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Universal deformations for thermo-elastic-plastic materials (*)

A. PAGLIETTI (CAGLIARI)

IN THE RANGE of large deformations, we consider the problem of determining the set of the deformations possible in every elastic-plastic material which is acted upon only by surface forces and by temperature changes. The thermo-plastic deformations are shown to render this set considerably wider than the analogous set relevant to perfectly elastic materials. The conditions under which a given thermally, elastically and plastically deformed state can be relaxed are also considered, and some problems connected with the choice of the measures of thermal, plastic and elastic deformations are discussed.

W zakresie dużych odkształceń rozważany jest problem określenia układu możliwych deformacji w każdym materiale sprężysto-plastycznym, oddziaływującym jedynie na zmiany powierzchniowe i temperaturowe. Wykazano, że dla odkształceń termoplastycznych układ ten jest znacznie szerszy niż analogiczny układ dla materiałów idealnie-sprężystych. Zbadano warunki, przy których dany stan odkształcony termicznie, sprężyście i plastycznie może być zrelaksowany oraz przedyskutowano kilka zagadnień związanych z wyborem miar dla odkształceń termicznych, plastycznych i sprężystych.

В области больших деформаций рассматривается проблема определения системы возможных деформаций в каждом упруго-пластическом материале, воздействующим только на поверхностные и температурные изменения. Доказано, что для термо-пластических деформаций эта система значительно более широкая, чем аналогичная система для идеально-упругих материалов. Исследованы условия, при которых данное состояние термически, упруго и пластически деформированное может релаксировать, а также обсуждено несколько вопросов, связанных с выбором мер для термических, пластических и упругих деформаций.

1. Introduction

THIS PAPER deals with some topics connected with the thermodynamic theory of elastic-plastic continuous materials. We shall in no way restrict the magnitude of the deformations which such materials can undergo. To point out that we are also considering thermal effects we shall add the prefix thermo- to the above elastic-plastic qualification. We shall give special attention to the problem of determining the universal solutions of the equilibrium equations of this class of materials and thus the universal deformations. By universal solution of the equilibrium equations we mean a solution possible for all elastic-plastic bodies, that is a solution not depending on their constitutive features; this solution, moreover, must be relative to a compatible deformation and must be attainable, after application of surface forces only, by means of a process of purely elastic deformation which, from an initial state free from external forces but generally, thermally and plastically deformed, brings the body to a final deformed configuration. In the following sections we shall set forth the exact meaning of the above elastic, plastic and thermal deformations.

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Let us note in the meantime that, in accordance with what we have said, we can put every state of thermo-plastic deformation in correspondence with a family of universal deformations, and that the families of universal deformations generally depend on the thermo-plastic deformations taken into consideration. The universal deformations here introduced generalize the analogous deformations studied by ERICKSEN [1, 2] only for perfectly elastic materials. The former can be reduced to those of ERICKSEN when the thermo-plastic deformations vanish or, in any case, they can easily be deduced from them when the thermo-plastic deformations do not produce any stress in the material, external forces not being present. In this case, in fact, we can assume the thermo-plastically deformed configuration as a natural state, and referring to it, consider the material perfectly elastic, since the process it must undergo to reach a universal deformation involves only elastic deformations. Thermo-plastic effects play an important role when dealing with universal deformations. To recognize this fact let us consider for instance that, as shown in this paper, the universal deformations for an elastic-plastic continuum, elastically isotropic and compressible, are not reduced to the homogeneous deformations as in the case of isotropic compressible and perfectly elastic continua, but constitute a wider set of deformations. This is true of course only if we admit that the thermo-plastic deformations can be to a large extent arbitrary.

In what follows, we shall also consider the problem of determining the conditions under which a state of thermo-elastic-plastic deformation may be relaxed, that is, may undergo a transformation which, keeping the thermo-plastic deformations constant, brings the body to a state in which the elastic deformations, and therefore the stresses, vanish. Both the study of the universal deformations and that of the relaxed states have a practical significance. The former, as known, helps us to establish which states of deformation can be attained by any elastic-plastic body when surface forces (the only ones available in laboratory) act upon it, and thus helps us also to proceed towards a correct experimental determination of the material constants. Contrary to what happens to the perfectly elastic materials considered in [2], for our materials the experiments do not have to be restricted only to homogeneous deformations. The study of relaxed states, on the other hand, will enable us to establish whether the amount of elastic energy in a deformed body can be wholly transferred to the surrounding world or whether, on the contrary, not all of it can. In the latter case the body, released from external forces, will not be able to reach an unstressed state, unless changes occur in the thermo-plastic deformations.

In this paper we shall frequently make references to the general theory of elastic-plastic continua proposed by GREEN & NAGHDI in [5 and 7]. A remarkable feature of this theory is the use in the range of finite strain of a simple summation rule to decompose the total strain into elastic and plastic components. When studying certain problems and only when dealing with certain kinds of elastic-plastic materials, as an alternative to the theory of GREEN & NAGHDI we could adopt with profit other theories which make use of such measures of strain as to simplify the constitutive equations specific to the materials they study, but which generally lose the disposability of a summation rule for the elastic and plastic components of the deformation. As we shall recognize in the course of our study, such theories are practical only when they concern those materials for which they have been principally conceived. Moreover, they do not offer any special advantage when they are employed in problems about the whole class of elastic-plastic materials. In a similar occasion, on the contrary, the general approach as in [5 and 7] is of greater advantage.

Section 2 is about the description of the kinematical quantities and of the coordinate systems we shall use. We shall also establish and discuss some relations occurring among the components of some of the tensorial quantities introduced to describe the deformation of the body. These relations will then be used in Sec. 6. In Sec. 3 we take into consideration the question of the decomposition of the total strain. The thermal strain, in addition to the elastic and the plastic, is explicitly introduced in order to be able to treat, as far as possible, the thermal effects in the same way as we would treat the plastic ones. Moreover, we give the reasons why the additive decomposition rule adopted does not restrict the generality of the theory, and we point out a kinematical interpretation which, together with the appropriate constitutive hypotheses, helps to us justify the names elastic, plastic and thermal strains, which we use for the quantities representing the partial deformations.

