\varepsilon), \mathbf{D}^\varepsilon\right) dx$$

$$- \sum_K \int_{V_K} \psi_K^\delta(x) t_{ij}^K \left(\frac{x}{\varepsilon}\right) e_{ij}(\mathbf{u}^\varepsilon)\, dx - \sum_K \int_{V_K} \psi_K^\delta(x) D_i^\varepsilon(x) E_i \left(\varphi^K\left(\frac{x}{\varepsilon}\right)\right) dx \Bigg\}$$

$$= \sum_K \liminf_{\varepsilon\to 0} \int_{V_K} \psi_K^\delta(x) U_{(\mathbf{t}^K, \mathbf{E}(\varphi^K))}\left(\frac{x}{\varepsilon}, \mathbf{e}(\mathbf{u}^\varepsilon), \mathbf{D}^\varepsilon\right) dx,$$

where

$$U_{(\mathbf{t}^K, \mathbf{E}(\varphi^K))}(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) = U(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) - \mathbf{t}^K : \boldsymbol{\varepsilon} - \mathbf{E}(\varphi^K)\cdot\boldsymbol{\varrho}.$$

Fenchel's inequality applied to $[(\mathbf{e}(\mathbf{u}^\varepsilon), \mathbf{D}^\varepsilon); (\boldsymbol{\sigma}^K, \mathbf{D}^{*K})]$ yields:

$$U^*_{(\mathbf{t}^K, \mathbf{E}^y(\varphi^K))}(y, \boldsymbol{\varepsilon}^*, \boldsymbol{\varrho}^*) \ge \boldsymbol{\sigma}^K : \mathbf{e}(\mathbf{u}^\varepsilon) + \mathbf{D}^{*K}\cdot\mathbf{D} - U_{(\mathbf{t}^K, \mathbf{E}^y(\varphi^K))}(y, \mathbf{e}(\mathbf{u}^\varepsilon), \mathbf{D}^\varepsilon).$$

Hence

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon)$$

$$\ge \sum_{K\in\mathcal{K}} \liminf_{\varepsilon\to 0} \int_{V_K} \psi_K^\delta \left[\boldsymbol{\sigma}^K : \mathbf{e}(\mathbf{u}^\varepsilon) + \mathbf{D}^{*K}\cdot\mathbf{D}^\varepsilon - U^*_{(\mathbf{t}^K, \mathbf{E}^y(\varphi^K))}(y, \boldsymbol{\sigma}^K, \mathbf{D}^{*K})\right] dx.$$

Standard calculation yields

$$U^*_{(\mathbf{t}^K, \mathbf{E}^y(\varphi^K))}(y, \boldsymbol{\sigma}^K, \mathbf{D}^{*K})$$

$$= \sup\left\{\boldsymbol{\sigma}^K : \boldsymbol{\varepsilon} + \mathbf{D}^{*K}\cdot\boldsymbol{\varrho} - U_{(\mathbf{t}^K, \mathbf{E}^y(\varphi^K))}(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \;\middle|\; \boldsymbol{\varepsilon} \in \mathbb{E}^3_s,\ \boldsymbol{\varrho} \in \mathbb{R}^3\right\}$$

$$= \sup\left\{(\boldsymbol{\sigma}^K + \mathbf{t}^K) : \boldsymbol{\varepsilon} + (\mathbf{D}^{*K} + \mathbf{E}^y(\varphi^K))\cdot\boldsymbol{\varrho} - U(y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \mid \boldsymbol{\varepsilon} \in \mathbb{E}^3_s,\ \boldsymbol{\varrho} \in \mathbb{R}^3\right\}$$

$$= U^*(y, \boldsymbol{\sigma}^K + \mathbf{t}^K(y), \mathbf{D}^{*K} + \mathbf{E}^y(\varphi^K)).$$

Thus we have

$$\langle U^*_{(\mathbf{t}^K, \mathbf{E}^y(\varphi^K))}(y, \boldsymbol{\sigma}^K, \mathbf{D}^{*K})\rangle = \langle U^*(y, \boldsymbol{\sigma}^K + \mathbf{t}^K(y), \mathbf{D}^{*K} + \mathbf{E}^y(\varphi^K(y)))\rangle. \tag{4.25}$$

