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On influence of viscosity on stability of train-track-systems

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THE PAPER is devoted to an analysis of behaviour of the train-track-systems modelled as a continuous or lumped subsystem moving with constant speed along an infinite Bernoulli–Euler-beam on a visco-elastic foundation. The critical velocities were determined on the basis of the stability criteria. It was proved that the stability regions are also influenced by a very small intensity of damping.

Pracę poświęcono analizie zachowania układu pojazd-tor modelowanego jako podukład ciągły lub dyskretny poruszający się ze stałą prędkością wzdłuż nieskończonej belki Bernouliego–Eulera na lepkosprężystym podłożu. Krytyczne prędkości określono na podstawie kryteriów stateczności. Wykazano zależność zakresów stateczności od lepkości nawet w przypadku bardzo małej intensywności tłumienia.

Работа посвящена анализу поведения системы транспортное средство–путь, как сплошной или дискретной подсистемы, движущейся с постоянной скоростью вдоль бесконечной балки Бернулли–Эйлера на вязкоупругом основании. Критические скорости определены на основе критериев устойчивости. Показана зависимость интервалов устойчивости от вязкости даже в случае очень малой интенсивности затухания.

1. Introduction

WITH INCREASING travelling speeds, the dynamic interaction between vehicles and guideway becomes more and more important. Thus, there is a need for simple but reliable models for such transportation systems in order to study the dynamical effects.

The models are useful to examine the vertical as well as the lateral motion. The track-subsystem is modelled as an infinite Bernoulli-Euler-beam on an elastic foundation, while the train-subsystem consists of different lumped or continuous models which are infinite in length, respectively. Both subsystems are in relative motion to each other with a constant velocity. The suspension is modelled by linear springs. The mathematical description of the different train-track-models depends on the modelling of the subsystems. It consists either of two coupled partial differential equations or of a set of ordinary differential equations coupled with a partial differential equation. The solution is obtained by applying the concept of travelling waves. Special attention is paid to the stationary solution and its stability.

The stability analysis of linear system is performed by investigating the roots of the resulting characteristic equation. Critical travelling speeds can be calculated depending on the system parameters. The results are obtained in the case of damping. The results obtained by means of comparatively simple models are believed to remain valid also for more complex systems and provide an insight into the problem of the dynamic stability of real train–track-systems.

There are various extensions of the classical problems towards more realistic models of railway tracks. A tensionless Winkler foundation was investigated in an analog-computer study by CRINER, MCCANN [1], where only small differences of the beam deflection were found compared to the classical model with the same loading. Another paper devoted to this subject is due to CHORUS, ADAMS [2]. Different beam models resting on a Pasternak foundation have been compared by SAITO, TERASAWA [3]. Though the Bernoulli–Euler beam theory compared to Timoshenko beam theory and the exact two-dimensional elastic theory gives extremely inconsistent results in front of the load for $U > U_{cr}$, it seems to be reliable for all velocities excluding the region mentioned. A periodic mass and stiffness distribution along the beam was investigated by POPP, MÜLLER [4] in order to approximate the effects of sleepers in a railway track. Again, for realistic system parameters the differences compared to the classical model turned out to be very small. Thus, the classical continuous model seems to be quite appropriate for the investigation of real railways tracks.

In contrast to the reviewed literature, the present paper, as an extension of [5], is devoted to models with more than one contact point. As a limiting case of a long train, an infinite moving beam will be investigated in detail. On the other hand, the case of a moving lumped system with two contact points will also be analyzed. Superposition of the solutions may provide insight into the dynamical bevaviour of trains of finite length.

