Pressure and surface-tension effects on surface-type waves in viscoelastic fluids

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IN OUR RECENT papers [1, 2] various properties of harmonic surface-type waves in compressible and incompressible viscoelastic fluids were discussed in greater detail. In the present paper the main attention is paid to the possible effects of the hydrodynamic pressure and the surface tension on the type of waves considered and the corresponding speeds of propagation.

W naszych ostatnich pracach [1, 2] przedyskutowano szczegółowo własności harmonicznych fal typu powierzchniowego w ściśliwych i nieściśliwych cieczach lepkosprężystych. W obecnej pracy zwrócono uwagę na możliwy wpływ ciśnienia hydrodynamicznego i napięcia powierzchniowego na typ rozważanych fal i odpowiednie prędkości propagacji.

В наших последних работах [1, 2] обсуждены подробно свойства гармонических волн поверхностного типа в сжимаемых и в несжимаемых вязкоупругих жидкостях. В настоящей работе обращено внимание на возможное влияние гидродинамического давления и поверхностного натяжения на тип рассматриваемых волн и соответствующие скорости распространения.

1. Introduction

THIS PAPER completes our previous considerations [1, 2] on various properties of harmonic surface-type waves in compressible and incompressible viscoelastic fluids. In those papers several examples of homogeneous fluids and two-layer immiscible fluids were discussed in greater detail but no surface-tension or interface-tension effects were taken into account. Now, our main attention is paid to the possible effects of the hydrodynamic pressure as well as the surface or interface tension on the type of waves considered and the corresponding speeds of propagation. To this end, after certain introductory remarks, two characteristic examples of waves in homogeneous and two-layer fluids are studied.

2. Governing equations and hydrodynamic pressure

The linearized constitutive equation of a viscoelastic compressible fluid, viz.

$$(2.1) \qquad \mathbf{T} = (-p + \lambda^* \mathrm{tr} \mathbf{D})\mathbf{1} + 2\eta^* \mathbf{D}, \qquad \mathbf{D} = \frac{1}{2}\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right),$$

where λ^* and η^* denote the frequency-dependent dynamic second (dilatational) and shear viscosities, respectively, leads to the following governing equations (cf. [1, 2]):

$$(2.2) \qquad \left(\nabla^2 - \frac{\varrho}{\lambda^* + 2\eta^*}\frac{\partial}{\partial t}\right)\Phi_1 = 0, \qquad \left(\nabla^2 - \frac{\varrho}{\eta^*}\frac{\partial}{\partial t}\right)\Phi_2 = 0,$$

where ϱ is the mass density of a fluid, and the scalar potentials (for plane waves) $\Phi_i(i = 1,2)$ are related to the velocity components as follows:

$$(2.3) \qquad u = \frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial z}, \qquad v = 0, \qquad w = \frac{\partial \Phi_1}{\partial z} - \frac{\partial \Phi_2}{\partial x}.$$

It should also be emphasized that the governing equations can be written in the form (2.2) only in the following cases.

1) If the hydrodynamic pressure p (undetermined in the constitutive equations) is a harmonic function, i.e. $\nabla^2 p = 0$; otherwise the governing equations are not homogeneous and the term $\nabla^2 p$ may be considered as a certain source distribution function.

2) If the thermodynamic pressure p is a barotropic function, i.e. $p = p(\varrho)$. In this case the linearized governing equations can be expressed again in the form (2.2), but $\lambda^* + 2\eta^*$ is replaced by $\lambda^* + 2\eta^* + \varrho_0/i\omega dp/d\varrho$ (cf. [2]). The barotropic function assumption is justified only for low-molecular gases and liquids. In general, we have

$$(2.4) \qquad p = -\frac{\partial \mathscr{A}}{\partial \left(\dfrac{1}{\varrho}\right)},$$

where $\mathscr{A}$ denotes the free energy function (or functional) and $1/\varrho$ is the specific volume. Thus the pressure p may depend not only on density and temperature but also on the state of deformation (or its history).

3) If the fluid considered is an incompressible one. The continuity condition $\nabla^2 \Phi_1 = 0$ ($\mathrm{div}\,\mathbf{v} = 0$) implies $\nabla^2 p = 0$ (cf. [2]), and since also $\lambda^* \to \infty$, only Eq. $(2.1)_2$ describing shear waves remains meaningful.

In our further analysis we use the solutions for surface-type waves in the form

$$(2.5) \qquad \begin{aligned} \Phi_i &= (A_i e^{\nu_i z} + B e^{-\nu_i z})\exp(\mu x + i\omega t), \\ p &= p_0(z)\exp(\mu x + i\omega t), \qquad i = 1, 2, \end{aligned}$$

where capital letters denote integration constants, and the parameters μ and ν_i, viz.

$$(2.6) \qquad \mu^2 = -k_x^2, \qquad \nu_i^2 = -k_{iz}^2 = -\left(\mu^2 + \frac{\varrho\omega^2}{G_i^*}\right), \qquad i = 1, 2,$$

are simply related to the components of the wave vectors.