The fundamental aspects of the general theory [5 and 7] will be summarized in Sec. 4. This theory will be slightly changed because of the explicit use of the thermal strain tensor. The consequences that the choice of the variables used to describe the deformation has on the form of the constitutive equations, will also be presented. In Sec. 5 we shall express, in terms of the variables which we have introduced, the conditions of compatibility for a state of thermo-elastic-plastic deformation. In this section, moreover, the conditions under which a deformed state can be relaxed will be laid down. Finally, in Sec. 6 we shall deduce the relations by which the universal deformations of elastic-plastic materials can be determined, and we shall show that the occurence of thermo-plastic deformations generally produces remarkable changes in the class of universal deformations, with respect to the case of perfectly elastic materials.

2. Some kinematical preliminaries

Let us consider a continuous body. To describe its state of deformation we have to make use of at least two systems of coordinates: the first is to label its material points, while the second is to assign to them a position in physical space. It will come in handy, however, to introduce further systems of coordinates, in addition to those strictly necessary, in order to be able to give to some quantities, fundamental in this paper, an interpretation by which we can more readily obtain the relations we are after.

In what follows, a bold-face letter will stand for a tensor or a vector. The same letter in light-face will be used as a kernel letter in component form notation. In this case the components of a tensor in different systems of coordinates will be generally distinguished by using different characters as indices without changing the kernel. We shall throughout adopt the summation convention, the device of distinguishing between covariant and controvariant components by means of the position of the indices, and all the other usual notations of tensorial calculus.

Let us, as usual, refer the motion of the body to a stationary spatial system of reference, not necessarily Cartesian, chosen in the three-dimensional Euclidean space where the body is. We shall indicate the coordinates of a generic point of this system by x^i, $(i = 1, 2, 3)$, or more concisely by $\mathbf{x}$, the base vectors by $\mathbf{b}_i$ and the metric tensor by $\mathbf{g}$. Let us suppose that at the initial instant the body is in a natural state at uniform temperature. Considering the body in this configuration, we shall indicate by $\mathbf{X}$ or X^L, $(L = 1, 2, 3)$, the coordinates of its points in a second system of reference, in general different from the first, which will be called a material coordinate system and whose metric tensor and base vectors will be indicated by $\mathring{\mathbf{g}}$ and $\mathring{\mathbf{b}}_L$, respectively. The deformation of the body at the generic instant t can thus be expressed as usual:

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t). \tag{2.1}$$

Let us now interpret the system of coordinates $\mathbf{X}$ as a system of convected coordinates. It will be useful ([1]) to focus our attention on three particular configurations which this system can assume after appropriate deformations in the body have occurred. The first configuration is that attained when the body, after a deformation process, assumes its final deformed state. In this configuration $\hat{\mathbf{b}}_L$ and $\hat{\mathbf{g}}$ will stand for the base vectors and the metric tensor of the system of convected coordinate system $\mathbf{X}$, respectively. On the other hand, the second and the third configuration of this system will be attained when the body undergoes two appropriate processes of deformation, not necessarily continuous for effectively attainable, which we precisely shall define after we have fully explained the meaning of elastic, plastic and thermal deformation. We shall call $\overset{*}{\mathbf{b}}_L$ and $\overset{*}{\mathbf{g}}$ respectively the base vectors and the metric tensor of the convected coordinate system $\mathbf{X}$ when it attains the second configuration, while we shall call $\bar{\mathbf{b}}_L$ and $\bar{\mathbf{g}}$ the analogous quantities relevant to the third configuration. Let us note at this point, that the systems of reference with base vectors $\overset{*}{\mathbf{b}}_L$ and $\bar{\mathbf{b}}_L$ might not be related to a Euclidean space, since we have admitted that the process we need to define them may not be continuous. On the contrary, the metric tensor $\hat{\mathbf{g}}$ is always related to a Euclidean space, because we are supposing the final configuration of the body attainable by means of a continuous deformation. By the way, let us also note that supposing $t = \hat{t}$, where $\hat{t}$ stands for the instant in which the final configuration is reached, (2.1) can be interpreted as a coordinate transformation from the system $\mathbf{X}$ to the system $\mathbf{x}$ ([2]). From what has just been said and from the law governing the change of tensorial components when the system of reference changes, it is a straightforward matter to deduce the relations between the components of our metric tensors in the system of reference $\mathbf{x}$ and those of the same tensors in the system of reference $\mathbf{X}$. For example, for the metric tensor $\hat{\mathbf{g}}$ we have ([3]):

$$\hat{g}_{LM} = {}^{\wedge}x^i{}_{,L}\, {}^{\wedge}x^l{}_{,M}\, \hat{g}_{il}, \tag{2.2}$$

([1]) Similarly to what has been done in [6, p. 230].

([2]) Since the two metric tensors $\hat{\mathbf{g}}$ and $\mathbf{g}$ are related to two Euclidean spaces, the relation $\mathbf{x} = \mathbf{x}(\mathbf{X}, \hat{t})$ which sets the two spaces in correspondence must be bi-univocal and bicontinuous; it can be thus interpreted, when necessary, as a transformation of coordinates in the space with metric tensor $\mathbf{g}$ (or $\hat{\mathbf{g}}$). Naturally, the components of $\mathbf{g}$ in the system $\mathbf{X}$ at instant $\hat{t}$ (or those of $\hat{\mathbf{g}}$ in the system $\mathbf{x}$) are different from the components of $\hat{\mathbf{g}}$ in $\mathbf{X}$ (or from the components of $\mathbf{g}$ in $\mathbf{x}$) unless at instant t we have $\mathbf{g} \equiv \hat{\mathbf{g}}$.