2. Stability of interaction of two infinite beams on viscoelastic foundation

We consider the model shown in Fig. 1 composed of two infinite continuous beams j, $j = 0, 1$, where beam 0 moves with a constant velocity U_0 relative to beam 1. Each

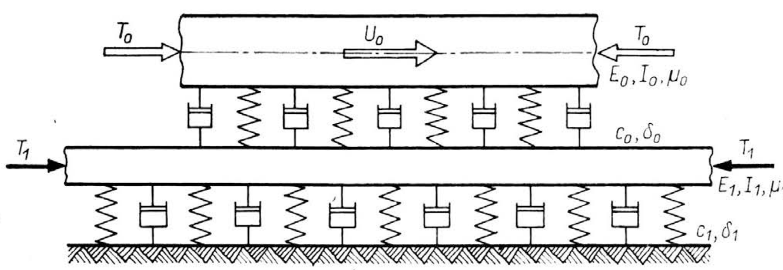

FIG. 1. Two continuous subsystems in relative motion.

beam (flexural rigidity $E_j I_j$, mass per unit length μ_j, longitudinal force T_j, $T_j > 0$ means compression and $T_j < 0$ means tension) is supported by a linear viscoelastic foundation (foundation constants δ_j, c_j). Two reference frames with coordinates (x_1, x_2) and $(\overset{*}{x}_1, \overset{*}{x}_2)$ are used, attached to beam 1 and 0, respectively. For an undisturbed motion, the beam displacements $w_j(x_1, t)$ in x_2-direction are assumed to be zero. The equations of motion with respect to the (x_1, x_2)-frame read

$$E_1 I_1 \frac{\partial^4 w_1}{\partial x_1^4} + T_1 \frac{\partial^2 w_1}{\partial x_1^2} + \mu_1 \frac{\partial^2 w_1}{\partial t^2} + p_f(x_1, t) - p_1(x_1, t) = 0, \tag{2.1}$$

$$p_f(x_1, t) = c_1 w_1(x_1, t) + \delta_1 \frac{\partial w_1(x_1, t)}{\partial t}, \tag{2.2}$$

$$E_0 I_0 \frac{\partial^4 w_0}{\partial x_1^4} + T_0 \frac{\partial^2 w_0}{\partial x_1^2} + \mu_0 \left(\frac{\partial^2 w_0}{\partial t^2} + 2U_0 \frac{\partial^2 w_0}{\partial x_1 \partial t} + U_0^2 \frac{\partial^2 w_0}{\partial x_1^2} \right) - p_0(x_1, t) = 0. \tag{2.3}$$

Here $p_j, j = 0, 1$, denotes the pressure which acts on beam i due to the disturbed motion of the beams. From the condition of compatibility it follows

$$p_1(x, t) = -p_0(x_1, t) = c_0(w_0 - w_1) + \delta_0 \left(\frac{\partial(w_0 - w_1)}{\partial t} + U_0 \frac{\partial(w_0 - w_1)}{\partial x} \right). \tag{2.4}$$

In order to simplify the analysis let us write Eq. (2.3) in the moving $(\overset{*}{x}_1, \overset{*}{x}_2)$-frame, where

$$x_1 - \overset{*}{x}_1 - U_0 t = 0, \tag{2.5}$$

$$x_2 - \overset{*}{x}_2 = 0. \tag{2.6}$$

Then Eq. (2.3) takes the form

$$E_0 I_0 \frac{\partial^4 \overset{*}{w}_0}{\partial \overset{*}{x}_1^4} + T_0 \frac{\partial^2 \overset{*}{w}_0}{\partial \overset{*}{x}_1^2} + \mu_0 \frac{\partial^2 \overset{*}{w}_0}{\partial t^2} + \delta_0 \frac{\partial(\overset{*}{w}_0 - \overset{*}{w}_1)}{\partial t} + c_0(\overset{*}{w}_0 - \overset{*}{w}_1) = 0. \tag{2.7}$$

To solve the set of Eqs. (2.1), (2.4), (2.7) together with the condition (2.5), we are looking for a steady-state solution in the form of travelling waves,

$$\begin{gathered} w_j = A_j e^{-ik_j(x_1 - v_1 t)}, \quad \overset{*}{w}_j = A_j e^{i\overset{*}{k}_j(\overset{*}{x}_1 - v_2 t)}, \\ -p_j = (-1)^j p e^{ik_j(x_1 - v_1 t)}, \quad j = 0, 1. \end{gathered} \tag{2.8}$$

