Arch. Mech., 36, 3, pp. 299–306, Warszawa 1984

Thermal creep in a spherical shell subjected to an elevated temperature field

J. C. MISRA, S. B. KAR and S. C. SAMANTA (KHARAGPUR)

THIS PAPER presents an analysis of the strain, strain rate and stress fields developed in a spherical shell under the combined influence of an internal pressure and a steady temperature field. A non-linear visco-elastic model represented by means of the Norton's law of creep is taken for the purpose of finite strain analysis. The analysis may be useful in estimating the failure criteria of spherical pressure vessels and other mechanical devices in which spherical geometries are of use. In the realm of biomechanics too, the present investigation would be of much practical interest in providing useful information on strain and stress fields generated in the human skull when subjected to the simultaneous action of mechanical and thermal loads.

Przeprowadzono analizę pól odkształceń, naprężeń i prędkości odkształceń w powłoce sferycznej poddanej działaniu ciśnienia wewnętrznego i ustalonego pola temperatury. Przyjęto nieliniowy lepkosprężysty model ciała spełniający prawo pełzania Nortona i umożliwiający uwzględnienie odkształceń skończonych. Wyniki analizy mogą znaleźć zastosowanie przy ocenie warunków zniszczenia sferycznych zbiorników ciśnieniowych lub innych konstrukcji o podobnej geometrii. Badania te mogą również mieć zastosowanie w biomechanice, dostarczając informacji o odkształceniach i naprężeniach w czaszce ludzkiej poddanej równoczesnemu działaniu obciążeń mechanicznych i cieplnych.

Проведен анализ полей деформаций, напряжений и скоростей деформаций в сферической оболочке подвергнутой действию внутреннего давления и установившегося поля температуры. Принята нелинейная вязкоупругая модель тела, удовлетворяющая закону ползучести Нортона и дающая возможность учета конечных деформаций. Результаты анализа могут найти применение при оценке условий разрушения сферических напорных резервуаров или других конструкций аналогичной геометрии. Эти исследования могут тоже иметь применение в биомеханике, доставляя информаций о деформациях и напряжениях в человеческом черепе, подвергнутом одновременному действию механических и термических нагрузок.

1. Introduction

MOST MODERN heavy industries like steel plants, power generators, jet engines and nuclear reactors work at considerably high pressure and temperature. In order to prevent functional damage and failure, the limiting strain rates for such systems must be known earlier. This is why an analysis for the creep of a material subjected to elevated temperatures is required.

Creep of isotropic and homogeneous tubes under internal pressure was studied by RIMROTT [1]. Replacing strain increments by strain rates in Levy–Mises equations for orthotropic materials, BHATNAGAR and GUPTA [2] formulated the constitutive equations for orthotropic visco-elastic materials. With these equations and the analysis of Rimrott several problems on cylindrical shells were analysed by BHATNAGAR *et al.* [3]. Creep behaviour of cylindrical and spherical shells at elevated temperature was discussed by

MISRA and SAMANTA [4, 5]. It is quite likely that temperature variation may have a significant role in the creep behaviour of a shell. The present investigation is devoted to the analysis of creep problems of spherical shells having internal pressure under the assumption of the validity of Norton's law. The effect of temperature variation is taken into account. The material of the shell is taken to be mechanically incompressible so that the dilatation is zero. The temperature field has been considered to be steady. The magnitudes of the strain and the strain-rate are computed for various time intervals.

2. Method of solution

Let us consider a spherical shell with a and b as the internal and external radii, respectively, subjected to a uniform normal pressure p on the inner surface. On account of symmetry there will be three nonzero stress components viz., σ_r, the radial component σ_θ and σ_φ, the two tangential components such that $\sigma_\theta = \sigma_\varphi = \sigma_t$ (say). If e_r, e_θ and e_φ are the normal components of the strain tensor in the spherical polar coordinates (r, θ, φ) and u, the radial displacement, the equation of equilibrium in the radial direction is given by

$$\frac{d\sigma_r}{dr} + \frac{2}{r}\exp(e_r - e_\theta)(\sigma_r - \sigma_t) = 0, \tag{2.1}$$

where

$$e_r = \log\left(1 + \frac{du}{dr}\right), \quad e_\theta = \log\left(1 + \frac{u}{r}\right), \tag{2.2}$$

so that

$$r\frac{de_\theta}{dr} = \exp(e_r - e_\theta) - 1. \tag{2.3}$$

This may be regarded as the equation of compatibility. Assuming incompressibility of the material of the shell, the constitutive equations are suitably modified in order to take the generated temperature field into account. Taking the principal axes of stress to be coincident with the axes of symmetry (which are further taken to be the axes of coordinates), and taking $G = H$ and $\sigma_\theta = \sigma_\varphi = \sigma_t$ for the present study, the strain-stress temperature relations may be put in the form

$$e_r = \frac{e}{2\sigma}[2G(\sigma_r - \sigma_t)] + \alpha_1 T \quad \text{and} \quad e_\varphi = e_\theta = \frac{e}{2\sigma}[G(\sigma_t - \alpha_r)] + \sigma_2 T, \tag{2.4}$$