([3]) A comma and a semicolon before an index will always be used to indicate respectively the partial derivation and the covariant derivation with respect to the indices which follow them.

where we have set $x^i{}_{,Q} = (\partial x^i/\partial X^Q)$ and used the symbol ^ to specify that the quantities to which it is appended are calculated at the instant $t = \hat{t}$. Since the system of reference $\mathbf{X}$ can be arbitrarily chosen, an appropriate choice can make the metric tensor $\hat{\mathbf{g}}$ coincide at instant $\hat{t}$ with $\mathbf{g}$, that is $\hat{g}_{il} \equiv g_{il}$. Of course only slight changes are necessary to refer to $\mathring{\mathbf{g}}$ what has been previously referred to $\hat{\mathbf{g}}$. For instance, similarly to what has been said about $\hat{\mathbf{g}}$, we can choose the system $\mathbf{X}$ in such a way that $\mathring{\mathbf{g}} \equiv \mathbf{g}$ at instant $\mathring{t}$. It should be clear, however, that if we exclude the trivial case of rigid motions, only one of the tensors $\hat{\mathbf{g}}$ and $\mathring{\mathbf{g}}$ can be made to coincide with $\mathbf{g}$.

Let us now consider the process that takes the body from the initial undeformed to the final deformed configuration. Let ds be the distance which, after the deformation, is assumed by two points of the body, $d\mathring{s}$ being their elementary distance before the deformation; and let $\mathbf{c}$ be the Cauchy's deformation tensor. In the spatial system of reference we shall then have the well-known relation:

$$(ds)^2-(d\mathring{s})^2 = (g_{ij}-c_{ij})dx^i dx^j. \tag{2.3}$$

On the other hand, in the convected coordinate system of reference we have:

$$(ds)^2-(d\mathring{s})^2 = (\hat{g}_{LM}-\mathring{g}_{LM})dX^L dX^M. \tag{2.4}$$

If we consider the components of the tensors appearing in this relation as relative to the system of reference $\mathbf{X}$ at instant $\hat{t}$, that is to the system with metric tensor $\hat{\mathbf{g}}$ and base vectors $\hat{\mathbf{b}}_L$, and if we interpret the relation $\mathbf{x} = \mathbf{x}(\mathbf{X}, \hat{t})$ which determines the configuration of the body at instant $\hat{t}$ as a coordinate transformation from the system $\mathbf{X}$ at $\hat{t}$ to the system $\mathbf{x}$, we shall get

$$(ds)^2-(d\mathring{s})^2 = (\hat{g}_{LM}-\mathring{g}_{LM})\,{}^\wedge X^L{}_{,i}\,{}^\wedge X^M{}_{,j}\,dx^i dx^j, \tag{2.5}$$

where ${}^\wedge X^P{}_{,l}$ stands for the components of the inverse of the deformation gradient, as defined by the relation ${}^\wedge x^l{}_{,R}\,{}^\wedge X^P{}_{,l} = \delta^P{}_R$. By comparing (2.5) with (2.3) we shall have:

$$c_{ij} = g_{ij}-(\hat{g}_{LM}-\mathring{g}_{LM})\,{}^\wedge X^L{}_{,i}\,{}^\wedge X^M{}_{,j}. \tag{2.6}$$

In a similar way we can obtain various relations analogous to (2.6) but involving different kinds of tensorial components. Among these we shall quote:

$$c_i{}^j = g_i{}^j-(\hat{g}_L{}^M-\mathring{g}_L{}^M)\,{}^\wedge X^L{}_{,i}\,{}^\wedge X_M{}^{,j} \tag{2.7}$$

and

$$c^{ij} = g^{ij}-(\hat{g}^{LM}-\mathring{g}^{LM})\,{}^\wedge X_L{}^{,i}\,{}^\wedge X_M{}^{,j}; \tag{2.8}$$

other relations may be obtained from these simply by changing the position of the lower-case indices with the aid of the metric tensor $\mathbf{g}$. It is important to note that (2.6), (2.7) and (2.8) do not represent the expressions in component form of one and the same tensorial relation. In fact it can easily be observed that if, for instance, we wanted to obtain (2.7) or (2.8) from (2.6), we would have to raise or lower the indices of the tensors which appear in (2.6) by using different metric tensors, namely $\mathbf{g}$ for c_{ij} and g_{ij}, $\bar{\mathbf{g}}$ for $\bar{g}_{LM}$, $\hat{\mathbf{g}}$ for $\hat{g}_{LM}$, and finally $\hat{\mathbf{g}}$ and $\mathbf{g}$ for ${}^\wedge X^L{}_{,i}$. However, the Eqs. (2.6) and similar ones are independent from the choice of the system of convected coordinates $\mathbf{X}$. Furthermore, since in the system of reference with base vectors $\hat{\mathbf{b}}_L$ it is always possible to define three different tensors in such a way that the covariant components of the first, the mixed components of the second and the controvariant components of the third coincide with $\mathring{g}_{LM}$, $\mathring{g}_L{}^M$ and $\mathring{g}^{LM}$, respectively, we can also interpret the expressions (2.6), (2.7) and (2.8) as representing three component form expressions, in the system **X** at instant $\hat{t}$, of three different tensorial relations between **c** and the tensors just defined. Naturally each of these tensorial relations has a component form expression coinciding with one of (2.6), (2.7), (2.8), only when expressed by means of components of appropriate variance in their capital indices ([4]), whilst, if we express them by components of a different kind, they will assume a different and generally more involved form. At this point it is worth noting that having interpreted (2.6) and its analogous as deriving from appropriate tensorial relations, we are allowed to perform a covariant derivation of each side of them and thus we can easily obtain further expressions for the covariant derivatives of the components of **c**. Moreover, if instead of the total deformation we want to consider that which the body undergoes in passing from the configuration with base vectors $\overset{*}{\mathbf{b}}_L$ or $\bar{\mathbf{b}}_L$ to the final configuration, by replacing $\mathring{\mathbf{g}}$ with $\overset{*}{\mathbf{g}}$ or $\bar{\mathbf{g}}$ and the line element $d\mathring{s}$ with $d\overset{*}{s}$ or $d\bar{s}$, we can in this case, for the components of the Cauchy's deformation tensor relative to the process of partial deformation considered, easily obtain formulas similar to (2.6), (2.7) and (2.8). We shall have for example:

