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Propagation of weak nonlinear long waves (*)

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THE LONG time behaviour of solutions of the Cauchy problem (1.2) and (1.3) is studied. Problems of the considered type arise in the theory of weak, nonlinear long waves in various branches of mechanics. The aim of the paper is a generalization of the method presented in [6]. It is found that, in general, it is necessary to solve systems of nonlinear partial differential equations in order to obtain approximate solutions in the far field. However, in many applications it is enough to solve either a few independent Burgers equations or a system of linear equations. This is the case of flow of a magnetoactive gas subject to a transversal magnetic field. This problem is solved to show an application of the theory.

Badamy zachowanie się po długim czasie rozwiązań zagadnienia Cauchy'ego (1.2), (1.3). Problemy powyższego typu powstają w teorii słabych, nieliniowych długich fal w różnych działach mechaniki. Celem pracy jest uogólnienie metody podanej w [6]. Stwierdza się, że na ogół trzeba rozwiązywać układy nieliniowych równań różniczkowych cząstkowych po to, aby otrzymać przybliżone rozwiązania dla czasów odległych od chwili początkowej. Jednakże w wielu zastosowaniach wystarczy rozwiązać albo kilka niezależnych równań Burgersa albo układ równań liniowych. Jest tak w przypadku przepływu gazu magnetoaktywnego w poprzecznym polu magnetycznym. Problem ten jest rozwiązany dla ilustracji ogólnej teorii.

Исследуем поведение, после длинных отрезков времени, решений задачи Коши (1.2), (1.3). Задачи вышеупомянутого типа возникают в теории слабых, нелинейных длинных волн в разных областях механики. Целью работы является обобщение метода приведенного в [6]. Констатируется, что в общем надо решать системы нелинейных дифференциальных уравнений в частных производных для того, чтобы получить приближенные решения для времен отдаленных от начального момента. Однако в многих применениях достаточно решить или несколько независимых уравнений Бюргерса, или систему линейных уравнений. Так состоит дело в случае магнитоактивного газа в поперечном магнитном поле, которая то задача решена для иллюстрации общей теории.

1. Introduction

WE STUDY problems concerning the formation and evolution of a weakly nonlinear motion of a physical system. This motion is assumed to be a small disturbance of a uniform state. In this paper we limit ourselves to such phenomena which can be treated as unsteady, i.e. time-dependent and spatially one-dimensional. Let t denote the nondimensional time ($t \geqslant 0$) and let x be the dimensionless space coordinate ($-\infty < x < \infty$).

Let

$$u = u(x, t) = \big(u_1(x, t), \ldots, u_n(x, t)\big)$$

represent the disturbance of an initially quiescent system. In many cases of interest it is a solution of the initial value problem

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$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + \varepsilon B(u)\frac{\partial n}{\partial x} = \varepsilon C\frac{\partial^2 u}{\partial x^2} + \varepsilon^2 V\left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \varepsilon\right), \tag{1.1}$$

$$u(x, 0) = u^{(0)}(x), \quad t \geqslant 0, \quad -\infty < x < \infty, \tag{1.2}$$

where A and C are constant $n \times n$ matrices, $B(u)$ is matrix linear in u i.e.

$$B(u) = \sum_{h=1}^{n} B_h u_h,$$

where B_n are constant matrices of the size $n \times n$.

Next, $\varepsilon > 0$ is a small parameter, $u^{(0)}(x)$ is a given vector field of initial data, and $V(x, y, z, \varepsilon)$ is a continuous function of its arguments.

We want to find an approximate solution $v(x, t, \varepsilon)$ of Eqs. (1.1) and (1.2) such that for

$$||u(x, t, \varepsilon) - v(x, t, \varepsilon)|| \leqslant K\varepsilon$$

for

$$0 \leqslant t \leqslant \frac{1}{\varepsilon}, \quad -\infty < x < \infty, \quad 0 < \varepsilon < \varepsilon_0.$$

Here $K > 0$, $\varepsilon_0 > 0$ are some positive constants. As a rule in such a situation, the term $\varepsilon^2 V\left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \varepsilon\right)$ in the right hand side of Eq. (1.1) is neglected. However, all terms which is multiplied by ε must be kept (cf. [1], [2]), hence we have to solve the following systems of equations:

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + \varepsilon B(u)\frac{\partial u}{\partial x} = \varepsilon C\frac{\partial^2 u}{\partial x^2}, \tag{1.3}$$

subject to the initial data (1.2).