(α_1, α_2 and α_3 denote the coefficients of linear thermal expansion in the radial, circumferential and azimuthal directions, respectively).

Due to mechanical incompressiblity, we have

$$e_r + e_\theta + e_\varphi = (\alpha_1 + \alpha_2 + \alpha_3)T.$$

For $\alpha_2 \doteq \alpha_3$, the equation reduces to $e_r + 2e_\theta = (\alpha_1 + 2\alpha_2)T$. This equation, on using the relations (2.2), assumes the form

$$\left(1+\frac{du}{dr}\right)\left(1+\frac{2u}{r}+\frac{u^2}{r^2}\right)=e^{(\alpha_1+2\alpha_2)T}.$$

Neglecting the terms involving the products and higher powers of the displacement u as well as its derivatives and also considering that $(\alpha_1+2\alpha_2)T$ is so small that its higher powers can be neglected, the above equation reduces to

$$\frac{du}{dr}+\frac{2u}{r}=(\alpha_1+2\alpha_2)T.$$

The general solution of this equation may be expressed as

$$r^2u=(\alpha_1+2\alpha_2)\int Tr^2dr+k, \tag{2.5}$$

where k is an arbitrary constant.

Now, if the inner surface is maintained at a temperature T_1 and the outer one at T_2, the temperature distribution is obtained in the form

$$T=\frac{aT_1}{r}+\frac{(bT_2-aT_1)(r-a)}{(b-a)r}. \tag{2.6}$$

From Eqs. (2.5) and (2.6) one obtains

$$r^2u=(\alpha_1+2\alpha_2)\left\{\frac{aT_1r^2}{2}+\frac{(bT_2-aT_1)}{(b-a)}\left(\frac{r^3}{3}-\frac{ar^2}{2}\right)\right\}+K, \tag{2.7}$$

which gives the displacement at any point of the shell under consideration.

The significant stress σ is given by

$$\sigma=1/\sqrt{2}\,[F(\sigma_\theta-\sigma_\varphi)^2+G(\sigma_\theta-\sigma_r)^2+H(\sigma_r-\sigma_\varphi)^2]^{1/2}. \tag{2.8}$$

In the present consideration, since

$$G=H \quad \text{and} \quad \sigma_\theta=\sigma_\varphi=\sigma_t,$$

the above relation takes the simple form

$$\sigma=\sqrt{G}\,(\sigma_t-\sigma_r). \tag{2.9}$$

From Eq. (2.4) we now have

$$e_r=\alpha_1T-e\sqrt{G}, \tag{2.10}$$

and

$$e_\theta=\alpha_2T+\frac{e\sqrt{G}}{2}, \tag{2.11}$$

so that

$$e_r-e_\theta=T(\alpha_1-\alpha_2)-\frac{3}{2}e\sqrt{G}. \tag{2.12}$$

Making use of Eqs. (2.11) and (2.12), one obtains from Eq. (2.3)

$$r\frac{\partial}{\partial r}\left(\alpha_2 T+\frac{e\sqrt{G}}{2}\right) = \exp\left\{T(\alpha_1-\alpha_2)-\frac{3}{2}e\sqrt{G}\right\}-1.$$

Multiplying this equation throughout by r^2 and integrating, we have

$$e = \frac{2(\alpha_1-\alpha_2)}{\sqrt{G}r^3}\left[\frac{aT_1r^2}{2}+\frac{bT_2-aT_1}{b-a}\left\{\frac{r^3}{3}-\frac{ar^2}{2}\right\}\right] - \frac{2\alpha_2}{\sqrt{G}r^3}\left[r^3\int\frac{\partial T}{\partial r}dr-3\int Tr^2dr\right]+\frac{B}{r^3},$$

B being a constant of integration.