Making use of relation (2.6) we find

$$A_j = \overset{*}{A}_j, \quad \overset{*}{k}_j = k_j = k, \quad \mathrm{Im}(k) = 0, \quad j = 0, 1, \tag{2.9}$$

$$v_1 - v_2 - U_0 = 0. \tag{2.10}$$

Introducing Eq. (2.8) into Eqs. (2.3), (2.5), (2.7) and using Eqs. (2.9) and (2.10) it follows

$$[\eta(R_1^2 - v_1^2) - \alpha^2] A_1 + \alpha^2 A_0 - i[(\delta_1 v_1 + \delta_0 v_2) A_1 - \delta_0 v_2 A_0] = 0, \tag{2.11}$$

$$(R_0^2 - v_2^2 + \alpha^2) A_0 - \alpha^2 A_1 - i\delta_0 v_2 (A_0 - A_1) = 0, \tag{2.12}$$

where

$$\begin{gathered} R_0^2 = \frac{1}{\mu_0}(E_0 I_0 k^2 - T_0), \quad R_1^2 = \frac{1}{\mu_1}\left(E_1 I_1 k^2 - T_1 + \frac{c_1}{k^2} \right), \\ \eta = \frac{\mu_1}{\mu_0}, \quad \alpha^2 = \frac{c_0}{\mu_0 k^2}. \end{gathered} \tag{2.13}$$

The condition of uniqueness of solution of the set (2.11)–(2.13) with respect to A_0, A_1 yields a relation between v_1 and v_2.

$$\begin{vmatrix} \eta(R_1^2 - \alpha^2 \eta^{-1} - v_1^2) - i(\delta_1 v_1 + \delta_0 v_2), & \alpha^2 - i\delta_0 v_2 \\ -\alpha^2 - i\delta_0 v_2, & R_0^2 + \alpha^2 - v_2^2 - i\delta_0 v_2 \end{vmatrix} = 0. \tag{2.14}$$

Eq. (3.14) together with Eq. (2.10) yields the characteristic equation of the problem under consideration

$$\Phi(v_1, v_2) = \Phi_R(v_1, v_2) + i\Phi_I(v_1, v_2) = 0, \quad v_1 - v_2 - U_0 = 0. \tag{2.15}$$

3. Stability analysis

In the present case with damping, the solution (2.8) may be unstable, stable and asymptotically stable, what depends on the value of the velocity U_0. Stability of the steady-state solution (2.8) requires Im $(kv_\nu) \leqslant 0$ or $\nu = 1, 2$ and in the case of instability Im $(kv)_\nu > 0$, $\nu = 1$ or $\nu = 2$. Now we will analyze the stability behaviour depending on the velocity U_0. First let us observe that the solution is stable if and only if there exist four complex roots $v_\nu^{(n)}$, $\nu = 1, 2, n = 1, 2, 3, 4$ of Eq. (2.15) such that

$$\mathrm{m}(kv_\nu^{(n)}) \leqslant 0. \tag{3.1}$$

If the inequality sign holds it follows asymptotic stability while in case where the equality sign holds it follows stability, when the real roots are different.

The regions S_I of U_0 for which

$$S_I = \{U_0 : v_\nu, \quad \mathrm{Im}(v_\nu) > 0, \ \Phi(v_\nu, U_0) = 0\}, \tag{3.2}$$

will be called instability regions. Since the analytical determination of critical parameters is complicated, let us use a geometrical approach. The critical values of U_0 at the boundaries of the instability region S_I are determined in the v_1, v_2-plane by the straight lines $v_2 = v_1 - U_0$ passing through the intersection points of the curve Re$\Phi = 0$ and Im$\Phi = 0$ Eq. (2.14). In the region $S = \{U_0 : U_0 \in [U_{1cr}, U_{2cr}]\}$, the solution (2.8) describes waves with amplitudes increasing in time. Beside this solution there exists also a trivial solution; thus, according to Lapunov's instability criterion, region S is the region of instability, $S = S_I$.