$G_1^* = i\omega(\lambda^* + 2\eta^*)$ and $G_2^* = i\omega\eta^*$ denote the corresponding dynamic moduli.

3. Surface-tension effects

The vertical displacements at the free surface $z = 0$ (or interface) are

$$(3.1) \qquad \zeta = D\exp(\mu x + i\omega t), \qquad \frac{\partial \zeta}{\partial t} = w(0).$$

Therefore the surface tension T directed along the z-axis, as being proportional to the curvature of the surface, can be written in the following approximate form:

$$(3.2) \qquad T = S \frac{\partial^2 \zeta}{\partial x^2} = -\mu^2 S \frac{v_1 A_1 - \mu A_2}{i\omega} = S\mu \frac{v_1 A_1 - \mu A_2}{C_R},$$

where $C_R^2 = -\omega^2/\mu^2$ is the complex speed of propagation along the x-axis, and S — the constant surface-tension coefficient.

4. Surface-type waves in homogeneous compressible fluids

Consider a homogeneous fluid contained in the lower half-space $z \leqslant 0$ (the z-axis is directed upwards). The boundary conditions at the free surface $z = 0$ ($T^{13} = 0$, $T^{33} + T = 0$ for $z = 0$) lead to the system of equations

$$(4.1) \qquad 2\mu v_1 A_1 + (v_2^2 - \mu^2) A_2 = 0,$$

$$-p_0(0) + [\lambda^*(v_1^2 + \mu^2) + 2\eta^* v_1^2] A_1 - 2\eta^* \mu v_2 A_2 + \mu \frac{S}{C_R} (v_1 A_1 - \mu A_2) = 0.$$

Therefore we have two equations and three quantities to be determined (A_1, A_2, $p_0(0)$). Without any loss of generality we put $p_0(0) = 0$, and then a solution of the system (4.1) exists if

$$(4.2) \qquad 4\mu^2 v_1 v_2 + (v_2^2 - \mu^2)^2 + \frac{\mu v_1 S}{\eta^* C_R} (v_2^2 + \mu^2) = 0$$

or

$$(4.3) \quad n[n^3 - 8n^2 + (24 - 16\vartheta)n - 16(1 - \vartheta)]$$
$$= \frac{2S\omega \sqrt{\varrho}}{G_2^{*3/2}} \sqrt{n - n^2 \vartheta} \, (n-2)^2 + \frac{S^2 \omega^2 \varrho}{G_2^{*3}} \sqrt{n} \, (1 - n\vartheta),$$

where

$$(4.4) \qquad n = -\frac{\omega^2 \varrho}{\mu^2 G_2^*} = \frac{C_R^2 \varrho}{G_2^*}, \qquad \vartheta = \frac{G_2^*}{G_1^*}.$$

In general, the secular equation (4.3) is a very complex algebraic equation and its solutions cannot be obtained in a straightforward way. Thus we restrict ourselves to the case of incompressible fluids ($G_1^* \to \infty$, $\vartheta \to 0$) under the additional assumption that the term on the right-hand side of Eq. (4.3) proportional to S^2 may be omitted for small surface tensions. This leads to the simplified equation

$$(4.5) \qquad n(n^3 - 8n^2 + 24n - 16) \simeq \frac{2S\omega \sqrt{\varrho}}{G_2^{*3/2}} \sqrt{n} \, (n-2)^2.$$

It was shown by CURRIE et al. [3, 4] that for incompressible fluids without surface-tension effects (then the secular equation is determined by the expression in parentheses on the left-hand side of Eq. (4.5)), only two roots are admissible; these are

$$(4.6) \qquad n_1 = 0.9126, \qquad n_2 = 3.5437 - i2.2303.$$

The real root n_1 corresponds to the so-called quasi-elastic waves, occurring also in purely elastic and elastic-like media. The complex root n_2 is characteristic of viscoelastic waves which may appear for sufficiently high loss angles (cf. [2, 4]).

On assuming that for small surface tensions any solution of Eq. (4.5) can be expressed as $n = n_0 + n'$, where n_0 — is the solution without surface-tension effects and n' — the corresponding linear increment, we arrive at

$$(4.7) \qquad n' = -\frac{2k\sqrt{n}\,(n-2)^2}{n(3n^2-16n+24)-k\dfrac{(n-2)^2}{\sqrt{n}}}, \qquad k = \frac{S\omega\sqrt{\varrho}}{G_2^{*3/2}}.$$

By way of illustration the values of n' and n calculated for elastic-like fluids (cf. [1, 2]), i.e. for $n_0 = 0.913$, are shown in Table 1. It is seen that the speeds of propagation for quasi-elastic waves increase with increasing values of the surface-tension coefficient S and frequency ω. For sufficiently high values of the parameter k this speed may be greater than the corresponding speed of shear waves C_2. It seems that similar results can be obtained for viscoelastic waves characterized by the root n_2.