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Consequently

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon,\mathbf{D}^\varepsilon) \geq \sum_{K\in\mathcal{K}} \int_{V_K} \psi_K^\delta(x)$$
$$\cdot \left\{ \boldsymbol{\sigma}^K : \mathbf{e}(\mathbf{u}) + \mathbf{D}^{*K}\cdot\mathbf{D} - \langle U^*[y,\boldsymbol{\sigma}^K + \mathbf{t}^K(y), \mathbf{D}^{*K} + \mathbf{E}^y(\varphi^K(y))]\rangle \right\} dx.$$

Passing to the supremum on the r.h.s. of the last inequality when $(\mathbf{t}^K,\varphi^K)$ runs over $\mathcal{S}_{\rm per}(Y)\times W^{1,q'}_{\rm per}(Y)$ we obtain

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon,\mathbf{D}^\varepsilon) \geq \sum_{K\in\mathcal{K}} \int_{V_K} \psi_K^\delta \left[\boldsymbol{\sigma}^K : \mathbf{e}(\mathbf{u}) + \mathbf{D}^{*K}\cdot\mathbf{D} - U_h^*(\boldsymbol{\sigma}^K,\mathbf{D}^{*K})\right] dx,$$

because $\sup(-\int) = -\inf\int$. Recall that U_h^* is given by (4.9). Since $\psi_K^\delta \geq 0$ and $\boldsymbol{\sigma}(x)$, $\mathbf{D}^*(x)$ are given by (4.23), therefore

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon,\mathbf{D}^\varepsilon) \geq \int_V \sum_{K\in\mathcal{K}} \psi_K^\delta(x)\,[\boldsymbol{\sigma}(x) : \mathbf{e}(\mathbf{u}(x)) + \mathbf{D}^*(x)\cdot\mathbf{D}(x)]\,dx$$
$$- \int_V \sum_{K\in\mathcal{K}} \psi_K^\delta(x) U_h^*(\boldsymbol{\sigma}(x),\mathbf{D}^*(x))\,dx.$$

The inequality

$$0 \leq \sum_{K\in\mathcal{K}} \psi_K^\delta \leq 1,$$

implies

$$0 \leq \sum_{K\in\mathcal{K}} \psi_K^\delta(x) U_h^*(\boldsymbol{\sigma}(x)),\mathbf{D}^*(x)) \leq U_h^*(\boldsymbol{\sigma}(x),\mathbf{D}^*(x)),$$

because $U_h^* \geq 0$. It follows that

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon,\mathbf{D}^\varepsilon)$$
$$\geq \int_V \sum_{K\in\mathcal{K}} \psi_K^\delta(x)\,[\boldsymbol{\sigma}(x) : \mathbf{e}(\mathbf{u}(x)) + \mathbf{D}^*(x)\cdot\mathbf{D}(x)]\,dx - \int_V U_h^*(\boldsymbol{\sigma}(x),\mathbf{D}^*(x))\,dx.$$

We pass now to the limit when $\delta\to 0$; $\sum_{K\in\mathcal{K}} \psi_K^\delta(x)$ tends to 1 for a.e. $x\in\Omega$ and consequently we obtain

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon,\mathbf{D}^\varepsilon)$$
$$\geq \int_V [\boldsymbol{\sigma}(x) : \mathbf{e}(\mathbf{u}(x)) + \mathbf{D}^*(x)\cdot\mathbf{D}(x)]\,dx - \int_V U_h^*(\boldsymbol{\sigma}(x),\mathbf{D}^*(x))dx.$$

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STEP 2. For each $\boldsymbol{\sigma} \in L^{p'}(V, \mathbb{E}^3_s)$ and $\mathbf{D}^* \in L^{q'}(V)^3$ there exist sequences $\{\boldsymbol{\sigma}^n\}_{n\in\mathbb{N}} \subset L^{p'}(V, \mathbb{E}^3_s)$ and $\{\mathbf{D}^{*n}\}_{n\in\mathbb{N}} \subset L^{q'}(V)^3$ of simple functions such that

$$\begin{aligned} \boldsymbol{\sigma}^n \to \boldsymbol{\sigma} \quad &\text{in } L^{p'}(V, \mathbb{E}^3_s) \text{ as } n \to \infty, \\ \mathbf{D}^{*n} \to \mathbf{D}^* \text{ in } &L^{q'}(V)^3 \quad \text{as } n \to \infty, \end{aligned}$$