Now let us determine the instability regions for certain particular cases. From the form of the characteristic curves in the v_1, v_2-plane it follows that for $\delta_0 = 0$, $\delta_1 > 0$, or $\delta_0 > 0$, $\delta_1 = 0$ and $R_0^2, R_1^2, \alpha^2 \in [0, \infty]$, the region of stability is bounded. The limit case $\alpha^2 = c_0 \mu_0 k^{-2}_{\to\infty}$ describes a stiff connection between beams 0 and 1. For this case the second critical velocity tends to infinity. The case $R^2 = E_0 I_0 k^2 - T_0 = 0$ corresponds to a moving beam $j = 0$ with rigidity equal to zero. For $T_0 = 0$, $E_0 I_0 = 0$, the case of a moving chain of densely distributed oscillators without mass interaction is obtained [8]. On the basis of the characteristic equations derived it can be found that the critical velocities depend on the products of the damping coefficient and their ratios. The characteristic results are illustrated in the (v_1, v_2)-plane. The curves representing the real part of the characteristic equations Re$\Phi(v_1, v_2) = 0$ depend on the product $\delta_1 \delta_0$ and for the case $\delta_1 \delta_0 \to 0$ they are identical with the elastic case. The imaginary part of the characteristic equations depends both on the product $\delta_1 \delta_0$ and the ratio $\delta_1 \delta_0^{-1}$. The curves representing the imaginary part Im$\Phi(v_1, v_2) = 0$ for $\delta_1 \delta_0^{-1} \approx 1$ are shown in Fig. 2, and for the cases $\delta_1 \delta^{-1} \to 0$ and $\delta_1 \delta_0^{-1} \to \infty$ they are shown in Fig. 3. It is easy to see that in the case of $\delta_1 \delta_0 \to 0$, the critical value of the motion velocity is not greater than that in the elastic case, and sometimes the difference is very pronounced (Fig. 3).

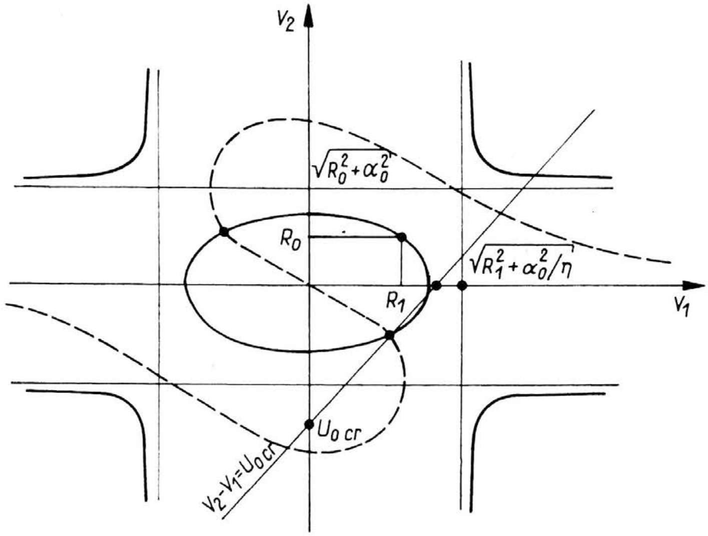

FIG. 2. Phase-velocities plane for $\delta_1 \delta_0^{-1} = 1$, $\delta_1 \delta_0 = 0$.

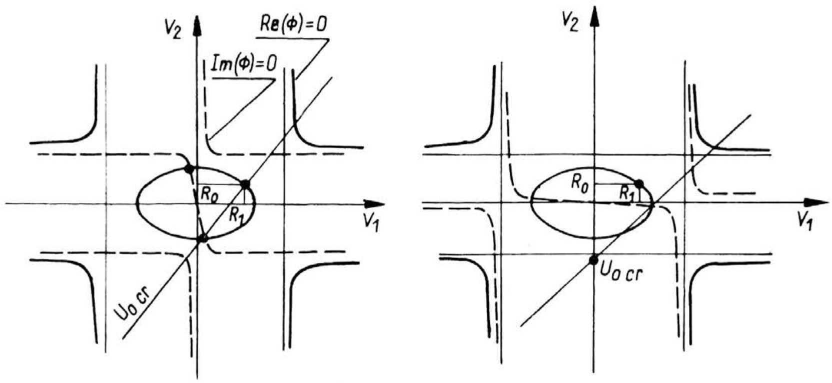

FIG. 3. Characteristic curves for $\delta_0 \delta_1 \to 0$ and a — $\delta_1 \delta_0^{-1} \to 0$, b — $\delta_1 \delta_0^{-1} \to \infty$.