$$(2.9) \qquad (ds)^2-(d\bar{s})^2 = (g_{ij}-c'_{ij})\,dx^i\,dx^j,$$

and

$$(2.10) \qquad (ds)^2-(d\bar{s})^2 = (\hat{g}_{LM}-\bar{g}_{LM})\,{}^\wedge X^L{}_{,i}\,{}^\wedge X^M{}_{,j}\,dx^i\,dx^j.$$

In (2.9) c'_{ij} represent the components of the Cauchy deformation tensor relative to the purely elastic process which takes place from the instant $\bar{t}$ to the instant $\hat{t}$. From (2.9) and (2.10) we obtain:

$$(2.11) \qquad c'_{ij} = g_{ij}-(\hat{g}_{LM}-\bar{g}_{LM})\,{}^\wedge X^L{}_{,i}\,{}^\wedge X^M{}_{,j}.$$

Let us finally remember, for completeness, that if we restrict our attention to the material system of reference, at the initial instant $\mathring{t}$, and to the spatial one, and if we interpre $\mathbf{x} = \mathbf{x}(\mathbf{X}, \hat{t})$, that is the relation (2.1) calculated at $t = \hat{t}$, as a transformation of coordinates between these two systems of reference, the tensor **c**, relative to the total deformation, will be expressed in its classical form:

$$(2.12) \qquad \mathbf{c} = {}^\wedge\overset{-1}{\mathbf{F}}{}^T\,\mathring{\mathbf{g}}\,{}^\wedge\overset{-1}{\mathbf{F}},$$

where ${}^\wedge\mathbf{F}$ stands for the deformation gradient, whose components are ${}^\wedge x^i{}_{,L}$, the symbol^{-1} means inverse [and therefore $({}^\wedge\overset{-1}{\mathbf{F}})^L{}_i = {}^\wedge X^L{}_{,i}$], and the symbol T means transpose. In (2.12), contrary to the preceding formulas, the capital index of ${}^\wedge X^L{}_{,i}$ is related to the system of reference with base vectors $\mathring{\mathbf{b}}_L$, instead of that with base vector $\hat{\mathbf{b}}_L$.

In what follows we shall also use Green's deformation tensor C, defined by the relation:

$$(2.13) \qquad \mathbf{C} = {}^\wedge\mathbf{F}^T\,\mathbf{g}\,{}^\wedge\mathbf{F},$$

([4]) As far as the lower-case indices are concerned, (2.6), (2.7) and (2.8) constitute three different tensorial relations, and therefore the position of these indices can be supposed arbitrary, so long as it is tensorially consistent.

or, in component form, in the material system of reference at the initial instant $\mathring{t}$:

$$C_{LM} = g_{lm}\, {}^{\wedge}x^l{}_{,L}\, {}^{\wedge}x^m{}_{,M}, \tag{2.14}$$

where, as for (2.12), the position of the capital indices can be changed by means of the metric tensor $\mathring{\mathbf{g}}$. Even if we can lay down for the components of $\mathbf{C}$ expressions analogous to (2.6), (2.7) and (2.8), we shall refrain from doing so, since such relations will not be used in what follows. If we have to consider the Cauchy deformation tensors $\overset{*}{\mathbf{C}}$ or $\bar{\mathbf{C}}$ related to the state which the body attains after the deformation process which takes it from the initial state to the deformed one with base vectors $\overset{*}{\mathbf{b}}_L$ or $\bar{\mathbf{b}}_L$, we can easily obtain for $\overset{*}{C}$ and $\bar{C}$ formulas similar to (2.13), by replacing t with $\overset{*}{t}$, or with $\bar{t}$, respectively. For instance:

$$\overset{*}{C} = {}^*\mathbf{F}^T \mathbf{g}\, {}^*F, \tag{2.15}$$

or, in component form, in the material system of reference at instant $\mathring{t}$:

$$\overset{*}{C}_{LM} = g_{lm}\, {}^*x^l{}_{,L}\, {}^*x^m{}_{,M}. \tag{2.16}$$

We shall also make frequent use of the Green-St. Venant strain tensor $\mathbf{E}$, defined by the relation:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathring{\mathbf{g}}), \tag{2.17}$$

which in covariant components, in the material system of reference at the initial instant, is given by:

$$E_{KL} = \frac{1}{2}(C_{KL} - \mathring{g}_{KL}). \tag{2.18}$$

Here again, if we want to consider only a part of the total deformation, for example that which takes the body from the initial state to that with base vectors $\overset{*}{\mathbf{b}}_L$, the relative tensor $\overset{*}{\mathbf{E}}$ can be immediately calculated by replacing $\mathbf{C}$ with $\overset{*}{\mathbf{C}}$ in (2.17).