Setting $\alpha_1+2\alpha_2 = P$, $\frac{bT_2-aT_1}{b-a} = Q$ and using the boundary condition viz., $e = e_a$ on $r = a$, we get

(2.13) $$e = \frac{P}{G}\left[\frac{aT_1}{r}+\frac{Q(2r-3a)}{3r}\right]-\frac{2\alpha_2}{\sqrt{G}}\left[\frac{aT_1}{r}+Q\left(1-\frac{a}{r}\right)\right] + e_a\frac{a^3}{r^3}-\frac{Pa^3}{\sqrt{G}r^3}\left[T_1-\frac{Q}{3}\right]+\left(\frac{2\alpha_2T_1a^3}{\sqrt{G}r^3}\right).$$

Differentiating this with respect to time, one obtains

(2.14) $$\dot{e} = \dot{e}_a\frac{a^3}{r^3},$$

since the other quantities involved in Eq. (2.13) have been assumed to be independent of time. In Eq. (2.14) $\dot{e}_a$ is the effective strain rate at $r = a$.

Therefore

(2.15) $$\frac{\partial\dot{e}}{\partial r} = -\frac{3a^3\dot{e}_a}{r^4}.$$

By Norton's law of creep

(2.16) $$\dot{e} = A\sigma^n,$$

we can write

(2.17) $$\left(\frac{\dot{e}}{A}\right)^{\frac{1}{n}} = \sigma = \sqrt{G}(\sigma_t-\sigma_r).$$

Now Eq. (2.1), with the help of Eq. (2.17), yields

(2.18) $$\frac{d\sigma_r}{dr} = \frac{2}{r}\left(\frac{\dot{e}}{A}\right)^{\frac{1}{n}}\frac{1}{\sqrt{G}}\left[1+(\alpha_1-\alpha_2)T-\frac{3\sqrt{G}}{2e}\right].$$

The above equation has been written on the assumption that $(\alpha_1-\alpha_2)T-\frac{3\sqrt{G}}{2e}$ is small and thereby higher powers of this quantity can be neglected.

Also, using Eqs. (2.6), (2.13) and (2.16), we have

$$\frac{d\sigma_r}{dr} = \frac{2(\dot{e}_a)^{\frac{1}{n}} a^{\frac{3}{n}}}{A^{\frac{1}{n}}\sqrt{G}} \left[\frac{1}{r^{\frac{n+3}{n}}} + P\left\{ \frac{aT_1}{r^{\frac{2n+3}{n}}} + Q\left(\frac{1}{r^{\frac{n+3}{n}}} - \frac{a}{r^{\frac{2n+3}{n}}} \right)\right\} - \frac{3P}{2}\left\{ \frac{aT_1}{r^{\frac{2n+3}{n}}} + Q \right.\right.$$

$$\left.\left. \times\left(\frac{2}{3r^{\frac{n+3}{n}}} - \frac{a}{r^{\frac{2n+3}{n}}} \right)\right\} + \frac{1}{r^{\frac{4n+3}{n}}} \left\{ \frac{3Pa^3\left(T_1 - \frac{Q}{3}\right)}{2} - \frac{3Ga^3 e_a}{2} - 3\alpha_2 T_1 a^3 \right\}\right].$$

Integrating this equation and making use of the boundary conditions $\sigma_r = -p$ on $r = a$ and $\sigma_r = 0$ on $r = b$, one finds

$$p = (\dot{e}_a)^{\frac{1}{n}} X[Y + (Ze_a + M)N], \tag{2.19}$$

where

$$X = \frac{2a^{\frac{3}{n}}}{\sqrt{G}A^{\frac{1}{n}}},$$

$$Y = -n/3\left(\frac{1}{b^{\frac{3}{n}}} - \frac{1}{a^{\frac{3}{n}}} \right) - P\left\{ \frac{aT_1 n}{n+3}\left(\frac{1}{b^{\frac{n+3}{n}}} - \frac{1}{a^{\frac{n+3}{n}}} \right) - \frac{Qan}{n+3} \right.$$

$$\left. \times\left(\frac{1}{b^{\frac{n+3}{n}}} - \frac{1}{a^{\frac{n+3}{n}}} \right)\right\} + \frac{3PaT_1 n}{2(n+3)}\left(\frac{1}{b^{\frac{n+3}{n}}} - \frac{1}{a^{\frac{n+3}{n}}} \right) - \frac{3PQan}{2(n+3)}\left(\frac{1}{b^{\frac{n+3}{n}}} - \frac{1}{a^{\frac{n+3}{n}}} \right),$$

$$Z = \frac{3\sqrt{G}a^3}{2},$$

$$M = 3\alpha_2 T_1 a^3 - 3Pa^3(T_1 - Q/3),$$

$$N = \frac{n}{3(n+1)}\left(\frac{1}{b^{\frac{3(n+1)}{n}}} - \frac{1}{a^{\frac{3(n+1)}{n}}} \right).$$