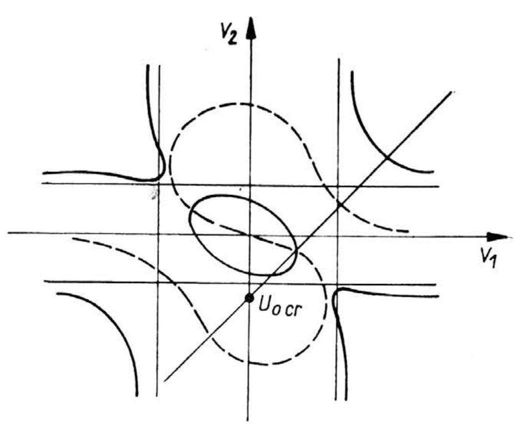

FIG. 4. Characteristic curves for the case of large damping — $\delta_0 \delta_1 \approx 1$.

In the case when the value of product $\delta_1 \delta_0$ is relatively great, the influence of damping on the shape of the curves representing the real part of the characteristic equation is essential, what is shown in Fig. 4.

4. Interaction of a moving lumped system with an infinite continuous system

Let us now consider the case similar to that shown in [5] but including damping. The system consists of a one degree of freedom vehicle (mass m, string contants c) with two contact points (distance L), moving along an infinite beam supported by linear elastic foundation or, in the case of $E_1 I_1 = 0$, $T < 0$, moving on a string under tension. The remaining parameters and the reference frames are specified as in Sect. 2. First let us analyze the case of a single periodic contact force $F(t)$ acting at point $x_1 = 0$, (cf. MATHEWS [5]). The corresponding equation of motion has the form

$$(4.1)\qquad E_1 I_1 \frac{\partial^4 \overset{*}{w}_1}{\partial \overset{*}{x}_1^4} + T_1 \frac{\partial^2 \overset{*}{w}_1}{\partial \overset{*}{x}_1} + \mu_1 \left(\frac{\partial^2 \overset{*}{w}_1}{\partial t^2} - 2U_0 \frac{\partial^2 \overset{*}{w}_1}{\partial \overset{*}{x}_1 \partial t} + U_0^2 \frac{\partial^2 \overset{*}{w}_1}{\partial \overset{*}{x}_1} \right) + b_1 \left(\frac{\partial \overset{*}{w}_1}{\partial t} - \frac{\partial \overset{*}{w}_1}{\partial \overset{*}{x}_1} \right) + c\overset{*}{w}_1 - p_1(\overset{*}{x}_1, t) = 0,$$

$$(4.2)\qquad p_1(\overset{*}{x}_1, t) = F(t)\,\delta(\overset{*}{x}_1) = Pe^{i\omega t}\delta(\overset{*}{x}_1).$$

The solution consists of two parts,

$$(4.3)\qquad \overset{*}{W}_1(x_1, t) = \overset{*}{W}_1(\overset{*}{x}_1, t)H(-\overset{*}{x}_1) + W_2(\overset{*}{x}_1, r)H(\overset{*}{x}_1),$$

where $H(\overset{*}{x})$ is the Heaviside step function, i.e. $H(\overset{*}{x}) = 1$ if $\overset{*}{x} > 0$ and $H(\overset{*}{x}) = 0$ if $\overset{*}{x} < 0$. The functions W_1 and W_2 fulfil the following compatibility condition at $\overset{*}{x}_1 = 0$:

$$(4.4)\qquad \begin{gathered} \frac{\partial^n \overset{*}{W}_1}{\partial \overset{*}{x}_1^n} = \frac{\partial^n \overset{*}{W}_2}{\partial \overset{*}{x}_1^n}, \quad n = 0, 1, 2, \\ E_1 I_1 \left(\frac{\partial^3 \overset{*}{w}_1}{\partial \overset{*}{x}_1^3} - \frac{\partial^3 \overset{*}{w}_2}{\partial \overset{*}{x}_1^3} \right) + Pe^{-i\omega t} = 0. \end{gathered}$$