3. Partial deformations associated with a given total deformation

To describe the state of deformation of a body, we shall use the strain tensor $\mathbf{E}$, as in the Green & Naghdi theory of elastic-plastic continua. It will be worth considering the decomposition:

$$\mathbf{E} = \mathbf{E}' + \mathbf{E}'' + \mathbf{E}''', \tag{3.1}$$

and calling the quantities $\mathbf{E}'$, $\mathbf{E}''$ and $\mathbf{E}'''$ tensor of elastic strain, tensor of plastic strain and tensor of thermal strain, respectively. These terms will be justified after having laid down the constitutive hypotheses for $\mathbf{E}'$, $\mathbf{E}''$ nad $\mathbf{E}'''$. In (3.1), contrary to what happens in the analogous decomposition in [5 and 7], we have introduced the tensor $\mathbf{E}'''$. In this way we have explicitly distinguished the thermal strain from the elastic one, without using, therefore, only the tensor of thermo-elastic strain. We can adduce two reasons for doing this. The first is that, as will subsequently be shown, the values of the total strain tensor in correspondence to a particular unstressed but thermically and plastically deformed state, play a fundamental role in the problems we are dealing with. This is the state imaginarily reached by the body before a purely elastic process (the only one that gives rise to stresses in the material) takes it to the final deformed state. The second reason is that, referring to the problems considered in this paper, it makes no difference whether the above mentioned unstressed state is reached by means of a thermal deformation, a plastic deformation, or a thermo-plastic deformation. In this respect plastic and thermal deformations are equivalent. It is then clear that we must keep the tensor of thermal deformation separate from that of elastic deformation, if, for a more immediate and unitary study, we want to treat, even from the analytical point of view, thermal deformations in the same way as plastic ones.

The device of introducing tensors tied with states of partial deformation, together with, or instead of, the tensor of total strain, is frequently used in the theories concerning elastic-plastic materials, principally to make the relations expressing their constitutive characteristics more workable. Naturally in order to measure deformations we cannot only use any one of the many possible definitions of strain but we can also change the definition according to the partial deformation considered. Each definition we choose may have its particular advantages towards obtaining a simple expression of the constitutive equations. Generally speaking, however, the quantities chosen to describe the deformation are not such to obey an additive rule (5) of the kind (3.1). Of course, the tensors $\mathbf{E}'$, $\mathbf{E}''$ and $\mathbf{E}'''$ in (3.1) are defined in the same way as $\mathbf{E}$, that is they are all Green-St. Venant strain tensors referred to the same reference configuration. In view of the frequent discussions as to whether a summation rule for the tensors measuring partial deformations is valid or not in the range of large deformations, we had better point out that the formal decomposition (3.1) can always be assumed. In fact, since $\mathbf{E}$ is a tensor, it can be considered as an element of a vectorial space which consists of all the tensors of the same rank as $\mathbf{E}$, defined in the same way as $\mathbf{E}$ but relative to different deformations. Therefore it can always be interpreted as composed of the sum of two or more tensors belonging to the same space.

From what has been said, it should be clear that the quantities $\mathbf{E}$, $\mathbf{E}'$, $\mathbf{E}''$ and $\mathbf{E}'''$ represent different processes which start from the same undeformed state and generally lead to different deformed configurations. On the other hand, it is often profitable to consider the total deformation as consequent to particular processes of partial deformation undergone by the body at successive instants. It is therefore better to introduce deformation tensors that are referred to deformed and generally different configurations. This is the case, for instance, when we consider the total deformation as produced by two ideal processes, the first of which starts from the initial state and generates only plastic deformations in the body, while the second, applied to the body plastically deformed by the preceding process, generates only elastic deformations and leads to the final configuration. Generalizing the observation attributed by GREEN & NAGHDI [8] to FOX, we shall now show that $\mathbf{E}'$, $\mathbf{E}''$

(5) Sometimes a summation rule is valid only for components of appropriate variance of the adopted strain tensors, cf. [3], whereas (3.1), being a tensorial relation, is valid for every kind of component.

and $\mathbf{E}'''$ can be interpreted ([6]) in the convected coordinate system $\mathbf{X}$ as strain tensors relative to three successive configurations of a deformation process that takes the body from the initial state to the final one, passing through two intermediate states which have respectively $\mathbf{E}''$ and $\mathbf{E}''+\mathbf{E}'''$ as the Green-St. Venant strain tensors referred to the initial state. To this end we shall refer to the convected coordinate system $\mathbf{X}$, whose metric tensor $\mathring{\mathbf{g}}$ at the initial instant $\mathring{t}$ will be from now on also indicated by $\mathbf{1}$, and we shall suppose that the body undergoes a process such that the metric tensor of the system $\mathbf{X}$ will successively assume the values:

$$\overset{*}{\mathbf{g}} = 2\mathbf{E}''+\mathbf{1} \tag{3.2}$$

at instant $\overset{*}{t}$,

$$\bar{\mathbf{g}} = 2(\mathbf{E}''+\mathbf{E}''')+\mathbf{1} \tag{3.3}$$

at instant $\bar{t}$, and

$$\hat{\mathbf{g}} = 2(\mathbf{E}'+\mathbf{E}''+\mathbf{E}''')+\mathbf{1} = 2\mathbf{E}+\mathbf{1} \tag{3.4}$$

at the final instant $\hat{t}$. We can see that in the process just imagined, the configuration reached by the body at the final instant coincides with that which it actually reaches after the real process. In fact, keeping in mind the meaning of Green's deformation tensor as well as the definition (2.17), we can deduce that, because of the real deformation process, a line element $d\mathring{s} = d\mathbf{X}^T\mathring{\mathbf{g}}d\mathbf{X}$ relative to the undeformed configuration becomes:

$$ds = d\mathbf{X}^T\,\mathbf{C}\,d\mathbf{X} = d\mathbf{X}^T(2\mathbf{E}+\mathbf{1})d\mathbf{X} = d\mathbf{X}^T\,\hat{\mathbf{g}}\,d\mathbf{X} \tag{3.5}$$

in the deformed configuration. Let us also note that, as far as the imaginary process is concerned, the deformation tensors in the convected coordinate system relative to the instants $\overset{*}{t}$, $\bar{t}$ and $\hat{t}$, and calculated by using as reference configurations the configurations attained by the body at instants $\mathring{t}$, $\overset{*}{t}$ and $\bar{t}$ respectively, coincide with the quantities $2\mathbf{E}''$, $2\mathbf{E}'''$ and $2\mathbf{E}'$, respectively. Thus, by using for instance covariant components ([7]), we get:

$$\overset{*}{g}_{KL}-\mathring{g}_{KL} = 2E''_{KL}, \tag{3.6}$$

$$\bar{g}_{KL}-\overset{*}{g}_{KL} = 2E'''_{KL} \tag{3.7}$$

and

$$\hat{g}_{KL}-\bar{g}_{KL} = 2E'_{KL}. \tag{3.8}$$

([6]) This interpretation will be a fundamental aid in what follows and, together with the constitutive hypotheses of the following section, will help us to justify the terms elastic, plastic and thermal deformations adopted respectively for $\mathbf{E}'$, $\mathbf{E}''$ and $\mathbf{E}'''$.