However, although Eq. (1.3) is much simpler that the original Eq. (1.1), still it remains nonlinear and it is hopeless to solve it explicitly. Therefore a perturbation technique must be used. As it is well known ([1, 2]), any regular perturbation method is out of use, therefore a more sophisticated argument must be applied. Usually either a technique of the group called methods of strained coordinates or that of multiple scales is used ([1, 2]). However, there are problems which cannot be solved by any of those methods when applied separately. An example is provided by the problem of reflexion of a weak shock wave from a plane wall ([3], [4]). The first who solved it were M. B. LESSER and R. SEEBASS [3]. In order to determine the approximation in the far field, they divided it into suitable subdomains, introduced a "slow" time variable

$$\tau = \varepsilon t \tag{1.4}$$

and used repeatedly the matching principle.

Thus, in a sense, their approach was a combination of the matched asymptotic expansion and multiple scale expansions (each type of those expansions is presented in [1, 2]). The use of such a sophisticated technique gave facilities for determining the correct incident and reflected shock structure as well as their trajectories.

Later, an alternative approach was presented by the present author [4] who used the slow time variable τ and strained both x and t; no division of the far field was used.

This technique was a combination of the strained coordinates method and two time expansions.

Next, the same technique was applied to determine uniformly valid approximation for the problem of regular reflexion of a weak shock wave from an inclined plane wall [5].

Recently, this method was extended by the same author to problems whose solutions are composed of more than two modes and the theoretical results were applied to the shock tube problem [6], exhibiting good a agreement with the other authors' results.

The aim of this paper is to weaken some of the assumptions under which our technique can work, also we reformulate the main idea what makes the calculations less tedious. Finally, the general scheme is applied to the problem of propagation of weak unsteady one-dimensional disturbances in a magnetoactive gas subject to a transversal magnetic field.

2. The general scheme

It will be easier to formulate assumptions and results of the general considerations by giving an abstract interpretation of the initial value problem (1.4) and (1.5).

We follow the general notation and terminology of the monograph by T. KATO [7], where further information can be found as well.

Let X be a finite-dimensional, normed linear space, let $n = \dim X$ and let $\|\cdot\|$ be the norm. Let A be a linear operator from X into itself. We assume that it is reducible, i.e. there linear subspaces $M_1, \ldots, M_m$ such that X can be represented as the direct sum

$$X = M_1 \oplus M_2 \oplus \ldots \oplus M_m,$$

and each to them is an invariant subspace of A, i.e.

$$AM_i \subset M_i, \quad i = 1, 2, \ldots, m.$$

Let $P_h (h = 1, 2, \ldots, m)$ be the projector from X into M_h. We assume that there are real numbers $\alpha_1, \ldots, \alpha_m$ such that

$$A = \sum_{h=1}^{m} \alpha_h P_h. \tag{2.1}$$

The projectors are given by the formula [7]

$$P_n = -\frac{1}{2\pi i} \int_{\Gamma_h} (A - \zeta I)^{-1} d\zeta, \tag{2.2}$$

where I is the identity operator, and the contour Γ_h encircles only one point $\zeta = \alpha_h$. The numbers $\alpha_1, \ldots, \alpha_m$ are eigenvalues of A and they are solutions of the algebraic equation

$$\det(A - \zeta I) = 0. \tag{2.3}$$

Next, let $B(u, v)$ denote a bilinear operator from $X^2 = X \times X$ into X. We assume that there are real linear functionals $\beta_1(u), \ldots, \beta_m(u)$ from X into the set of real numbers R such that for every fixed $u \in X$ and arbitrary $v \in X$

$$P_h B(u, P_h, v) = \beta_h(u) P_h v, \quad h = 1, \ldots, m. \tag{2.4}$$

Finally, let C be another linear operator from X into itself. We assume that for every $h = 1, 2, \ldots, m$ the eigenvalues of the operators

$$C_{hh} = P_h C P_h \tag{2.5}$$

considered as linear operators from M_h into M_h, have a positive real part.

Let $u^{(0)}(x)$ be a twice continuously differentiable function with values from X. The aim of this chapter is to construct an approximation to the solution of the following abstract initial value problem:

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + \varepsilon B\left(u, \frac{\partial u}{\partial x}\right) = \varepsilon C\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t \geqslant 0 \tag{2.6}$$

subject to the initial condition

$$u(x, 0) = u^{(0)}(x), \tag{2.7}$$

where the operators A, $B(\cdot, \cdot)$ and C have the properties formulated above, and ε is a small positive parameter.

Introducing the notation

$$P_h u = u_h,$$

we can replace Eq. (2.6) by a system of m coupled equations

$$\frac{\partial}{\partial t} u_h + \alpha_h \frac{\partial u_h}{\partial x} + \varepsilon \sum_{j=1}^{m} \sum_{k=1}^{m} P_h B\left(u_j, \frac{\partial u_k}{\partial x}\right) = \varepsilon \sum_{j=1}^{m} P_h C \frac{\partial^2 u_j}{\partial x^2}, \tag{2.8}$$

$$u_h(x, 0) = u_h^{(0)}(x) = P_h u^{(0)}(x), \quad h = 1, 2, \ldots, m. \tag{2.9}$$

Now the problem (2.8) and (2.9) is formally similar to that considered in [6].