Setting $Y + ZNe_a + MN = x$, we have

$$\dot{x} = ZN\dot{e}_a.$$

Equation (2.19) may now be rewritten in the form

$$\frac{X^n x^n}{ZNp^n} dx = dt$$

which, when integrated between the limits at $t = 0$

$$x = Y + MN,$$

and at $t = t$

$$x = x,$$

gives

$$\frac{X^n}{ZNp^n}\left\{ \frac{x^{n+1} - (Y+MN)^{n+1}}{n+1} \right\} = t.$$

Further setting $\frac{X^n}{ZN_p^n} = Q_1$ we have

$$e_a = \frac{\left[\frac{(n+1)t}{Q_1} + (Y+MN)^{n+1}\right]^{1/n+1} - (Y+MN)}{ZN}. \tag{2.20}$$

Also, from Eq. (2.19) one gets

$$\dot{e}_a = \frac{p^n}{\{(Y+MN)+ZNe_a\}^n X^n}. \tag{2.21}$$

Equation (2.20) can be used to calculate the strain e_a on the inner surface at any time t. Once e_a is known, we can calculate $\dot{e}_a$ from Eq. (2.21). e and $\dot{e}$ can be found from Eqs. (2.13) and (2.14). Using this we can calculate e_r and e_θ from Eqs. (2.10) and (2.11) so that the strain field at any given point can be completely specified. The radial stress σ_r may also be similarly obtained from Eq. (2.18) and the radial displacement from Eq. (2.7).

3. Numerical results and conclusions

Let us now try to illustrate the applicability of the above analysis by considering a specific example. For this purpose we take (cf. MISRA and SAMANTA [4]), $T_1 = 1200°K$, $T_2 = 300°K$,

$$a = 0.10 \text{ m}, \quad b = 0.50 \text{ m}, \quad p = 2.75\times 10^8 \frac{\text{N}}{\text{m}^2},$$

$$n = 6, \quad A = 4.647\times 10^{-53}\text{day}^{-1}\left(\frac{\text{N}}{\text{m}^2}\right)^{-6}, \quad F = 0.5, \quad G = H = .75,$$

$$\alpha_1 = 25\times 10^{-6} \text{ per } °\text{K}, \quad \alpha_2 = \alpha_3 = 26\times 10^{-6} \text{ per } °\text{K}.$$

The values of the strain and the strain rate on the inner surface of the shell obtained through numerical computation of the expressions (2.20) and (2.21) are presented in Table 1.

Table 1.

Time (t) in days	e_a(in units of 10^{-2} m/m)	$\dot{e}_a$ (in units of 10^{-4} per day)
0	0	0.786
20	0.15	0.788
40	0.31	0.791
60	0.47	0.793
80	0.63	0.796
100	0.79	0.799
120	0.95	0.801
140	1.11	0.804

A comparison with the computed values presented in [5], shows that the effect of temperature variation on the generated strain field in the shell is considerable.

In conclusion, it may be pointed out that although the analysis presented here could be further improved by considering the temperature dependence of the material parameters and that of the constant 'A' involved in Norton's law, the present study possesses the potential of providing a reasonably good estimate of the concerned values.

References

1. F. P. S. RIMROTT, *Creep of thick-walled tubes under internal pressure considering large strain*, J. Appl. Mech., **26**, 271, 1959.
2. N. S. BHATNAGAR, R. P. GUPTA, *On the consititutive equations of the orthotropic theory of creep*, J. Phys. Soc. Japan, **21**, 1003, 1966.
3. N. S. BHATNAGAR *et al.*, *Large strain creep analysis of thick-walled cylinders*, Int. J. Nonlinear Mech., **9**, 127–140, 1974.
4. J. C. MISRA, S. C. SAMANTA, *Finite creep in thick walled cylindrical shells at elevated temperatures*, Acta Mech. **41**, 149–155, 1981.
5. J. C. MISRA, S. C. SAMANTA, *Kriechverhalten der Kugel schalen*, Beton und Stahlbetonbau, **76**, 143–145 1981.

DEPARTMENT OF MATHEMATICS,
INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR, INDIA.

Received April 9, 1983.