respectively. Here

$$\begin{aligned} \boldsymbol{\sigma}^n(x) &= \sum_{K(n)} \chi^{\delta_n}_{V_{K(n)}}(x)\boldsymbol{\sigma}^{K(n)}, \quad \boldsymbol{\sigma}^{K(n)} \in \mathbb{E}^3_s, \\ \mathbf{D}^{*n}(x) &= \sum_{K(n)} \chi^{\delta_n}_{V_{K(n)}}(x)\mathbf{D}^{*K(n)}, \quad \mathbf{D}^{*K(n)} \in \mathbb{R}^3, \end{aligned}$$

and $\delta_n = 1/n$, $\operatorname{diam} V_{K(n)} \le \delta_n$, $\overline{V} = \bigcup_{K(n)} \overline{V}_{K(n)}$. We conclude by the previous step that

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \ge \int_V [\boldsymbol{\sigma}^n(x) : \mathbf{e}(\mathbf{u}(x)) + \mathbf{D}^{*n}(x)\cdot\mathbf{D}(x)]\, dx - \int_V U^*_h(\boldsymbol{\sigma}^n(x), \mathbf{D}^{*n}(x))\, dx.$$

A passage to the limit on the r.h.s. of the last inequality when $n \to \infty$ finally gives:

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \ge \int_V [\boldsymbol{\sigma}(x) : \mathbf{e}(\mathbf{u}(x)) + \mathbf{D}^*(x)\cdot\mathbf{D}(x) - U^*_h(\boldsymbol{\sigma}(x), \mathbf{D}(x))]\, dx.$$

II. We pass now to demonstrate that for any $(\mathbf{u}, \mathbf{D}) \in W^{1,p}(V)^3 \times L^q(V)^3$ there exists a sequence $\{\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon\}_{\varepsilon>0} \subset W^{1,p}(\Omega)^3 \times L^q(\Omega)^3$ such that

$$\begin{aligned} \mathbf{u}^\varepsilon \rightharpoonup \mathbf{u} \quad &\text{in} \quad W^{1,p}(V)^3 \quad \text{weakly}, \\ \mathbf{D}^\varepsilon \rightharpoonup \mathbf{D} \quad &\text{in} \quad L^q(V)^3 \quad \text{weakly} \end{aligned}$$

when $\varepsilon \to 0$ and

$$J_h(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \ge \limsup_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon). \tag{4.26}$$

Obviously, the convergence of $\mathbf{u}^\varepsilon$ to $\mathbf{u}$ is strong in $L^p(V)^3$.

STEP 3. We take

$$u_i(x) = \varepsilon_{ij}x_j + a_i\,, \qquad \boldsymbol{\varepsilon} \in \mathbb{E}^3_s\,, \qquad a_i \in \mathbb{R}, \tag{4.27}$$

whereas $\mathbf{D}$ is an arbitrary element of $\mathbb{R}^3$ treated as a constant function of $L^q(V)^3$.

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Next we set

$$\text{(4.28)}\qquad \begin{aligned} \mathbf{u}^\varepsilon(x) &= \mathbf{u}(x) + \varepsilon\widetilde{\mathbf{v}}\left(\frac{x}{\varepsilon}\right), \\ \mathbf{D}^\varepsilon(x) &= \mathbf{D}(x) + \widetilde{\mathbf{d}}\left(\frac{x}{\varepsilon}\right), \end{aligned}$$

where $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{d}})$ solves the local problem. Hence we conclude that

$$\begin{aligned} \mathbf{u}^\varepsilon &\to \mathbf{u} \quad \text{in} \quad L^p(V)^3 \quad \text{strongly}, \\ \mathbf{D}^\varepsilon &\rightharpoonup \mathbf{D} \quad \text{in} \quad L^q(V)^3 \quad \text{weakly}, \end{aligned}$$

as $\varepsilon \to 0$.