Utilizing the condition of radiation and the Eqs. (4.3), (4.4) we obtain a relation between the beam deflection $w_1(\overset{*}{x}_1, t)$, force $F(t)$, velocity U_0 and frequency ω. In the steady-state case it follows that

$$(4.5)\qquad \left. \frac{\overset{*}{w}_1(\overset{*}{x}_1, t)}{F(t)} \right|_{t\to\infty} = \frac{\overset{*}{w}_{1\infty}(\overset{*}{x}_1)}{P} = G(\overset{*}{x}_1, \omega, U_0);$$

in the case of elastic waves the relation between the wave velocity v and wave number k is

$$(4.6)\qquad v^2 = (E_1 I_1 k^4 - Tk^2 + c)/\mu k^2,$$

what leads to the critical velocity U_{cr},

$$(4.7)\qquad \left. \frac{\partial v}{\partial k} \right|_{v=U_{cr}} = 0 \Rightarrow U_{cr}^2 = \sqrt{\frac{4c_1 E_1 I_1}{\mu_1^2}} - \frac{T_1}{\mu_1}.$$

Now let us analyze the model shown in Fig. 5. Since the symmetry is assumed, the equation of motion of the mass m takes the simple form

$$m\frac{d^2y}{dt^2}+F_c(t)=0, \tag{4.8}$$

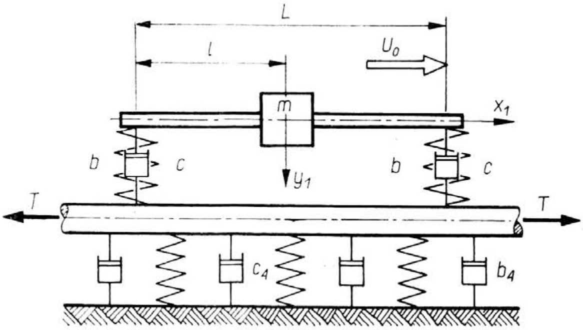

FIG. 5. Hybrid system.

$$F_c(t)=c\left[2y(t)-\overset{*}{w}_1\left(-\frac{L}{2},t\right)-\overset{*}{w}_1\left(\frac{L}{2},t\right)\right] \\ +b\left[2\frac{dy}{dt}=\frac{d\overset{*}{w}_1\left(-\frac{L}{2}\right)-d\overset{*}{w}_1\left(+\frac{L}{2}\right)}{dt}\right], \tag{4.9}$$

where y characterizes the mass displacement, and $F_c/2$ denotes the contact force in the x_2-direction. Equation of the continuous system is given by Eq. (4.1), where the load has the form

$$p_1(\overset{*}{x}_1,t)=\frac{1}{2}F_c(t)\,\delta\left(\overset{*}{x}-\frac{L}{2}\right)+\frac{1}{2}F_c(t)\,\delta\left(\overset{*}{x}+\frac{L}{2}\right). \tag{4.10}$$

