

Non-isothermic large elastic-plastic deformations (*)

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WITHIN the frame of the phenomenological theory of non-isothermic large elastic-plastic deformations we treat "elementary processes", which may be considered as a sequence of equilibrium states. Therefore we can correlate a description by thermodynamic state equations to the usual description as thermo-mechanical process. This is shown in general and for a special material. Some possible generalisations are discussed.

W ramach fenomenologicznej teorii nieizotermicznych dużych odkształceń sprężysto-plastycznych ogrzewamy "procesy elementarne", które mogą być rozważane jako ciąg stanów równowagi. Dlatego możemy sprowadzać opis przez termodynamiczne równania stanu do zwykłego opisu jako procesu termo-mechanicznego. Wykazano to w przypadku ogólnym i dla szczególnego materiału. Przedyskutowano niektóre możliwe uogólnienia.

В рамках феноменологической теории неизоотермических больших упруго-пластических деформаций нагреваем „элементарные процессы”, которые могут рассматриваться как последовательность состояний равновесия. Поэтому можем свести описание через термодинамические уравнения состояния к обыкновенному описанию как термо-механического процесса. Это доказано в общем случае и для частного материала. Обсуждены некоторые возможные обобщения.

1. Introduction

WE CONFINE ourselves to the phenomenological theory of elastic-plastic bodies. The constitutive law of these bodies is assumed to be independent of the scale of time, i.e. rate-independent. The bodies are considered as classical continua, that is, the kinematics of these bodies is completely described by the displacements as functions of space and time.

We restrict ourselves to processes which are quasi-static and homogeneous throughout the body. We call these processes "elastic-plastic elementary processes". Processes of this kind can be considered as a sequence of equilibrium states. Therefore it must be possible to describe such processes by means of thermodynamic state equations. This representation is correlated to the usual description of such elastic-plastic deformations as a thermo-mechanical process using process variables, which are not necessarily state variables in the thermodynamic sense.

In two earlier papers [1, 2] the first author has pointed out the differences and the correlations between these two methods of description. These considerations shall be extended here in two directions. At first, we want to show the application of the theoretical results to a real material in a special process. Secondly, we make some remarks on a possible generalization of the theory.

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For brevity, we refer to the cited papers [1, 2], which contain many references to other papers connected with the subject treated here. We repeat only the main results in Chaps. 2 to 4.

2. Kinematics and fundamental assumptions

The total deformations of a body can be completely described by the changes of the metric g_{ik} of the body-fixed coordinate-system ξ^i . The metric of the undeformed state is

$$(2.1) \quad g_{ik}(\xi^r, t_0) = \overset{\circ}{g}_{ik}(\xi^r).$$

In the deformed state we have

$$(2.2) \quad g_{ik} = g_{ik}(\xi^r, t).$$

In our case of homogeneous deformations the metric g_{ik} becomes independent of ξ^r throughout time, if the metric $\overset{\circ}{g}_{ik}$ is independent. We use the deformed state as the reference state in our considerations.

The changes of the metric are given by the metric transformation tensor q_k^i in the following way:

$$(2.3) \quad \begin{aligned} g_{ik} &= q_i^r \overset{\circ}{g}_{rk} = \overset{\circ}{g}_{ir} q_k^r, \\ \overset{\circ}{g}_{ik} &= (q^{-1})_i^r g_{rk} = g_{ir} (q^{-1})_k^r, \end{aligned}$$

with

$$(2.4) \quad q_k^i = \overset{\circ}{g}^{ir} g_{rk}, \quad (q^{-1})_k^i = g^{ir} \overset{\circ}{g}_{rk}.$$

Every strain tensor which describes the total strain may be expressed as a function of the metric transformation tensor or its inverse

$$(2.5) \quad \varepsilon_k^i = \varepsilon_k^i(q_s^r) = \varepsilon_k^i((q^{-1})_s^r).$$

Likewise, the total strain rate is derivable from this tensor. We get

$$(2.6) \quad d_k^i = \frac{1}{2} g^{ir} \dot{g}_{rk} = \frac{1}{2} (q^{-1})_s^i (\dot{q})_{.k}^s = -\frac{1}{2} (q^{-1})^{i.s} \dot{q}_k^s.$$