Applying Th. 1.5 of DACOROGNA [20, Chap. 2], we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} J_h(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) = \lim_{\varepsilon\to 0} \int_V U\left[\frac{x}{\varepsilon}, \boldsymbol{\varepsilon} + \mathbf{e}(\widetilde{\mathbf{v}})\left(\frac{x}{\varepsilon}\right), \mathbf{D} + \widetilde{\mathbf{d}}\left(\frac{x}{\varepsilon}\right)\right] dx \\ = \int_V \langle U[y, \boldsymbol{\varepsilon} + \mathbf{e}^y(\widetilde{\mathbf{v}}(y)), \mathbf{D} + \widetilde{\mathbf{d}}(y)]\rangle dx \\ = \int_V U_h(\boldsymbol{\varepsilon}, \mathbf{D}) dx = \int_V U_h\left[\mathbf{e}(\mathbf{u}(x)), \mathbf{D}(x)\right] dx, \end{aligned}$$

since $\mathbf{u}$ is given by (4.27) and $\mathbf{D} \in L^q(V)^3$ is a constant function.

STEP 4. Let now $\mathbf{u}$ be a continuous affine function as an element of $W^{1,p}(V)^3$ and $\mathbf{D}$ a simple function in the space $L^q(V)^3$:

$$\text{(4.29)}\qquad \mathbf{u}(x) = \boldsymbol{\varepsilon}^K x + \mathbf{a}^K, \qquad x \in V_K,$$

$$\text{(4.30)}\qquad \mathbf{D}(x) = \sum_K \chi_{V_K}(x)\mathbf{D}^K, \qquad \mathbf{D}^K \in \mathbb{R}^3,$$

where $\boldsymbol{\varepsilon}^K \in \mathbb{E}^3_s$, $\mathbf{a}^K \in \mathbb{R}^3$ and $\{V_K\}_{K\in\mathcal{K}}$ is a finite partition of V formed by polyhedral sets.

We set

$$V_K^\delta = \{x \in V_K \mid \operatorname{dist}(x, \partial V_K) > \delta\}, \qquad \delta > 0.$$

Let $\psi_K^\delta \in \mathcal{D}(V_K)$ be such that $0 \le \psi_K^\delta \le 1$ and $\psi^\delta_{K|V_K^\delta} = 1$. With every family of functions $(\mathbf{v}^K, \mathbf{d}^K) \in W^{1,p}_{\text{per}}(Y)^3 \times \Delta_{\text{per}}(Y)$ we link the following sequences:

$$\text{(4.31)}\qquad \mathbf{u}^{\varepsilon,\delta}(x) = \mathbf{u}(x) + \varepsilon \sum_{K\in\mathcal{K}} \psi_K^\delta(x)\mathbf{v}^K\left(\frac{x}{\varepsilon}\right),$$

$$\text{(4.32)}\qquad \mathbf{D}^{\varepsilon,\delta}(x) = \mathbf{D}(x) + \sum_{K\in\mathcal{K}} \psi_K^\delta(x)\mathbf{d}^K\left(\frac{x}{\varepsilon}\right).$$

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It is evident that

$$\mathbf{u}^{\varepsilon,\delta} \to \mathbf{u} \quad \text{in} \quad L^p(V)^3 \quad \text{strongly},$$
$$\mathbf{D}^{\varepsilon,\delta} \rightharpoonup \mathbf{D} \quad \text{in} \quad L^q(V)^3 \quad \text{weakly},$$

as $\varepsilon \to 0$.

We recall that for $N_0 \in \mathbb{N}$ sufficiently large one has:

$$\int_{V_K} \left| \mathbf{v}^K \left(\frac{x}{\varepsilon}\right)\right|^p dx \le N_0 \int_Y |\mathbf{v}^K(y)|^p\, dy.$$

Take $0 < t < 1$. Since $t\psi_K^\delta + t(1-\psi_K^\delta) + (1-t) = 1$, therefore by convexity of $U((x/\varepsilon),\cdot,\cdot)$ we obtain