Table 1.

k	0	0.01	0.05	0.1	0.2	0.5
n'	0	0.002	0.011	0.021	0.043	0.110
n	0.913	0.915	0.924	0.934	0.956	1.023

5. Surface-type waves in two-layer incompressible fluids

Consider a two-layer incompressible fluid with the upper layer of thickness h and the lower fluid contained in the half-space $z \leqslant 0$. We assume, moreover, that the layers can slide freely at the interface. The following boundary conditions should be satisfied at the free surface and the interface:

$$(5.1) \qquad \begin{aligned} T^{13} &= 0, \quad T^{33}+T = 0, \quad \text{for} \quad z = h, \\ T^{33} &= \overline{T}^{33}+\overline{T}, \quad w = \overline{w}, \quad \text{for} \quad z = 0, \end{aligned}$$

where T denotes the surface tension and the overbars refer to quantities in the lower fluid. Being interested exclusively in shear waves of the surface-type (cf. [1]), we arrive at the system of equations

$$(\nu_2^2-\mu^2)(A_2 e^{\nu_2 h}+B_2 e^{-\nu_2 h}) = 0,$$

$$-p_0(h)-2\eta^*\mu\nu_2(A_2 e^{\nu_2 h}-B_2 e^{-\nu_2 h})-\frac{\mu^2 S}{C_R}(A_2 e^{\nu_2 h}+B_2 e^{-\nu_2 h}) = 0,$$

(5.2) $A_2 + B_2 - \bar{A}_2 = 0,$

$$-\left(p_0(0) - \bar{p}_0(0)\right) - 2\eta^* \mu \nu_2 (A_2 - B_2) + 2\bar{\eta}^* \mu \bar{\nu}_2 A_2 + \frac{\mu^2 S}{C_R} \bar{A}_2 = 0,$$

where S and $\bar{S}$ are the surface- and interface-tension coefficients, respectively.

Therefore, we have four equations and five quantities to be determined ($A_2, B_2, \bar{A}_2,$ $p_0(h), p_0(0) - \bar{p}_0(0)$). If we put $p_0(0) = 0$ (then the pressure difference at the interface results from Eq. (5.2)$_4$), the secular equation will be exactly the same as that obtained previously, when surface-tension effects were absent (cf. [1]). On the other hand, if we put $p_0(0) = \bar{p}_0(0)$ (then the pressure at the free surface results from Eq. (5.2)$_2$), the secular equation will take the following form:

(5.3) $$\mathrm{th}\nu_2 h = -\frac{2\eta^* \nu_2}{2\bar{\eta}^* \bar{\nu}_2 + \mu \bar{S}/C_R}.$$

In general, the above transcendental equation is a very complex one and its solutions cannot be obtained in a straightforward way. However, under the additional assumptions that $\mathrm{th}\nu_2 h \to 1$ (high-frequency approximation) and $G_2^* \simeq \bar{G}_2^*$ (similar viscoelastic characteristics of both fluids), we formally arrive at

(5.4) $$C_R^2 = \frac{G_2^*}{2\varrho} \left(1 + \sqrt{1 + \frac{\varrho^3 \omega^2 \bar{S}^2}{G_2^{*3}} \frac{1}{(\varrho + \varrho)^2}}\right),$$

where $\varrho, \bar{\varrho}$ denote the mass densities of both fluids. In the case of elastic-like fluids (cf. [1, 2]), the above relation simplifies to

(5.5) $$C_R^2 = \frac{C_2^2}{2} \left(1 + \sqrt{1 + \frac{\omega^2 \bar{S}^2}{C_2^3} \frac{1}{(\varrho + \varrho)^2}}\right), \quad C_2^2 = \frac{G_2}{\varrho}.$$

This example shows that shear waves of the surface-type in a two-layer incompressible fluid with interface-tension effects propagate faster than ordinary shear waves in a bulk fluid.

6. Final remaks

On the basis of the above considerations we can formulate the following conclusions.

1) The hydrodynamic pressure function plays an essential role in the surface-type wave propagation. The values of pressure assumed either at the free surface or interface may affect the type and speed of the waves considered.

2) Surface or interface tensions in homogeneous as well as two-layer fluids enlarge, in principle, the corresponding speeds of propagation. Under certain conditions these speeds may be higher than those determined in the cases in which surface- or interface-tension effects are not taken into account.

3) A similar analysis applied to various cases of waves in two-layer fluids (cf. [1]) shows that surface- or interface-tension effects are meaningful only for fluids with free outer surfaces.

References

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