The change of the metric may be split into its elastic and its plastic component in the following manner:

$$(2.7) \quad \begin{aligned} q_k^i &= \underbrace{\overset{\circ}{g}^{ir} \overset{*}{g}_{rm}}_{q_{P,m}^i} \underbrace{\overset{*}{g}^{ms} \overset{\circ}{g}_{sk}}_{q_{E,k}^m}, \\ (q^{-1})_k^i &= \underbrace{g^{ir} \overset{*}{g}_{rm}}_{(q^{-1})_m^i} \underbrace{\overset{*}{g}^{ms} \overset{\circ}{g}_{sk}}_{(q^{-1})_{.k}^m}. \end{aligned}$$

This multiplicative splitting of the metric change leads to an additive splitting of the strain rate according to

$$\begin{aligned}
 d_k^i &= \text{sym} \frac{1}{2} \{ (q^{-1})^i_r (\dot{q})^r_k \} + \text{sym} \frac{1}{2} \{ (q^{-1})^i_r (\dot{q})^r_m q_k^m \}, \\
 (2.8) \quad &= -\text{sym} \frac{1}{2} \{ (\dot{q}^{-1})^i_r q_k^r \} - \text{sym} \frac{1}{2} \{ (q^{-1})^i_r (\dot{q}^{-1})^r_m q_k^m \}, \\
 &= d_k^i \quad + \quad d_k^i.
 \end{aligned}$$

We assume that the constitutive law of the elastic component of the deformations is isotropic and independent of the plastic deformations, because otherwise the splitting of the total deformation into its elastic and its plastic component would be useless. In this case the stress-strain relations of the elastic component of the deformation can be written in the general form

$$(2.9) \quad s_k^i = s_k^i(q_s^r, T) = s_k^i((q^{-1})^r_s, T),$$

where s_k^i means the Kirchhoff stress tensor related to the true stress σ_k^i by

$$(2.10) \quad s_k^i = \frac{\rho}{\rho} \sigma_k^i \quad (\rho = \text{mass density}).$$

Thus we may derive stress-strain relations in an incremental form

$$(2.11) \quad d_k^i = d_k^i \{ s_k^i, (\dot{s})^i_k, T, \dot{T}; g_{ik}, d_k^i \}.$$

An example may illustrate this. From the frequently used elastic stress-strain relation

$$(2.12) \quad \varepsilon_k^i = \frac{1}{2} \{ \delta_k^i - (q^{-1})^i_k \} = \frac{1}{2G} \left\{ s_k^i - \frac{\nu}{1+\nu} s_r^r \delta_k^i \right\} + \alpha (T - \dot{T}) \delta_k^i,$$

we get

$$(2.13) \quad d_k^i = \frac{1}{2G} \left\{ \text{sym} [q_k^r (\dot{s})^i_r] - \frac{\nu}{1+\nu} q_k^i (\dot{s}_m^m) \right\} + \alpha \dot{T} \delta_k^i,$$

which is approximately equal to

$$(2.14) \quad d_k^i = \frac{1}{2G} \left\{ s_k^i|_0 - \frac{\nu}{1+\nu} s_m^m|_0 \delta_k^i \right\} + \alpha \dot{T} \delta_k^i,$$

where

$$(2.15) \quad s_k^i|_0 = \text{sym} (\dot{s})^i_k$$

means the so-called covariant derivation with respect to time or the Jaumann-derivation. Regarding the plastic deformations, we make the usual assumption that plastic deformations occur only if the stress and the temperature fulfill a given yield condition

$$(2.16) \quad F(s_k^i, T; k^2, \dots, \alpha_k^i, \dots, A_{ks}^i, \dots) = 0,$$

and the loading condition

$$(2.17) \quad \frac{\partial F}{\partial s_k^i} s_k^i|_0 + \frac{\partial F}{\partial T} \dot{T} > 0.$$

Furthermore, we assume that the plastic strain rate is related to the yield and the loading condition by the approach

$$(2.18) \quad d_p^i = \dot{\lambda} \frac{\partial F}{\partial s_k^i} + \kappa_{ks}^{ir} t_r^s|_0,$$

where t_k^i is the stress deviator. The first term corresponds to the theory of the so-called plastic potential. The second additional term represents a correction which is generally quite small.