([7]) We observe that, since (3.2), (3.3) and (3.4) are valid in the convected coordinate system $\mathbf{X}$, their expressions (3.6), (3.7) and (3.8) in component form do not correspond with that of three tensorial relations in a stationary system of reference. They are to be interpreted as non-tensorial relations between the covariant components of the quantities which appear in them. For this reason we cannot use only one metric tensor to raise their indices and thus to obtain from (3.6), (3.7) and (3.8) analogous relations involving mixed and controvariant components. See also what has already been said on this point when dealing with (2.6), (2.7) and (2.8) in Sec. 2.

Since in passing from the configuration at instant $\bar{t}$ to that at the final instant $\hat{t}$ we can suppose $\mathbf{E}''$ and $\mathbf{E}'''$ to be constant and equal to the values they assume in (3.1), it follows from (3.8) that we can interpret the Green-St. Venant strain tensor $\mathbf{E}'$, to within an inessential multiplying factor, as a strain tensor in the convected coordinate system. This tensor is relative to the purely elastic ideal process that occurs between instant $\bar{t}$ and instant $\hat{t}$, and is calculated with respect to the state reached by the body when its plastic and thermal strain tensors assume values equal to those actually assumed in the final deformed state. By means of (3.6) and (3.7) we can easily give similar interpretations to the strain tensors $\mathbf{E}''$ and $\mathbf{E}'''$, considering the processes that occur from instants $\mathring{t}$ to $\overset{*}{t}$, and from $\overset{*}{t}$ to $\bar{t}$, respectively.

As already observed, the deformed states imagined at instants $\overset{*}{t}$ and $\bar{t}$ may not be reached by the body by means of a continuous deformation, not only because they could correspond to a non-compatible deformation, but also because they could produce values of $\mathbf{E}''$ and $\mathbf{E}'''$ that, in an unstressed state, cannot be attained by the body without violating the constitutive hypotheses made for it. Even if the latter occurrence does not formally invalidate our results, it will nevertheless be excluded in order to give easier interpretations. On the contrary, we shall admit that the deformed states at instants $\overset{*}{t}$ and $\bar{t}$ can be non-compatible, even if we conclude, in view of the usual assumptions of continuity for the real deformation of the body, that they can actually be reached.

4. Results of a general theory of elastic-plastic materials

In this section we shall summarize the principal results of the general theory of elastic-plastic continua proposed by GREEN & NAGHDI [5, 7], and we shall introduce in this theory some slight changes due to our explicit use of the thermal strain tensor $\mathbf{E}'''$. Even if in the following sections we shall not make use of all the relations laid down in this section, they are nevertheless reported here for completeness.

Let us first of all focus our attention on the constitutive hypotheses for the strain tensor $\mathbf{E}'''$. These do not appear in the Green and Naghdi theory, because in this theory the thermal strain tensor is included in the thermo-elastic strain tensor. We shall call ϑ the temperature and we shall suppose that $\mathbf{E}'''$ is given by a relation of the kind:

$$\mathbf{E}''' = \mathbf{E}'''(\mathbf{E}'', \vartheta, \mathbf{X}, \mathring{\mathbf{b}}_K), \tag{4.1}$$

where the dependence on $\mathbf{X}$ takes into account the eventual non-homogeneity of the material, and that on the base vectors $\mathring{\mathbf{b}}_K$, here used as material descriptors, takes the eventual anisotropies. In (4.1) $\mathbf{E}''$ appears among the independent variables, but $\mathbf{E}'$ does not. The presence of $\mathbf{E}''$ is not surprising because in the range of large deformations the constitutive expression for a particular strain depends in general on the values assumed by the other partial strains which, in addition to it, are necessary to determine the total strain ([8]). This observation, however, should be applicable also to $\mathbf{E}'$, and there-

([8]) Naturally (4.1) must be such that when ϑ coincides with the temperature of the initial undeformed state, $\mathbf{E}'''$ vanishes whatever the value assumed by E''.

fore also $\mathbf{E}'$ should appear among the independent variables in the above expression for $\mathbf{E}'''$. If it doesn't, it is because by (4.1) we have implicity restricted the possible constitutive expressions for $\mathbf{E}'''$ ([9]). This restriction is helpful because, thanks to it, we can allow the body to undergo processes, like those imagined in the last section between instants $\bar{t}$ and $\hat{t}$, where only $\mathbf{E}'$ varies, while $\mathbf{E}''$ and $\mathbf{E}'''$ are kept constant. Our strain measures can be criticized as rendering the constitutive equations too complicated, and in fact (4.1) for instance, may be said to have a rather cumbrous form. Similar objections can, however, be dismissed merely by noting that for our study we shall find the decomposition rule (3.1) useful, while we shall obtain no advantage from a simple form of the constitutive relations.