$$\begin{aligned}(4.33)\qquad J_\varepsilon(t\,\mathbf{u}^{\varepsilon,\delta}, t\,\mathbf{D}^{\varepsilon,\delta}) &= \sum_K \int_{V_K} U\Big[\frac{x}{\varepsilon}, t\,\psi_K^\delta\left(\boldsymbol{\varepsilon}^K + \mathbf{e}(\mathbf{v}^K)\left(\frac{x}{\varepsilon}\right)\right) + t\,(1-\psi_K^\delta)\boldsymbol{\varepsilon}^K \\ &\quad + (1-t)\frac{\varepsilon\, t}{1-t}\left(\psi^\delta_{K,(i}(x) v^K_{j)}\left(\frac{x}{\varepsilon}\right)\right), t\,\psi_K^\delta\left(\mathbf{D}^K + \mathbf{d}^K\left(\frac{x}{\varepsilon}\right)\right) \\ &\quad + t(1-\psi_K^\delta)\mathbf{D}^K + (1-t)\mathbf{0}\Big]dx \\ &\le \sum_K \Bigg\{ \int_{V_K} U\left[\frac{x}{\varepsilon}, \boldsymbol{\varepsilon}^K + \mathbf{e}(\mathbf{v}^K)\left(\frac{x}{\varepsilon}\right), \mathbf{D}^K + \mathbf{d}^K\left(\frac{x}{\varepsilon}\right)\right] dx \\ &\quad + c_1\Big(|\boldsymbol{\varepsilon}^K|^p + |\mathbf{D}^K|^q\Big)\int_{V_K}(1-\psi_K^\delta)dx + c_1(1-t)\int_{V_K}\frac{\varepsilon\, t}{1-t}\left|\left(\psi^\delta_{K,(i}(x) v^K_{j)}\left(\frac{x}{\varepsilon}\right)\right)\right|^p dx\Bigg\}.\end{aligned}$$

We recall that $U \ge 0$.

Let now ε tend to zero. Then we get

$$\begin{aligned}\limsup_{\varepsilon\to 0} J_\varepsilon(t\,\mathbf{u}^{\varepsilon,\delta}, t\,\mathbf{D}^{\varepsilon,\delta}) &\le \sum_K \Big\{ |V_K| \langle U\left(y, \boldsymbol{\varepsilon}^K + \mathbf{e}^y(\mathbf{v}^K(y)), \mathbf{D}^K + \mathbf{d}^K(y)\right)\rangle \\ &\quad + c_1\left(|\boldsymbol{\varepsilon}^K|^p + |\mathbf{D}^K|^q\right)\int_{V_K}(1-\psi_K^\delta)dx\Big\}.\end{aligned}$$

Next, let $t \to 1^-$ and $\delta \to 0$. The sequence $\sum_K \psi_K^\delta(x)$ converges to 1 almost everywhere when $\delta \to 0$. We conclude that

$$\begin{aligned}(4.34)\qquad &\limsup_{\substack{\delta\to 0\\ t\to 1^-}}\ \limsup_{\varepsilon\to 0} J_\varepsilon(t\,\mathbf{u}^{\varepsilon,\delta}, t\,\mathbf{D}^{\varepsilon,\delta}) \\ &\qquad \le \sum_K |V_K| \langle U\left(y, \boldsymbol{\varepsilon}^K + \mathbf{e}^y(\mathbf{v}^K(y)), \mathbf{D}^K + \mathbf{d}^K(y)\right)\rangle.\end{aligned}$$

To proceed further we shall exploit the following lemma due to ATTOUCH [10].

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LEMMA 3. Let $\{a_{A,B} \mid A \in \mathbb{N}, B \in \mathbb{N}\}$ be a doubly indexed family in $\overline{\mathbb{R}}$-the extended reals. Then there exists a mapping $A \to B(A)$, increasing to $+\infty$, such that

$$\limsup_{A\to\infty} a_{A,B(A)} \le \limsup_{B\to\infty} \limsup_{A\to\infty} a_{A,B}. \qquad \square$$

Applying this lemma we construct a mapping $\varepsilon \to (t(\varepsilon), \delta(\varepsilon))$ with $(t(\varepsilon), \delta(\varepsilon)) \to (1^-, 0)$ such that by setting

$$\mathbf{u}^\varepsilon = t(\varepsilon)\, \mathbf{u}^{\varepsilon,\delta}, \qquad \mathbf{D}^\varepsilon = t(\varepsilon)\, \mathbf{D}^{\varepsilon,\delta},$$

we obtain from (4.34)

$$\limsup_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) \le \sum_K |V_K| \left\langle U\left(y, \boldsymbol{\varepsilon}^K + \mathbf{e}^y(\mathbf{v}^K(y)), \mathbf{D}^K + \mathbf{d}^K(y)\right)\right\rangle. \tag{4.35}$$