3. Description of elastic-plastic elementary processes as thermo-mechanical processes

At first we must choose a suitable set of independent process variables as, for example, the stress s_k^i and the temperature T . The problem is to describe the changes of all other process variables, i.e. of all dependent process variables, according to the history of the independent variables. So, under the usual assumptions of uniqueness, we get the following process description:

independent process variables: s_k^i, T
 dependent process variables (a): $k^2 \dots, \alpha_k^i \dots, A_{ks}^{ir} \dots,$
 (b): $\epsilon_k^i(q_s), q, w \dots,$

if $F(s_k^i, T; k^2 \dots, \alpha_k^i \dots, A_{ks}^{ir}) = 0$ (yield condition),

and $\frac{\partial F}{\partial s_k^i} s_k^i|_0 + \frac{\partial F}{\partial T} \dot{T} > 0$ (loading condition),

$$(3.1) \quad \begin{aligned} d_k^i &= \frac{1}{2} g^{ir} \dot{g}_{rk} = d_k^i + d_k^i \\ &= d_k^i \{s_k^i, s_k^i|_0, T, \dot{T}; g_{ik}, k^2 \dots, \alpha_k^i \dots, A_{ks}^{ir} \dots\} \\ (k^2) &= (k^2) \{ \dots \dots \dots \} \\ \alpha_k^i|_0 &= \alpha_k^i|_0 \{ \dots \dots \dots \} \\ A_{ks}^{ir}|_0 &= A_{ks}^{ir}|_0 \{ \dots \dots \dots \} \end{aligned}$$

otherwise

$$d_k^i = d_k^i \{s_k^i, s_k^i|_0, T, \dot{T}; g_{jk}, d_k^i\}.$$

The dependent process variables may be split into two groups. The first group contains those quantities which are necessary to fix the state of the material. The second group includes all other dependent variables as, for example, the total strain ϵ_k^i , the applied heat q , or the mechanical work w , etc.

The independent process variables s_k^i and T in this description, may be replaced by another suitable set, for example, by the total strains ϵ_k^i and the applied heat q . The scheme of the process description remains unaltered, in principle, by this change. The only difference is that in this case the group of dependent variables (a) has to be enlarged for two further quantities, the stresses s_k^i and the temperature T , or for another equivalent set of state variables.

4. Description of elastic-plastic elementary processes by state equations

We use as thermodynamic state variables the elastic strain, represented by q_k^i , the absolute temperature T , and a number of other state variables ($h \dots, \beta_k^i \dots, B_{ks}^{ir} \dots$) which may be scalars or tensors of even order. The plastic strain and the total strain are unsuitable as state variables since, in general, they do not uniquely define the state of the material.

The thermodynamics of the elastic-plastic elementary processes are governed by

(a) a thermic state equation

$$(4.1)_1 \quad s_k^i = s_k^i(q_k^i, T, \dots),$$

(b) a caloric state equation for the specific internal energy

$$(4.1)_2 \quad u = u(q_k^i, T, \dots),$$

(c) a statement about the entropy production \dot{s} .

The two state equations may be combined by introducing the specific free energy (Helmholtz-function)

$$\varphi = \varphi(q_k^i, T, \dots).$$

In a simplified, though based on our assumptions, sufficient form, the first law of thermodynamics states

$$(4.2) \quad \dot{u} = \dot{q} + \dot{w} = \dot{q} + \dot{w}_E + \dot{w}_P = \dot{q} + \frac{1}{\varrho} s_k^i d_i^k = \dot{q} + \frac{1}{\varrho} s_k^i d_i^k + \frac{1}{\varrho} s_k^i d_i^k.$$

The plastic work must be separated into the component \dot{w}_H , which changes the hardening state of the material and into another component \dot{w}_D , denoting the energy dissipation:

$$(4.3) \quad \dot{w}_P = \dot{w}_H + \dot{w}_D.$$

Only the second part \dot{w}_D enters the entropy production

$$(4.4) \quad \dot{s} = \frac{1}{T} \{ \dot{q} + \dot{w}_D \}.$$

The second law of thermodynamics requires

$$(4.5) \quad \dot{w}_D \geq 0.$$

Since the elastic part of the deformations, according to our assumptions, does not depend on the plastic deformations, we may divide the free energy into two different components