Let us now consider the plastic strain tensor ([10]). By $\mathbf{S}$ we shall indicate the symmetric Piola stress tensor, tied with Cauchy's stress tensor $\mathbf{t}$ by means of:

$$\mathbf{S} = \mathbf{S}^T = (\det\mathbf{F})\overset{-1}{\mathbf{F}}\mathbf{t}\overset{-1}{\mathbf{F}}{}^T; \tag{4.2}$$

and we shall introduce the loading surface:

$$f(S, \mathrm{E}'', \vartheta) = k, \tag{4.3}$$

where k is a scalar function which depends on the whole story of the motion of the body, and whose time rate we can suppose to be expressed by:

$$\dot{k} = k^{KL}(S, E'', \vartheta)\dot{E}''_{KL} = \mathrm{tr}\,\{\mathbf{h}\dot{\mathbf{E}}''\} \tag{4.4}$$

(a dot above a quantity indicates the material time derivation). We admit that the constitutive law for the plastic strain rate is given by ([11]):

$$\dot{\mathbf{E}}'' = \mathbf{G}(\mathbf{S}, \dot{\mathbf{S}}, k, \mathbf{E}'', \vartheta, \dot{\vartheta}), \tag{4.5}$$

with the condition that the Eqs. (4.4) and (4.5) are valid when the conditions:

$$\begin{cases} f = k \\ \mathrm{tr}\left\{\dfrac{\partial f}{\partial \mathbf{S}}\dot{\mathbf{S}}\right\} + \dfrac{\partial f}{\partial \vartheta}\dot{\vartheta} \geqslant 0 \end{cases} \tag{4.6}$$

are satisfied, while when (4.6) are not satisfied we have:

$$\dot{k} = 0 \tag{4.7}$$

and

$$\dot{\mathbf{E}}'' = 0 \tag{4.8}$$

instead of (4.4) and (4.5).

As for the elastic strain tensor, we shall admit that it can be expressed by:

$$\mathbf{E}' = \mathbf{E}'(\mathbf{S}, \mathbf{E}'', k, \vartheta), \tag{4.9}$$

([9]) Note that this restriction does not set any limitation to the family of elastic-plastic materials here considered. In fact, it does not restrict the constitutive form of $\mathbf{E}$, because, as will be seen, $\mathbf{E}'$ can assume a very general form.

([10]) For simplicity, in what follows we shall suppose this tensor symmetric.

([11]) In (4.5) as well as in (4.1) $\mathbf{E}'$ does not explicitly appear among the independent variables. For this reason we can repeat here observations similar to those made for (4.1) above.

which we shall suppose smoothly invertible with respect to $\mathbf{S}$, so that it will have the unique inverse:

$$\mathbf{S} = \mathbf{S}(\mathbf{E}', \mathbf{E}'', k, \vartheta). \tag{4.10}$$

Indicating by ψ, η and $\mathbf{q}_R$ respectively the specific Helmholtz free energy, the specific entropy and the heat flux vector measured per unit of time and unit of area of the body in the undeformed configuration, we shall finally admit that they have the following form:

$$\psi = \psi(\mathbf{E}', \mathbf{E}'', k, \vartheta), \tag{4.11}$$

$$\eta = \eta(\mathbf{E}', \mathbf{E}'', k, \vartheta), \tag{4.12}$$

and

$$\mathbf{q}_R = \mathbf{q}_R(\mathbf{E}', \mathbf{E}'', k, \vartheta, \nabla\vartheta), \tag{4.13}$$

where ∇ means gradient with respect to the material coordinates $\mathbf{X}$. Let us note that in all the constitutive relations so far expressed, the tensor $\mathbf{E}'''$ does not appear among the independent variables. The reason is that by means of (4.1) we can eliminate it and replace it by $\mathbf{E}''$ and ϑ. The same cannot be done for $\mathbf{E}''$. In fact $\mathbf{E}''$ is defined by the nonholonomic relation (4.5) and therefore cannot be eliminated by the group of variables that are strictly necessary to determine the state of the body.

Following a procedure analogous to that in [5 and 7], we can express respectively the first and the second principle of thermodynamics in the form:

$$\varrho_0 r - \operatorname{tr}\left\{\varrho_0\left(\frac{\partial\psi}{\partial\mathbf{E}'}\dot{\mathbf{E}}' + \frac{\partial\psi}{\partial\mathbf{E}''}\dot{\mathbf{E}}'' + \frac{\partial\psi}{\partial k}\mathbf{h}\dot{\mathbf{E}}''\right)\right\} + \varrho_0\left(\frac{\partial\psi}{\partial\vartheta} + \eta\right)\dot{\vartheta} + \varrho_0\dot{\eta}\vartheta - \operatorname{div}\mathbf{q}_R + \operatorname{tr}\{\mathbf{S}(\dot{\mathbf{E}}' + \dot{\mathbf{E}}'' + \dot{\mathbf{E}}''')\} = 0 \tag{4.14}$$

and

$$\operatorname{tr}\left\{\left(\mathbf{S} - \varrho_0\frac{\partial\psi}{\partial\mathbf{E}'}\right)\dot{\mathbf{E}}'\right\} + \operatorname{tr}\left\{\left[\mathbf{S} - \varrho_0\left(\frac{\partial\psi}{\partial\mathbf{E}''} - \frac{\partial\psi}{\partial k}\mathbf{h}\right)\right]\dot{\mathbf{E}}''\right\} + \operatorname{tr}\{\mathbf{S}\dot{\mathbf{E}}''\} - \varrho_0\left(\frac{\partial\psi}{\partial\vartheta} + \eta\right)\dot{\vartheta} - \operatorname{tr}\left\{\frac{1}{\vartheta}(\mathbf{q}_R\nabla\vartheta)\right\} \geqslant 0, \tag{4.15}$$

where ϱ_0 stands for the mass density of the body in the initial configuration and r for the heat supply function per unit of mass. From the second principle we can deduce the relations:

$$\mathbf{S} = \varrho_0\frac{\partial\psi}{\partial\mathbf{E}'}, \tag{4.16}$$

$$S = \frac{\partial\psi}{\partial\vartheta} + \operatorname{tr}\{\mathbf{SD}\},$$

and

$$-\operatorname{tr}\{\mathbf{q}_R\nabla\vartheta) \geqslant 0. \tag{4.18}$$

In (4.17) **D** indicates the quantity:

$$\mathbf{D} = \frac{\partial \mathbf{E}'''}{\partial \vartheta}, \tag{4.19}$$

therefore $\mathbf{D}\dot{\vartheta}$ represents the value $\dot{\mathbf{E}}'''$ assumed when $\dot{\mathbf{E}}' = \dot{\mathbf{E}}'' = 0$ ([12]). Because of the introduction of the tensor $\mathbf{E}'''$ in our study, in the right side of (4.17) we have the term $\mathrm{tr}\{\mathbf{SD}\}$ which represents explicitly that part of entropy related to the thermal deformation of the material. Let us finally note that (4.16) justifies the name elastic deformation adopted for $\mathbf{E}'$.