Taking the infimum on the right-hand side of the last inequality when $(\mathbf{v}^K, \mathbf{d}^K)$ run over $W^{1,p}_{\text{per}}(Y)^3 \times \Delta_{\text{per}}(Y)$, we get

$$\begin{aligned} J_s(\mathbf{u}, \mathbf{D}) \le \limsup_{\varepsilon\to 0} J_\varepsilon(\mathbf{u}^\varepsilon, \mathbf{D}^\varepsilon) &\le \sum_K |V_K| U_h(\boldsymbol{\varepsilon}^K, \mathbf{D}^K) \\ &= \int_\Omega U_h(\mathbf{e}(\mathbf{u}),\, \mathbf{D})dx = J_h(\mathbf{u}, \mathbf{D}), \end{aligned} \tag{4.36}$$

where J_s stands for the $\Gamma\left[(\text{w}-W^{1,p}(V)^3) \times (\text{w}-L^q(V)^3)\right]-\limsup_{\varepsilon\to 0} J_\varepsilon$. In (4.36) we have used the fact that

$$\begin{aligned} &\mathbf{u}^\varepsilon \to \mathbf{u} && \text{strongly} && \text{in} && L^p(V)^3, \\ &\mathbf{D}^{\varepsilon,\delta} \rightharpoonup \mathbf{D} && \text{weakly} && \text{in} && L^q(V)^3, \end{aligned}$$

when $\varepsilon \to 0$.

STEP 5. From Sec. 3 we know that the convexity of J_ε is preserved by the Γ-limit superior. By virtue of the property (ii) of U_h we write

$$J_s(\mathbf{u}, \mathbf{D}) \le c_1 \int_V \left(|\mathbf{e}(\mathbf{u}(x))|^p + |\mathbf{D}(x)|^q\right)\, dx,$$

where $\mathbf{u} \in W^{1,p}(V)^3$, $\mathbf{D} \in L^q(V)^3$. Being convex and finite, the functional J_s is continuous on the space $W^{1,p}(V)^3 \times L^q(V)^3$. Exploiting the properties of the homogenized potential U_h we readily conclude that J_h is also a convex and continuous functional on this space. By density of piecewise affine continuous functions in $W^{1,p}(V)$ and simple functions in $L^q(V)$, cf. [17], the inequality $J_s(\mathbf{u}, \mathbf{D}) \le J_h(\mathbf{u}, \mathbf{D})$ is readily extended to $W^{1,p}(V)^3 \times L^q(V)^3$, see [23, 24]. This completes the proof. □

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REMARK 5. The proof of Th. 2 remains valid for $\mathbf{u} \in W^{1,p}(V)^3$ with $\mathbf{ı} = \mathbf{0}$ on γ_0 and $\mathbf{D} \in \mathbf{W}(\mathrm{div}, V)$ satisfying $\mathrm{div}\,\mathbf{D} = 0$ in V. For the the sake of sinplicity, let us assume that $\gamma_3 = \emptyset$, hence $\gamma_2 = \partial V$. Then we take $\mathbf{u}$ as in (4.29), with $\mathbf{u} = \mathbf{0}$ on γ_0 and

$$\mathbf{D}^{\varepsilon,\delta}(x) = \mathbf{D}^{\varepsilon}(x) = \mathbf{D}(x) + \mathbf{d}^{K}\left(\frac{x}{\varepsilon}\right), \qquad \mathbf{d}^{K} = \mathbf{d}.$$

Here $\mathbf{d}$ is an element in $\Delta_{\mathrm{per}}(Y)$. Since $\mathrm{div}_y \mathbf{d}(y) = 0$ in Y therefore $(\mathrm{div}\,\mathbf{d})(x/\varepsilon)$ vanishes in V. Instead of $J_\varepsilon(t\,\mathbf{u}^{\varepsilon,\delta}, t\,\mathbf{D}^{\varepsilon,\delta})$ we now consider $J_\varepsilon(t\,\mathbf{u}^{\varepsilon,\delta}, \mathbf{D}^{\varepsilon})$