$$(4.6) \quad \varphi = \varphi_E(q_k^i, T) + \varphi_H(T, h \dots, \beta_k^i \dots, B_{ks}^{ir} \dots),$$

where the first component refers to the elastic deformations, the second to the changes of the hardening state. From the Eqs. (4.2), (4.3) and (4.4) we derive

$$(4.7) \quad \dot{\varphi} = -s\dot{T} + \dot{w}_E + \dot{w}_H.$$

On the other side, we get from the Eq. (4.6)

$$(4.8) \quad \dot{\varphi} = \frac{\partial \varphi}{\partial q_k^i} \dot{q}_k^i |_0 + \frac{\partial (\varphi + \varphi)}{\partial T} \dot{T} + \frac{\partial \varphi}{\partial h} \dot{h} + \dots + \frac{\partial \varphi}{\partial \beta_k^i} \dot{\beta}_k^i |_0 + \dots + \frac{\partial \varphi}{\partial B_{ks}^{ir}} \dot{B}_{ks}^{ir} |_0 + \dots$$

By comparison of these two equations we may conclude

$$(4.9) \quad s = - \frac{\partial \varphi}{\partial T} = - \frac{\partial (\varphi + \varphi)}{\partial T},$$

$$s_k^i = \dot{q}_k^i \frac{\partial \varphi}{\partial q_k^i},$$

$$\dot{w}_H = \frac{\partial \varphi}{\partial h} \dot{h} + \dots + \frac{\partial \varphi}{\partial \beta_k^i} \dot{\beta}_k^i |_0 + \dots + \frac{\partial \varphi}{\partial B_{ks}^{ir}} \dot{B}_{ks}^{ir} |_0 + \dots$$

The thermodynamic equations have to be completed by a statement about the entropy production. We assume this statement in the general form:

$$(4.10) \quad \dot{w}_D = C_{ks}^{ir} s_k^i d_P^r > 0$$

$$\text{with } C_{ks}^{ir} = C_{ks}^{ir}(q_k^i, T, h, \dots, \beta_k^i, \dots, B_{ks}^{ir} \dots).$$

The Eqs. (4.9) and (4.10) are the governing thermodynamic equations of non-isothermic elastic-plastic elementary processes. But in general we cannot derive the complete set of constitutive equations from these thermodynamic equations. We need more information about the material behaviour. Such specifications as, for example, the yield condition, are not state equations. They belong to the domain process description.

5. Correspondence between thermodynamic relations and the process description for an isotropic hardening material

For a carbon steel C45 (DIN 1720) in a pure tension test at a moderate temperature and strain rate, we find the material behaviour which is shown in Fig. 1 [3]. From this we may derive the stress-strain-temperature relations for loading in pure tension in the form

$$(5.1) \quad \sigma = \sigma(\varepsilon, T).$$

For our purpose it is more useful to write this relation in the form

$$(5.2) \quad \sigma = \sigma(w, T) = \frac{c_1(T)w_P}{c_2(T) + w_P} + \sigma_0(T).$$

In our special case we get

$$\begin{aligned}
 c_1(T) &= 72.42 - 36.03 \cdot 10^{-3} T \frac{kp}{\text{mm}^2}, \\
 c_2(T) &= 7.35 - 8.04 \cdot 10^{-3} T \frac{kp}{\text{mm}^2}, \\
 \sigma_0(T) &= 47.41 - 38.9 \cdot 10^{-3} T \frac{kp}{\text{mm}^2},
 \end{aligned}
 \tag{5.3}$$

with T in $^{\circ}\text{K}$.

We may consider the carbon steel approximately as an isotropic work-hardening material obeying the v . Mises-Hill yield condition and the theory of plastic potential. Furthermore,