For a steady-state motion from Eqs. (4.8), (4.9) and (4.5) we obtain the characteristic equation in the form

$$G_0(\omega)+2G(0,\omega,U_0)+G(L,\omega,U_0)+G(-L,\omega,U_0)=0. \tag{4.11}$$

For $U_0^2<U_{cr}^2$ the function $G(x_1,\omega,U_0)$ is

$$G(\overset{*}{x}_1,\omega,U_0)=[G_R(|\overset{*}{w}_1|,\omega,U_0)+iG_I(|\overset{*}{x}_1|,\omega,U_0)]\,H(\overset{*}{x}_1) \\ +G_R(|\overset{*}{x}_1|,\omega,U_0)-iG_I(|\overset{*}{x}_1|,\omega,U_0)]\,H(-\overset{*}{x}_1), \tag{4.12}$$

where the forms of functions G_R and G_I in the case of a beam are given in [7], and in the case of such a simple continuous systems as a string, Eq. (4.11) takes the form

$$\left[\frac{1}{2}a^3-\beta_1 f(-\Lambda,U_0)\right]\Lambda^2-[\varkappa_1 f(-\Lambda,U_0)-\beta_1 a^3]\Lambda+\varkappa_1 a^3=0 \quad \text{for } U_0>a. \tag{4.13}$$

In the case of a string the function $f(-\Lambda, U_0)$ is given by the formula

$$(4.14)\quad f(-\Lambda, U_0) = \frac{\exp\dfrac{-\Lambda}{U_0-a} - \exp\dfrac{-\Lambda}{U_0+a}}{(U_0^2-a^2)^{-1}2\gamma_0},$$

$$\Lambda = (\varepsilon+i\omega)L = \sigma+i\theta, \quad \varkappa = \alpha_0 L^2, \quad \alpha_0 = cm^{-1}, \quad \beta_1 = bm^{-1}L,$$

$$\gamma_0 = 4\varrho L m^{-1}.$$

Separating the real and imaginary parts of the characteristic equation (4.13) we obtain

$$(4.15)\quad \left[F_c\beta_1 + \frac{a^3}{2}\right]\sigma^2 + [F_c\varkappa_1 - \beta_1 a^3]\sigma - \left[F_c\beta_1 + \frac{a^3}{2}\right]\theta + [F_s\varkappa_1]\theta - [2F_s\beta_1]\sigma\theta - \varkappa_1 a^3 = 0,$$

$$(4.16)\quad -[F_s\beta_1]\sigma^2 - [F_s\varkappa_1]\sigma + [F_s\beta_1]\theta^2 + [F_c\varkappa_1 + \beta_1 a^3]\theta + 2\left[F_c\beta_1 + \frac{a^3}{2}\right]\sigma\theta = 0.$$

where the following abreviation are used

$$(4.17)\quad F_c = F_c(\sigma, \theta) = \frac{V^2-1}{2\gamma_0} e^{\frac{-\sigma}{V+1}}\left[\cos\frac{\Omega}{V+1} - e^{\frac{-2\sigma a}{V^2-1}}\cos\frac{\Omega}{V-1}\right],$$

$$F_s = F_s(\sigma, \theta) = i\frac{V^2-1}{2\gamma_0} e^{\frac{-\sigma}{V+1}}\left[\sin\frac{\Omega}{V+1} - e^{\frac{-2\sigma a}{V^2-1}}\sin\frac{\Omega}{V-1}\right],$$

$$\Omega = \theta a^{-1}, \quad V = U_0 a^{-1}.$$

Now, using Eqs. (4.15)–(4.17) and the instability condition, the boundaries of the regions where instability is possible can be plotted in the V, Ω-plane.