We assume that solution for Eqs. (2.8) and (2.9) can be written in the form

$$u_h(x, t, \varepsilon) = v_h(\xi_k, \tau) + \varepsilon\omega_h(\xi_1 . \ldots, \xi_m, \tau) + \ldots, \quad v_h, \omega_h \in M_h, \tag{2.10}$$

$$x - \alpha_h t = \xi_h + \varepsilon\varphi_h(\xi_1, \ldots, \xi_m, \tau) + \ldots, \tag{2.11}$$

$$\tau = \varepsilon t, \quad h = 1, 2, \ldots, m. \tag{2.12}$$

Substituting the expansions (2.10)–(2.12) into Eqs. (2.9) and equating the terms multiplied by the same powers of ε, we see that v_h can be arbitrary and therefore we pass to the approximation of order ε to Eqs. (2.8).

This is a linear partial differential equation which can have a bounded solution provided that [6]

(2.13)

i) $$\sup_{\substack{-\infty < \xi_h < \infty \\ \tau \geqslant 0}} ||v_h(\xi_h, \tau)|| < \infty, \quad h = 1, 2, \ldots, m,$$

ii) $$\sup_{\substack{-\infty < \xi_h < \infty \\ \tau \geqslant 0}} \left\| \frac{\partial v_h(\xi_h, \tau)}{\partial \xi_h} \right\| < \infty, \quad h = 1, 2, \ldots, m,$$

iii) $$\sup_{\substack{\tau \geqslant 0 \\ -\infty < x < \infty \\ -\infty < y < \infty}} \int_x^y \left\| \frac{\partial}{\partial \xi_h} v_h(\xi_h, \tau) \right\| d\xi_h < \infty, \quad h = 1, 2, \ldots, m,$$

iv) the functions $\varphi_h(\xi_1, \ldots, \xi_m, \tau)$ are given by

$$\varphi_h(\xi_1, \ldots, \xi_m, \tau) = \sum_{\substack{j=1 \\ j\neq h}}^{m} \frac{1}{\alpha_h - \alpha_j} \int_{\xi_h}^{\xi_j} \beta_h\big((v(\sigma, \tau)\big) d\sigma, \tag{2.14}$$

v) the vector fields $r_1(\xi_1, \tau), \ldots, r_h(\xi_h, \tau)$ are solutions of the following initial value problems: find the vector fields $r_h(\xi_h, \tau) \in M_h$ such that

$$\frac{\partial v_h(\xi_h, \tau)}{\partial \tau} + \beta_h\big(v_h(\xi_h, \tau)\big) \frac{\partial}{\partial \xi_h} v_h(\xi_h, \tau) = C_{hh} \frac{\partial^2 v_h(\xi_h, \tau)}{\partial \xi_h^2}, \tag{2.15}$$

$$v_h(\xi_h, 0) = u_h^{(0)}(x)|_{x=\xi_h} \in M_h, \quad h = 1, 2, \ldots, m. \tag{2.16}$$

All calculations are omitted here because they can be carried out exactly in the same manner which is given in full extense in [6].

We must point out an essential difference between Eqs. (2.15) and very similar in form equations obtained in [6]. If for some $h = h_0$ $\dim M_{h_0} = 1$, then Eq. (2.15) for $h = h_0$ is in fact the Burgers equation or the diffusion equation. However, if $\dim M_{h_0} \geqslant 2$, then Eqs. (2.15) for $h = h_0$ form in general a system of $\dim M_{h_0}$ nonlinear partial differential equations. Thus, in the case of multiple eigenvalues of the operator A, it is much more difficult to determine the asymptotics of the initial value problem (2.6) and (2.7).

Let us note that if we assume additionally that

vi) $$\sup_{\substack{-\infty < x < \infty \\ -\infty < y < \infty \\ \tau \geqslant 0}} \left\| \int_x^y v_h(\xi_h, \tau) d\xi_h \right\| < \infty,$$

then it is not necessary to strain the variable, i.e. we can assume that

$$\xi_h = x - \alpha_h t.$$

Indeed,

$$v_h(\xi_h, \tau) = v_h(x - \alpha_h t + \varepsilon\varphi + \ldots, \tau) = v_h(x - \alpha_h t, \tau) + \varepsilon \frac{\partial v_h(x - \alpha_h t, \tau)}{\partial \xi_h} \varphi_h + O(\varepsilon^2)$$

using here the expression (2.14) for φ_h, we see that if ii) and vi) are satisfied, then $\frac{\partial}{\partial \xi_h} r_h(x - \alpha_h t, \tau)\varphi_h = O(1)$ and therefore we can write

$$u_h(x, t, \tau) = v_h(x - \alpha_h t, \tau) + O(\varepsilon)$$

instead of Eq. (2.10).