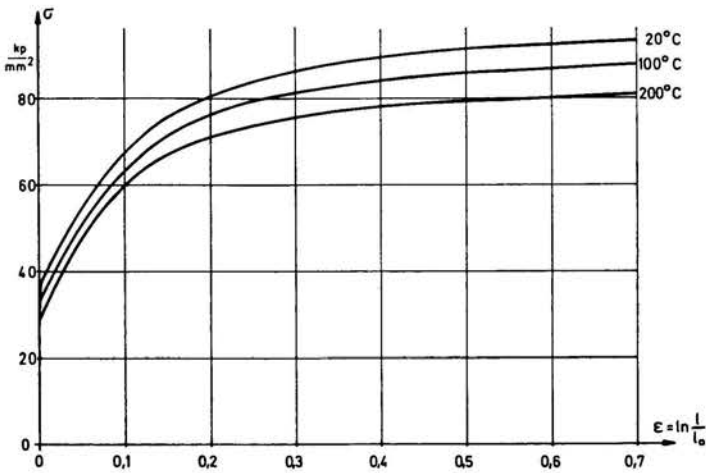


FIG. 1.

we assume that a constant ratio of 90% of the plastic work is dissipated. With these assumptions we get the following general *process-description* for the material under consideration [Eq. (3.1)]

- independent process variables: s_k^i, T
- dependent process variables: (a) w_P or $k^2(w, T)$, respectively,
- (b) $q_k^i, q \dots$

yield condition:

$$\begin{aligned}
 F(s_k^i, T, w) &= t_k^i t_i^k - k^2(w, T) = 0, \\
 k^2(w, T) &= \frac{2}{3} \left\{ \sigma_0(T) + \frac{c_1(T) w_P}{c_2(T) + w_P} \right\}^2,
 \end{aligned}
 \tag{5.4}_1$$

loading condition:

$$\frac{\partial F}{\partial s_k^i} s_k^i |_{t_0} + \frac{\partial F}{\partial T} \dot{T} = 2 t_k^i t_i^k |_{t_0} - \frac{\partial k^2}{\partial T} \dot{T} > 0,
 \tag{5.4}_2$$

elastic strain rate (approximation according to the Eq. (2.14))

$$(5.4)_3 \quad d_E^i = \frac{1}{2G} \left\{ s_k^i |_0 - \frac{\nu}{1+\nu} s_m^m |_0 \delta_k^i \right\} + \alpha \dot{T} \delta_k^i,$$

with $\alpha = 11.9 \cdot 10^{-6} K^{-1}$,

plastic strain rate:

when (5.4)₁ and (5.4)₂ are fulfilled:

$$(5.4)_4 \quad d_P^i = \dot{\lambda} \frac{\partial F}{\partial s_i^k} = \frac{2t_s^* t_r^s |_0 - \frac{\partial k^2}{\partial T} \dot{T}}{k^2 \frac{\partial k^2}{\partial w_P}},$$

otherwise:

$$(5.4)_5 \quad d_P^i = 0,$$

rate of plastic work:

$$(5.4)_6 \quad \dot{w}_P = s_k^i d_P^k,$$

rate of applied heat [approximation; the exact formulation is given later in the Eq. (5.8)]:

$$(5.4)_7 \quad \dot{q} = c \dot{T} - \xi \dot{w}_P,$$

$$\text{with } \xi = 0.9 = \text{const}, \quad c = 465 \frac{J}{\text{kgK}} \text{ (heat capacity).}$$

When we use the total strain $\varepsilon_k^i(q_s^*)$ and the applied heat as independent process variables, then we must interchange the roles of s_k^i and T against $\varepsilon_k^i(q_s^*)$ and q , which causes no difficulty.

In the process description the hardening parameter k^2 depends uniquely on the plastic work w_P and the temperature T . This fact leads to the following approach for the *corresponding thermodynamic relations* to make sure that the plastic work is equivalent to thermodynamic state variables:

free energy:

$$(5.5)_1 \quad \varphi = \varphi(q_k^i, T) + h \quad \text{with} \quad h = \varphi_H;$$

entropy production:

$$(5.5)_2 \quad \dot{w}_D = \xi(h) s_k^i d_P^k = \xi(h) \dot{w}_P.$$

From this approach, and in connection with the Eqs. (4.3) and (4.9)₃, we get

$$\dot{w}_P = \dot{w}_H + \dot{w}_D = \dot{h} + \xi(h) \dot{w}_P = \frac{\dot{h}}{1 - \xi(h)}.$$