REMARK 6. By exploiting the assumption (A) it is not difficult to slow that Th. 1 applies and consequently

$$\inf\{J_h(\mathbf{u},\mathbf{D}) - L(\mathbf{u},\mathbf{D}) \mid (\mathbf{u},\mathbf{D}) \in X\} = \lim_{\varepsilon\to 0}\left(\inf\{J_\varepsilon(\mathbf{u},\mathbf{D}) - L(\mathbf{u},\mathbf{D}) \mid (\mathbf{u},\mathbf{D}) \in X\}\right)$$

where

$$X = \Big\{(\mathbf{u},\mathbf{D}) \in W^{1,p}(V)^3 \times \mathbf{W}(\mathrm{div}, V) \mid \mathbf{u} = 0 \ \text{ on } \gamma_0, \quad D_{i,i} = 0 \ \text{ in } V, \quad D_i n_i = 0 \ \text{ on } \gamma_3\Big\}. \tag{4.37}$$

5. Comments on non-uniform homogenization

From the point of view of homogenization, the coercivity condition appearing in (2.3) can be significantly weakened. In fact, let now

$$U = U(x, y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{E}^3_s \times \mathbb{R}^3 \to [0, +\infty),$$

be a measurable function, Y-periodic in y, continuous in x and convex in $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varrho}$. This function is assumed to satisfy the following conditions, cf. [25–27]:

(i) $|U(x, y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) - U(x', y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho})| \le \omega(|x - x'|)\,(a(y) + U(x, y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}))$,

(ii) $0 \le U(x, y, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) \le b(x)\,(a(y) + |\boldsymbol{\varepsilon}|^p + |\boldsymbol{\varrho}|^q)$,

for each $x \in V$, $y \in Y$, $\boldsymbol{\varrho} \in \mathbb{R}^3$ and each $\boldsymbol{\varepsilon} \in \mathbb{E}^3_s$. Here $a \in L^1_{loc}(\mathbb{R}^3)$ is a Y periodic function, $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function, continuous at zero and such that $\omega(0) = 0$ and $b : \mathbb{R}^3 \to \mathbb{R}$ is a continuous and nonnegative functon. The assumption (ii) admits internal energies, which are not strictly convex. Thus it may then happen that $U(x, y, \mathbf{e}(\mathbf{u}), \mathbf{D}) = 0$ though either $\mathbf{e}(\mathbf{u})$ or/and $\mathbf{D}$ do not disappear. The dependence of U on the macroscopic variable x means that after homogenization, the effective potential U_h still depends on x (nonuniform homogenization). Indeed, the latter potential takes the form:

$$U_h(x, \boldsymbol{\varepsilon}, \boldsymbol{\varrho}) = \inf\left\{\frac{1}{|Y|}\int_Y U\left(x, y, \mathbf{e}^y(\mathbf{v}) + \boldsymbol{\varepsilon}, \mathbf{d}(y) + \mathbf{D}\right) dy \mid \mathbf{v} \in W^{1,p}_{\mathrm{per}}(Y)^3,\ \mathbf{d} \in \Delta_{\mathrm{per}}(Y)\right\}. \tag{5.1}$$

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The macroscopic variable x changes slowly while the local variable y characterizes fast local changes. More general case of the stored energy function was considered in [26, 27]. The last papers were inspired by application to geometrically nonlinear structures such like plates and shells, yet the general nonuniform homogenization procedure can be adapted to our case of nonlinear piezoelectric composites. We observe that for microperiodic composites, U in (5.1) does not depend on the macroscopic variable $x \in V$. The assumption (i) is then trivially satisfied. We conclude from (ii) that though then the problem becomes noncoercive, yet the Γ-convergence yields the same results as for the coercive problem studied in Sec. 4.

6. Final remarks

To model the behaviour of piezoelectric composites with periodic structure and subjected to stronger electric fields, nonlinear homogenization has been used. Our considerations are confined to small deformations. Such a case has its practical value, cf. [8]. Theorem 2 justifies also the homogenization results obtained by the first author in [1] for linear piezocomposites as well as the homogenization formulae used by BISEGNA and LUCIANO [3–5], cf. also [2]. The primal and dual effective potential cannot be explicitly found, except in particular cases. For instance, such is the case of layered composites. Hence the need follows for bounding the effective potential U_h from below and from above. To this end, the nonlinear bounding techniques developed by TALBOT and WILLIS [28] can be applied. We observe that bounding techniques for linear piezoelectric composites have been used in [3, 21].

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