Hence, under Assumptions i), ii) iii) and v), vi) it is enough to apply the multiple scale method in order to get a uniformly valid approximation. This remark is in accordance with the discussion given in [2], Chapter 5.1.

However, if vi) is not satisfied, the straining of coordinates along with the multiple scales must be applied.

Assuming that v_h depends on more variables, say some $\xi_h, \eta_h, \ldots, \zeta_h$ and τ, it is possible to weaken the assumption (2.4), however, it remains an open question how to construct an approximation to the initial value problem (2.6) and (2.7) uniformly valid up to the time of order ε^{-1} in the case of an arbitrary bilinear operator and multiple eigenvalues of the operator A. Let us notice, however, that in the extreme case when

$$A = \alpha I, \qquad \alpha \in R$$

with the help of the transformation of independent variables

$$\xi = x - \alpha t, \qquad \tau = \varepsilon t$$

and substitution

$$u(x, t, \varepsilon) = v(\xi, \tau),$$

we reduce the problem (2.6) and (2.7) to the equivalent form

$$\frac{\partial v}{\partial \tau} + B\left(v, \frac{\partial v}{\partial \xi}\right) = C\frac{\partial^2 v}{\partial \xi^2}, \tag{2.17}$$

$$v(\xi, 0) = u^{(0)}(\xi) \in X. \tag{2.18}$$

Here the small parameter ε is not present and no further simplification can be attained. Note also that it is not necessary to impose any assumptions for the bilinear operator B, (the eigenvalues of C must have a positive real part).

3. Application to magnetogasdynamics

Equations of flow of a viscous and heat conducting fluid susceptible to electromagnetic forces consist of the usual equations of conservation of mass, momentum and energy with the magnetic force $\mathbf{J} \times \mathbf{B}$ ($\mathbf{J}$ is the current density, $\mathbf{B}$ is the magnetic induction) induced in the momentum equation and the Joule heat $\sigma^{-1}\mathbf{J}^2$ (σ is the electrical conductivity) introduced into the energy equation; thus the system is [8]

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{V}) = 0, \tag{3.1}$$

$$\varrho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V}\right)\mathbf{V} + \nabla p = \mathbf{J} \times \mathbf{B} + \nabla \cdot \mathbf{P}, \tag{3.2}$$

$$\varrho\left(\frac{\partial e}{\partial t} + \mathbf{V} \cdot \nabla e\right) + p\nabla \cdot \mathbf{V} = \sigma^{-1}\mathbf{J}^2 + \mathbf{P}:\nabla \mathbf{V} - \nabla \boldsymbol{\theta}, \tag{3.3}$$

where ϱ is the density, $\mathbf{V}$ is the fluid velocity, $\mathbf{P}$ is the stress tensor, e is the internal energy, $\boldsymbol{\theta}$ is the heat flux.

To these Maxwell's equations and Ohm's law are added

$$\frac{\partial B}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{3.4}$$

$$\frac{1}{\mu}\nabla \times \mathbf{B} = \mathbf{J}, \tag{3.5}$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}), \tag{3.6}$$

where μ is the magnetic permeability, $\mathbf{E}$ is the electric field.

In order to close the system of equations (3.1)–(3.6), we add

$$\mathbf{P} = -\frac{2}{3}\mu_1 \nabla \cdot \mathbf{V}\mathbf{U} + \mu_1[\nabla\mathbf{V} + (\nabla\mathbf{V})^*], \tag{3.7}$$

$$\boldsymbol{\theta} = -\lambda \nabla T \tag{3.8}$$

and limit ourselves to the ideal, polytropic gas

$$p = R\varrho T, \tag{3.9}$$

$$e = c_v T, \tag{3.10}$$

$$R = c_p - c_v. \tag{3.11}$$

where T is the temperature, μ_1 is the viscosity, $\mathbf{U}$ is the unit matrix, λ is the coefficient of heat conductivity, c_p and c_v are specific heat constants, finally $\mathbf{A}$ is a quadratic matrix and $\mathbf{A}^*$ denotes its transposition.