This means, as required,

$$(5.6) \quad h = h(w)_P$$

In our special case, it holds that $\xi(h) = 0.9 = \text{const}$. Therefore the (latent) hardening energy $\varphi = h$ becomes (with $h_0 = 0$)

$$(5.7) \quad h = (1 - \xi)w = 0.1 w_P$$

From our approach, (5.5)₁, we may also derive for this case, the exact relation between the rate of applied heat, the time derivative of temperature and the strain rates. Using the Eqs. (4.4) and (4.9)₁ we get

$$\begin{aligned} \dot{q} + \dot{w}_D &= \dot{q} + \xi \dot{w}_P = T \dot{s} = T \left\{ \frac{\partial s}{\partial q_k^i} q_k^i|_0 + \frac{\partial s}{\partial T} \dot{T} + \frac{\partial s}{\partial h} \dot{h} \right\} \\ &= -T \left\{ \frac{\partial^2 \varphi}{\partial q_k^i \partial T} q_k^i|_0 + \frac{\partial^2 \varphi}{\partial T^2} \dot{T} + \frac{\partial^2 \varphi}{\partial h \partial T} \dot{h} \right\}, \end{aligned}$$

i.e.

$$(5.8) \quad \dot{q} + \xi \dot{w}_P = -T \left\{ \frac{\partial^2 \varphi}{\partial T^2} \dot{T} + \frac{\partial^2 \varphi}{\partial q_k^i \partial T} q_k^i|_0 \right\}$$

In this equation

$$(5.9) \quad -T \frac{\partial^2 \varphi}{\partial T^2} = c(q_k^i, T),$$

represents the heat capacity at constant strain ($d_k^i = d_P^i = 0$). We may consider this heat capacity as constant:

$$c(q_k^i, T) = c.$$

Furthermore, we know from experimental results that the second term on the right-hand side of the Eq. (5.8) can be neglected in most cases. So we can replace the Eq. (5.8) by

$$(5.10) \quad \dot{q} + \xi \dot{w}_P = c \dot{T}.$$

This is identical to the Eq. (5.4)₇ which we used as an approximation in the process description.

Looking back, we may state once more that we need the corresponding thermodynamic relations to make sure, that the constitutive law of the process description is compatible with the basic laws of thermodynamics. On the other hand, we can see that the thermodynamic relations alone, even in the simple case here, are not sufficient to derive the complete set of constitutive equations of the process description. We need further information such as statements regarding the yield condition, the form of the plastic stress-strain relations (plastic potential), etc.

6. The isotropic and adiabatic behaviour of an isotropic hardening material in simple shear

We take the same material as in Chapt. 5 and consider simple shear processes as shown in Fig. 2. We denote this material as material *A*. The processes will be carried out, on the one hand isothermically and, on the other hand, adiabatically. We find the solution of the problems by numerically integrating a system of first-order differential equations originating from the Eqs. (5.4)₁–(5.4)₇. In the first case (isothermic process), the total strain

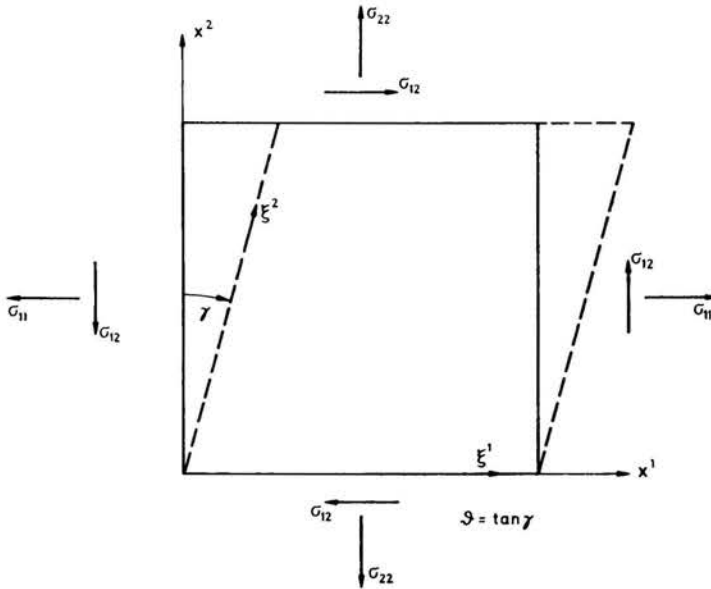


FIG. 2.

rate d_k^i and the temperature T_0 are given, and in the second case (adiabatic process) the total strain rate and the vanishing of the applied heat are prescribed.