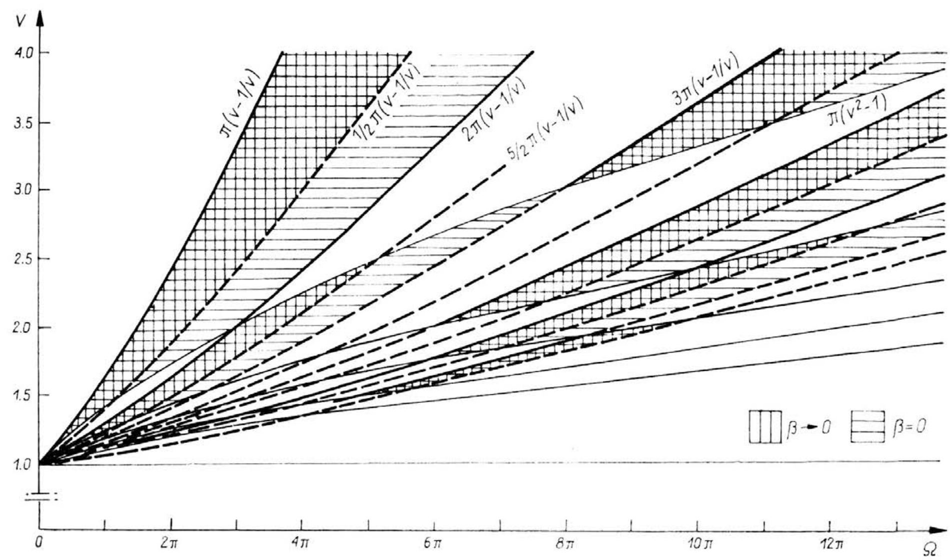

FIG. 6. Estimates of instability regions for the case of vanishing intensity of damping and the case without damping.

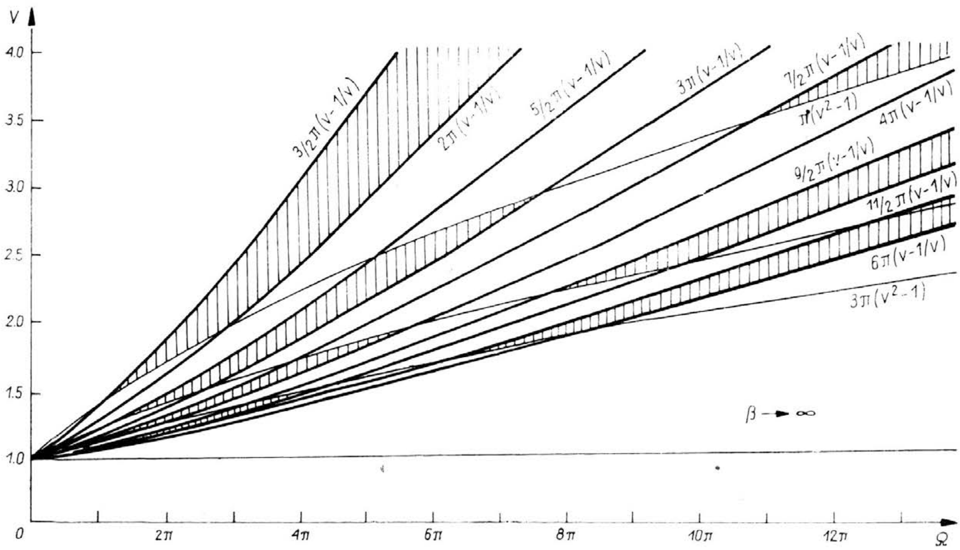

FIG. 7. Estimates of instability regions for the case of large damping.

The results calculated for various values of damping are shown in Fig. 6 and Fig. 7. It is interesting to note that, also in the case of two points of interaction, the regions of instability for damping equal zero [9] are different from those in the case of damping tending to zero (Fig. 6). If the value of the generalized damping parameter β_1 increases, the regions of instability are changed and for very large values of β_1 the configuration takes the form shown in Fig. 7.

The above estimates obtained from the characteristic equations (4.14), (4.16) and based on the stability condition (3.1) give only a qualitative information on the influence of damping in the hybrid systems in relative motion.

It is interesting to note that for the estimates shown in Fig. 6 and Fig. 7 the set of such regions is countable but infinite. The first critical velocity is equal to the velocity of elastic waves in the continuous subsystem.

The results of more accurate calculations show that the fist critical velocity depends also on the value of damping. Further results concerning the instability region, as well as a discussion of the influence of the parameters $\varkappa$ and γ on the instability regions of a damped system, will be given in a separate paper.

5. Concluding remarks

In the case of dynamic analysis of an infinite beam resting on an elastic foundation and subject to the action of a moving infinite beam or a moving lumped subsystem of finite length, the wave approach is appropriate and yield results of practical importance. The methods applied and the results obtained for comparatively simple models can be extended to more complex systems and, thus, provide additional insight into the problem of the dynamic stability of real train-track-systems.

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