In this paper we bound ourselves to one-dimensional motion in the x direction with speed $v(x, t)$. Then only a transverse magnetic field is possible, and therefore we may put

$$\mathbf{V} = (v, 0, 0), \quad \mathbf{B} = (0, 0, B), \quad \mathbf{E} = (0, E, 0), \quad \mathbf{J} = (0, E - vB, 0)$$

with all quantities being functions of x and t only. Under these conditions we obtain from Eqs. (3.1)–(3.11).

$$\frac{\partial \varrho}{\partial t} + v\frac{\partial \varrho}{\partial x} + \varrho\frac{\partial v}{\partial x} = 0, \tag{3.12}$$

$$\varrho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) + R\frac{\partial(\varrho T)}{\partial x} = \sigma B(E - vB) + \frac{4}{3}\frac{\partial}{\partial x}\left(\mu_1\frac{\partial v}{\partial x}\right), \tag{3.13}$$

$$\varrho C_v\left(\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x}\right) + p\frac{\partial v}{\partial x} = \sigma(E - vB)^2\frac{4}{3}\mu_1\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right), \tag{3.14}$$

$$\frac{\partial B}{\partial t} + \frac{\partial E}{\partial x} = 0, \tag{3.15}$$

$$\frac{1}{\mu}\frac{\partial B}{\partial x} + \sigma(E - vB) = 0. \tag{3.16}$$

We shall consider small disturbances of a uniform flow which is described by the constant density ϱ_0, temperature T_0, magnetic induction B_0, vanishing velocity $v_0 = 0$, and electric field $E_0 = 0$. Thus we take

$$\varrho = \varrho_0(1 + \varepsilon\bar{\varrho}), \tag{3.17}$$

$$T = T_0(1 + \varepsilon\bar{T}), \tag{3.18}$$

$$B = B_0(1 + \varepsilon\bar{B}), \tag{3.19}$$

$$v = a_0\varepsilon\bar{v}, \tag{3.20}$$

where

$$a_0 = \sqrt{\gamma R T_0} \tag{3.21}$$

is the speed of sound of the basic flow, γ is the ratio of specific heats.

From Eq. (3.16) we get

$$E = \varepsilon a_0 B_0 \left[\bar{v} - \frac{1}{a_0 \mu\sigma}\frac{\partial \bar{B}}{\partial x} + \varepsilon \bar{v}\bar{B}\right]. \tag{3.22}$$

Additionally, we assume that the small disturbances vary slowly, therefore we may assume that the viscosity coefficient μ_1, the coefficient of heat conductivity λ, the electrical conductivity σ and the magnetic permeability μ are constant. The nondimensional space variable $\bar{x}$ and nondimensional time $\bar{t}$ are defined by

$$x = \frac{\bar{x}}{a_0 \varepsilon\mu\sigma}, \tag{3.23}$$

$$t = \frac{\bar{t}}{a_0^2 \varepsilon\mu\sigma}. \tag{3.24}$$

The new unknown functions $\bar{\varrho}$, $\bar{T}$, $\bar{B}$, $\bar{v}$ satisfy the following system of equations (the bar over the dimensionless quantities is omitted):

$$\frac{\partial\varrho}{\partial t} + \frac{\partial v}{\partial x} + \varepsilon\frac{\partial}{\partial x}(\varrho v) = 0, \tag{3.25}$$

$$\frac{\partial v}{\partial t} + \frac{1}{\gamma}\frac{\partial\varrho}{\partial x} + \frac{1}{\gamma}\frac{\partial T}{\partial x} + \frac{1}{M_A^2}\frac{\partial B}{\partial x} + \varepsilon\left[v\frac{\partial v}{\partial x} + \frac{1}{\gamma}(T-\varrho)\frac{\partial\varrho}{\partial x} + \frac{1}{M_A^2}(B-\varrho)\frac{\partial B}{\partial x} - \frac{4}{3}P_m\frac{\partial^2 v}{\partial x^2}\right] = O(\varepsilon^2), \tag{3.26}$$

$$\frac{\partial T}{\partial t} + (\gamma-1)\frac{\partial v}{\partial x} + \varepsilon\left[v\frac{\partial T}{\partial x} + (\gamma-1)T\frac{\partial v}{\partial x} - \frac{\gamma}{\Pr}P_m\frac{\partial^2 T}{\partial x^2}\right] = O(\varepsilon^2), \tag{3.27}$$

$$\frac{\partial B}{\partial t} + \frac{\partial v}{\partial x} + \varepsilon\frac{\partial}{\partial x}(vB) = \varepsilon\frac{\partial^2 B}{\partial x^2}, \tag{3.28}$$

where M_A is the Alfvén number

$$M_A = a_0\sqrt{\frac{\mu\varrho_0}{B_0}}, \tag{3.29}$$

Pr is the Prandtl number

$$\Pr = \frac{c_p\mu_1}{\lambda}, \tag{3.30}$$

and P_m is the magnetic Prandtl number

$$P_m = \frac{\mu\mu_1\sigma}{\varrho_0}. \tag{3.31}$$

We assume that

$$\frac{1}{M_A^2} = O(1),$$

$$\frac{1}{\Pr} = O(1),$$

and

$$P_m = O(1).$$

We close Eqs. (3.25)–(3.28) with the initial conditions

(3.31) $$\varrho(x, 0) = \varrho^{(0)}(x),$$

(3.32) $$T(x, 0) = T^{(0)}(x),$$

(3.33) $$v(x, 0) = v^{(0)}(x),$$

(3.34) $$B(x, 0) = B^{(0)}(x).$$

In order to have an approximate solution to the initial value problem (3.25)–(3.28) and (3.31)–(3.34), it is enough to use the general theory developed in the previous Chapter.