For comparison, we introduce, furthermore, a theoretical material whose yield condition is unaffected by temperature. This means, for this material, the hardening parameter k^2 is

$$k^2 = k_p^2(w, T_0).$$

In isothermic processes this material (denoted as material *B*) shows the same behaviour as material *A*. But in adiabatic processes we have differences. For material *A*, the temperature influences the yield condition as well as the elastic part of deformation. For material *B*, only the elastic components of the deformations are changed by temperature.

Regarding this we must distinguish three cases:

- (I): isothermic processes with material *A* or *B*,
- (II A): adiabatic processes with material *A*
(hardening rule depending on temperature),
- (II B): adiabatic processes with material *B*
(hardening rule independent of temperature).

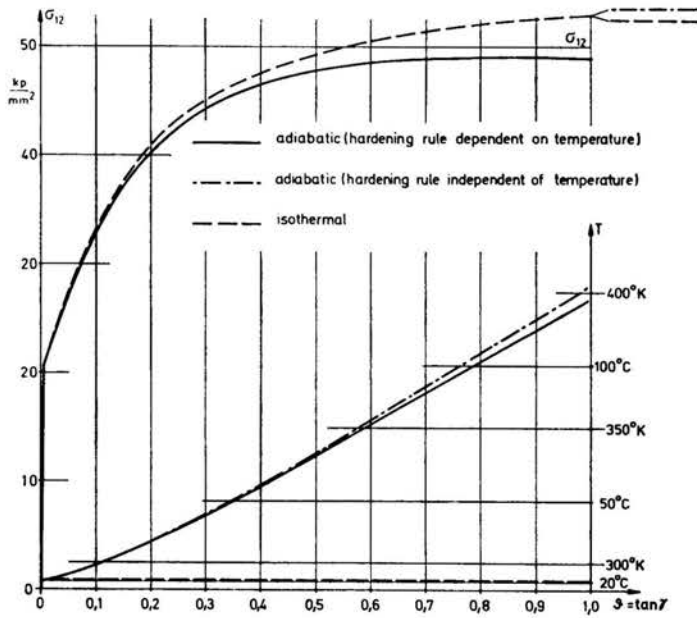


FIG. 3.

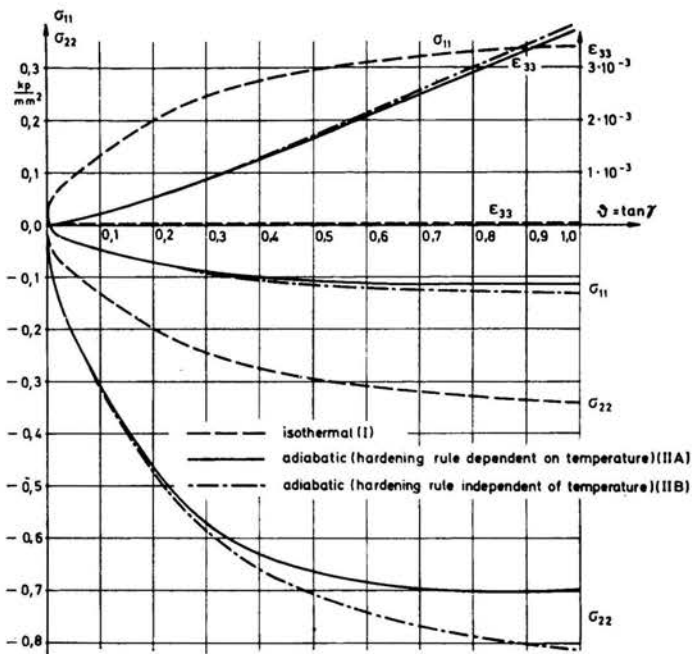


FIG. 4.

The results for the shear stresses and the temperature are shown in Fig. 3. We see that the differences between the shear stresses in the isothermic and in the adiabatic processes are mainly influenced by the dependence of the yield condition upon temperature. The differences between the cases (I) and (II*B*) are negligible, but not the differences between II*A* and II*B*. It should be remarked that in the case (II*A*), we get a maximum shear stress for $\vartheta = 0.87$. So for larger deformation we find in this case a softening effect due to the increasing temperature. With respect to the temperature the differences between the adiabatic cases, (II*A*) and (II*B*), are rather small, since the differences in the plastic work, in both these cases, are not so important.

The second-order effects are more influenced by the temperature than the first-order effects. This can be seen from Fig. 4. The effects are partially changed in the opposite direction (see stress σ_{11}). This is due to the strong influence of the temperature on the elastic deformations. We may conclude this from the fact, that the differences between the cases II*A* and II*B* are less than the differences between I to II*A* or II*B*, respectively.

From other experiments, however, we know that the observable second-order effects cannot be explained by the influence of the elastic part of the deformation alone. We get more realistic results when we use stress-strain relations derived from the Eq. (2.18) with a small correction concerning the theory of plastic potential (see [4]). In this case the difference between the second-order effects in isothermic and in adiabatic processes may be slightly less. But in any case, the influence of temperature on second-order effects is more important than the influence on first-order effects.

7. Some possible generalizations

In order to generalize the theoretical approach for isotropic materials we may assume free energy:

$$(7.1)_1 \quad \varphi = \varphi \left(q_k^i, T \right) + \varphi \left(h, T \right),$$

and entropy production:

$$(7.1)_2 \quad \dot{w}_D = \xi \left(h, T \right) \dot{w}_P.$$

From this approach we derive in the same way which led to the Eq. (5.8),

$$(7.2) \quad \dot{q} + \xi \left(h, T \right) \dot{w}_P = -T \left\{ \frac{\partial^2 \varphi}{\partial q_k^i \partial T} q_k^i \Big|_0 + \frac{\partial^2 (\varphi + \varphi)}{\partial T^2} \dot{T} + \frac{\partial^2 \varphi}{\partial h \partial T} \dot{h} \right\}.$$

The heat capacity at constant strain and constant h is

$$c = -T \frac{\partial^2 (\varphi + \varphi)}{\partial T^2}.$$

If we assume, that this heat capacity does not depend on the hardening state characterized by h , then it follows that

$$\frac{\partial^2 \varphi}{\partial T^2} = f(T) \quad \text{i.e.} \quad \varphi(h, T) = a(T) + b(h)T + g(h).$$

We may link the first term $a(T)$ to φ_E , since it does not depend on h . So we get

$$(7.3) \quad \varphi_H(h, T) = b(h)T + g(h).$$

Putting this in the Eq. (7.2) we obtain

$$(7.4) \quad \dot{q} + \xi(h, T) \dot{w}_P = -T \left\{ \frac{c\dot{T}}{T} + \frac{\partial^2 \varphi}{\partial q_k^i \partial T} q_k^i \Big|_0 + \underbrace{\frac{\partial^2 \varphi}{\partial h \partial T}}_{\frac{db(h)}{dh}} \dot{h} \right\}.$$

This relation allows for interactions between the state variables h , q_k^i and T , which are not included in our former approach in Chapt. 5.

Furthermore, from (7.1)₂ we may, with respect to the Eqs. (4.3), (4.9)₃ and (7.3), derive

$$(7.5) \quad \dot{w}_P = \frac{1}{1 - \xi(h, T)} \underbrace{\left\{ \frac{db(h)}{dh} T + \frac{dg(h)}{dh} \right\}}_{\frac{\partial \varphi}{\partial h}} \dot{h}.$$

The integration of this equation depends in general upon the slope of the process. Therefore we cannot construct a unique relation $w_P(h, T)$ in general. For the same reason, a hardening rule of the form $k^2 = k^2(w_P, T)$ is incompatible with an approach according to the Eqs. (7.1)₁ and (7.1)₂. The only exception is when

$$(7.6) \quad \frac{\frac{db(h)}{dh} T + \frac{dg(h)}{dh}}{1 - \xi(h, T)} = f(h).$$

Then we get

$$h = h(w_P).$$

It should be remarked that in this case, $\xi(h, T)$ cannot be kept constant as in the example in Chapt. 5.

We may extend our approaches to anisotropic hardening materials by assuming [see the Eq. (4.6)]

$$\varphi = \varphi_E(q_k^i, T) + \varphi_H(T, h, \beta_k^i, B_{ks}^{ir} \dots).$$

In such cases the relations become still more complicated. Examples of this kind shall be treated at another time.

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