As the linear space X we take the set R_4 of all ordered quadruples $u = (\varrho, T, v, B)$, where ϱ, T, v, B are real, with the usual operations in the Euclidean space. The linear operator A from R_4 into itself is given by the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \gamma-1 & 0 \\ \dfrac{1}{\gamma} & \dfrac{1}{\gamma} & 0 & \dfrac{1}{M_A^2} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(We denote linear operators and their matrices with the same symbols. It will not lead to confusion because we will not make any changes of basis in R_4).

The eigenvalues of A are

$$\alpha_1 = -\sqrt{1+\frac{1}{M_A^2}}, \quad \alpha_2 = 0, \quad \alpha_3 = \sqrt{1+\frac{1}{M_A^2}},$$

where α_1 and α_3 are single, and α_2 is double.

There are three invariant subspaces of A:

M_1 consists of all ordered quadruples of the form

$$a\left(1, \gamma-1, -\sqrt{1+\frac{1}{M_A^2}}, 1\right), \quad a \in R_1, \quad \dim M_1 = 1;$$

M_2 consists of all ordered quadruples $(a+b,\ (\gamma-1)a-b,\ 0,\ -M_A^2 a)$, $a, b \in R_1$, therefore

$$\dim M_2 = 2;$$

and, finally M_3 is a one-dimensional linear subspace given by

$$a\left(1, \gamma-1, +\sqrt{1+\frac{1}{M_A^2}}, 1\right), \quad a \in R_1$$

It is clear that

$$R_4 = M_1 \oplus M_2 \oplus M_3.$$

The corresponding projectors P_h: $R_4 \to M_h$ are given by the following matrices:

$$P_1 = \begin{pmatrix} \frac{M_A^2}{2\gamma(1+M_A^2)}, & \frac{M_A^2}{2\gamma(1+M_A^2)}, & -\frac{M_A}{2\sqrt{1+M_A^2}}, & \frac{1}{2(1+M_A^2)} \\ \frac{M_A^2(\gamma-1)}{2\gamma(1+M_A^2)}, & \frac{M_A^2(\gamma-1)}{2\gamma(1+M_A^2)}, & -\frac{M_A(\gamma-1)}{2\sqrt{1+M_A^2}}, & \frac{\gamma-1}{2(1+M_A^2)} \\ -\frac{M_A}{2\gamma\sqrt{1+M_A^2}}, & -\frac{M_A}{2\gamma\sqrt{1+M_A^2}}, & \frac{1}{2}, & -\frac{1}{2M_A\sqrt{1+M_A^2}} \\ \frac{M_A^2}{2\gamma(1+M_A^2)}, & \frac{M_A^2}{2\gamma(1+M_A^2)}, & -\frac{M_A}{2\sqrt{1+M_A^2}}, & \frac{1}{2(1+M_A^2)} \end{pmatrix}.$$

We have

$$u_1 \equiv P_1 u = r\left(1, \gamma-1, -\sqrt{1+\frac{1}{M_A^2}}, 1\right), \tag{3.35}$$

where

$$r = \frac{M_A^2(\varrho+T)-\gamma M_A\sqrt{1+M_A^2}\,v+\gamma B}{2\gamma(1+M_A^2)}, \tag{3.36}$$

$$P_2 = \begin{pmatrix} \frac{M_A^2(\gamma-1)+\gamma}{\gamma(1+M_A^2)}, & -\frac{M_A^2}{\gamma(1+M_A^2)}, & 0, & -\frac{1}{1+M_A^2} \\ -\frac{M_A^2(\gamma-1)}{\gamma(1+M_A^2)}, & \frac{M_A^2+\gamma}{\gamma(1+M_A^2)}, & 0, & -\frac{\gamma-1}{1+M_A^2} \\ 0, & 0, & 0, & 0 \\ -\frac{M_A^2}{\gamma(1+M_A^2)}, & -\frac{M_A^2}{\gamma(1+M_A^2)}, & 0, & \frac{M_A^2}{1+M_A^2} \end{pmatrix}$$

and

$$u_2 \equiv P_2 u = c_1(1, \gamma-1, 0, -M_A^2)+c_2(1, -1, 0, 0), \tag{3.37}$$

where

$$c_1 = \frac{\varrho+T-\gamma B}{\gamma(1+M_A^2)}, \tag{3.38}$$

$$c_2 = \frac{(\gamma-1)\varrho-T}{\gamma}. \tag{3.39}$$

Finally

$$P_3 = \begin{pmatrix} \frac{M_A^2}{2\gamma(1+M_A^2)}, & \frac{M_A^2}{2\gamma(1+M_A^2)}, & \frac{M_A}{2\sqrt{1+M_A^2}}, & \frac{1}{2(1+M_A^2)} \\ \frac{M_A^2(\gamma-1)}{2\gamma(1+M_A^2)}, & \frac{M_A^2(\gamma-1)}{2\gamma(1+M_A^2)}, & \frac{M_A(\gamma-1)}{2\sqrt{1+M_A^2}}, & \frac{\gamma-1}{2(1+M_A^2)} \\ \frac{M_A}{2\gamma\sqrt{1+M_A^2}}, & \frac{M_A}{2\gamma\sqrt{1+M_A^2}}, & \frac{1}{2}, & \frac{1}{2M_A\sqrt{1+M_A^2}} \\ \frac{M_A^2}{2\gamma(1+M_A^2)}, & \frac{M_A^2}{2\gamma(1+M_A^2)}, & \frac{M_A}{2\sqrt{1+M_A^2}}, & \frac{1}{2(1+M_A^2)} \end{pmatrix}$$

and

(3.41) $$u_3 \equiv P_3 u = s\left(1, \gamma-1, \sqrt{1+\frac{1}{M_A^2}}, 1\right),$$

where

(3.42) $$s = \frac{M_A^2(\varrho+T)+\gamma M_A\sqrt{1+M_A^2}\,v+\gamma B}{2\gamma(1+M_A^2)}.$$

It is easy to check that

$$A = -\sqrt{1+\frac{1}{M_A^2}}\,P_1+0\cdot P_2+\sqrt{1+\frac{1}{M_A^2}}\,P_3$$

and, additionally,

(3.43) $$\varrho = r+c_1+c_2+s,$$

(3.44) $$T = (\gamma-1)(r+c_1+s)-c_2,$$

(3.45) $$v = \sqrt{1+\frac{1}{M_A^2}}\,(s-r),$$

(3.46) $$B = r-M_A^2 c_1+s.$$

Thus all conditions imposed upon the operator A in the previous chapter are satisfied in the case under consideration, hence we can pass to a study of the bilinear operator B. For any fixed $u \in R_4$ the operator $B(u,\ \cdot): R_4 \to R_4$ is given by the following matrix:

$$B(u) = \begin{pmatrix} v & 0 & \varrho & 0 \\ 0 & v & (\gamma-1)T & 0 \\ \dfrac{T-\varrho}{\gamma} & 0 & v & \dfrac{B-\varrho}{M_A^2} \\ 0 & 0 & B & v \end{pmatrix}.$$

It is a problem of simple calculations to check that

$$P_1 B(u) P_1 = \beta_1(u) P_1,$$

where

(3.47) $$\beta_1(u) = \frac{\varrho-M_A^2 T+2\sqrt{1+M_A^2}\,M_A v-2B}{2M_A\sqrt{1+M_A^2}};$$

next

$$P_2 B(u) P_2 = \beta_2(u) P_2,$$

where

(3.48) $$\beta_2(u) = v$$

and finally

$$P_3 B(u) P_3 = -\beta_1(u) P_3.$$

Thus we have checked that the assumptions (2.4) are satisfied as well.

The following formulae will be used later:

$$\beta_1(u_1) = -\frac{M_A^2(\gamma+1)+3}{2M_A\sqrt{1+M_A^2}}\,r, \tag{3.49}$$

$$\beta_1(u_2) = \frac{1-(\gamma-3)M_A^2}{2M_A^3\sqrt{1+M_A^2}}\,c_1 + \frac{M_A}{2\sqrt{1+M_A^2}}\,c_2, \tag{3.50}$$

$$\beta_1(u_3) = \frac{1-(\gamma-3)M_A^2}{2M_A\sqrt{1+M_A^2}}\,s, \tag{3.51}$$

$$\beta_2(u_1) = -\sqrt{1+\frac{1}{M_A^2}}\,r, \tag{3.52}$$

$$\beta_2(u_2) = 0, \tag{3.53}$$

$$\beta_2(u_3) = \sqrt{1+\frac{1}{M_A^2}}\,s, \tag{3.54}$$

where r, c_1, c_2 and s are defined by Eqs. (3.36), (3.38), (3.39) and (3.42), respectively.

The final assumptions that must be checked concern the operator C which in the present case is given by the matrix (see Eqs. (3.25)–(3.28))

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{\gamma P_m}{\mathrm{Pr}} & 0 & 0 \\ 0 & 0 & \dfrac{4}{3}P_m & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is a matter of simple calculations to show that

$$C_{11} = P_1 C P_1 = \omega P_1 \tag{3.55}$$

where

$$\omega = \frac{1}{2}\left[\left(\frac{4M_A^2(\gamma-1)}{3\gamma(1+M_A^2)} + \frac{\gamma}{\mathrm{Pr}}\right)P_m + \frac{1}{1+M_A^2}\right] = \mathrm{const}. \tag{3.56}$$

Therefore the operator P_1CP_1 considered as an operator from M_1 into M_1 has only one eigenvalue which is positive.

Next we have

$$P_2CP_2u = \eta(1, \gamma-1, 0, -M_A^2) + \xi(1, -1, 0, 0), \tag{3.58}$$

where

$$\eta = \left(\frac{4}{3}\,\frac{P_m(\gamma-1)}{\gamma(1+M_A^2)} + \frac{M_A^2}{1+M_A^2}\right)c_1 - \frac{4}{3}\,\frac{P_m}{\gamma(1+M_A^2)}\,c_2, \tag{3.59}$$

$$\xi = -\frac{4}{3}\,\frac{P_m(\gamma-1)}{\gamma}\,c_1 + \frac{4}{3}\,\frac{P_m}{\gamma}\,c_2. \tag{3.60}$$

The eigenvalues of the operator P_2CP_2 considered as an operator from M_2 into itself, are those of the matrix

(3.61) $$D = \begin{pmatrix} \frac{4}{3}\frac{P_m(\gamma-1)}{\gamma(1+M_A^2)}+\frac{M_A^2}{1+M_A^2}, & -\frac{4}{3}\frac{P_m}{\gamma(1+M_A^2)} \\ -\frac{4}{3}\frac{P_m(\gamma-1)}{\gamma}, & \frac{4}{3}\frac{P_m}{\gamma} \end{pmatrix}.$$

It can be readily proved that they are positive provided that $P_m > 0$.
Finally

(3.62) $$P_3CP_3 = \omega P_3,$$

what proves that the last of the imposed conditions is satisfied.

Thus we can use the results of the previous chapter, that is we must solve the following equations:

(3.63) $$\begin{cases} \frac{\partial r}{\partial \tau} - \frac{M_A^2(\gamma+1)+3}{2M_A\sqrt{1+M_A^2}} r \frac{\partial r}{\partial \xi_1} = \omega \frac{\partial^2 r_1}{\partial \xi_1^2}, \\ r_1(\xi_1, 0) = r_1^{(0)}(\xi_1), \end{cases}$$

(3.64) $$\begin{cases} \frac{\partial c_1}{\partial \tau} = \left(\frac{4}{3}\frac{P_m(\gamma-1)}{\gamma(1+M_A^2)}+\frac{M_A^2}{1+M_A^2}\right)\frac{\partial^2 c_1}{\partial \xi_2^2} - \frac{4}{3}\frac{P_m}{\gamma(1+M_A^2)}\frac{\partial^2 c_2}{\partial \xi_2^2}, \\ \frac{\partial c_2}{\partial \tau} = -\frac{4}{3}\frac{P_m(\gamma-1)}{\gamma}\frac{\partial^2 c_1}{\partial \xi_2^2} + \frac{4}{3}\frac{P_m}{\gamma}\frac{\partial^2 c_2}{\partial \xi_2^2}, \\ c_1(\xi_2, 0) = c_1^{(0)}(\xi_2), \\ c_2(\xi_2, 0) = c_2^{(0)}(\xi_2) \end{cases}$$

and

(3.65) $$\frac{\partial s}{\partial \tau} + \frac{M_A^2(\gamma+1)+3}{2M_A\sqrt{1+M_A^2}} s \frac{\partial s}{\partial \xi_3} = \omega \frac{\partial^2 s}{\partial \xi_3^2},$$

$$s(\xi_3, 0) = s^{(0)}(\xi_3),$$

where $r^{(0)}$, $c_1^{(0)}$, $c_2^{(0)}$ and s^0 are obtained by substitution of $\varrho^{(0)}$, $T^{(0)}$, $v^{(0)}$ and $B^{(0)}$ into Eqs. (3.36), (3.38), (3.39) and (3.42) instead of ϱ, T, v, B.

Equations (3.63) and (3.65) are the Burgers equations and therefore they can be solved explicitly [9]. Also, Eqs. (3.64) can be solved explicitly since it is a system of linear equations.

Once the functions r, c_1, c_2 and s have been determined, we can find φ_1, φ_2 and φ_3 as a result of integration. Thus the asymptotics is determined. We do not go into those details because it is quite a routine work (for the details, results, and graphs see [6]).

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