5. Compatibility conditions and existence of relaxed states

Let us now consider a generic state of deformation determined by the tensors $\mathbf{E}'$, $\mathbf{E}''$ and $\mathbf{E}'''$. We intend to find the conditions that $\mathbf{E}''$ and $\mathbf{E}'''$ have to fulfil in order that the states of deformation in which $\mathbf{E}$ coincides with $\mathbf{E}''$ or with $\mathbf{E}'''$ or with $\mathbf{E}''+\mathbf{E}'''$ be compatible. Of course, such states of deformation are unstressed, since their elastic deformation vanishes; for this reason they will be called relaxed states. If, to the conditions of compatibility, we add the condition that in hyperspace $\{\mathbf{S}, \vartheta\}$ the loading surfaces (4.3) have to contain within them the point determined by $\mathbf{S}$ equal to $\mathbf{0}$ and by ϑ equal to the value required to produce the thermal deformation which is taken into consideration, then the above states of partial deformation can actually be reached by the body by means of a continuous process. From a deformed state in which $\mathbf{E} = \mathbf{E}'+\mathbf{E}''+\mathbf{E}'''$, this process brings the body to another deformed state in which $\mathbf{E} = \mathbf{E}''$ (or $\mathbf{E} = \mathbf{E}''+\mathbf{E}'''$) without varying the plastic deformations (or both the plastic and the thermal deformations). In such circumstances we shall say that the body can be relaxed.

Let us now express the conditions of compatibility for the state of total deformation by using the tensors $\mathbf{E}'$, $\mathbf{E}''$ and $\mathbf{E}'''$. Indicating the Riemann-Christoffel tensor relative to a generic second rank tensor $\mathbf{A}$ by $\mathbf{R}^{(\mathbf{A})}$, the conditions of compatibility expressed in the material system of reference at the initial instant are well known ([13]):

$$R^{(\mathbf{C})}_{KMPQ} = 0, \tag{5.1}$$

where $\mathbf{C}$ is the Green deformation tensor relative to the total deformation. By (2.17) and (3.1), we can express (5.1) in the form:

$$R^{(2\mathbf{E}'+2\mathbf{E}''+2\mathbf{E}''')}_{KMPQ} = 0. \tag{5.2}$$

If we introduce the quantities ([14]):

$$\overset{\mathbf{A}}{\Gamma}_{LPQ} = \frac{1}{2}(A_{LQ;P}+A_{PQ;L}-A_{LP;Q}), \tag{5.3}$$

([12]) **0** indicates the null tensor.

([13]) Cf. [4, p. 272].

([14]) We suppose that the covariant derivation is here based upon $\mathring{g}$.

we can express (5.1) or (5.2) in unfolded form:

$$(5.4)\quad R^{(C)}_{KMPQ} = E'_{KP;MQ}+E''_{KM;PQ}+E'''_{KM;PQ}+E'_{PQ;KM}+E''_{PQ;KM}+E'''_{PQ;KM} -E'_{KP;MQ}-E''_{KP;MQ}-E'''_{KP;MQ}-E'_{MQ;KP}-E''_{MQ;KP}-E'''_{MQ;KP} +\overset{-1}{C}{}^{RS}(\overset{2E'}{\Gamma}_{KMR}+\overset{2E''}{\Gamma}_{KMR}+\overset{2E'''}{\Gamma}_{KMR})(\overset{2E'}{\Gamma}_{PQS}+\overset{2E''}{\Gamma}_{PQS}+\overset{2E'''}{\Gamma}_{PQS})-(\overset{2E'}{\Gamma}_{MQR}+\overset{2E''}{\Gamma}_{MQR} +\overset{2E'''}{\Gamma}_{MQR})(\overset{2E'}{\Gamma}_{PKS}+\overset{2E''}{\Gamma}_{PKS}+\overset{2E'''}{\Gamma}_{PKS}) = 0.$$

Of these relations only the six which correspond to the sets of values: 1212, 1313, 2323, 1213, 2123, 3132, assumed respectively by the indices $KMPQ$, are independent. Incidentally, let us now consider the conditions of compatibility relative to three states of deformation, all three referred to the same initial undeformed state and characterized respectively by the values $\mathbf{E}'$, $\mathbf{E}''$ and $\mathbf{E}'''$ of the Green-St. Venant strain tensor. The conditions of compatibility for these deformations can be obtained from (5.4) by setting respectively $\mathbf{E}'' = \mathbf{E}''' = \mathbf{0}$ and $\mathbf{C} = 2\mathbf{E}'+\mathbf{1}$, or $\mathbf{E}' = \mathbf{E}''' = \mathbf{0}$ and $\mathbf{C} = 2\mathbf{E}''+\mathbf{1}$, or $\mathbf{E}' = \mathbf{E}'' = \mathbf{0}$ and $\mathbf{C} = 2\mathbf{E}'''+\mathbf{1}$. Since (5.4) is a non-linear relation, it may be satisfied even if these three partial deformations are not compatible. Moreover, even if one or more are compatible, it does not necessarily follow that the total deformation is compatible too.

In this paper we are interested in those particular states of partial deformation associated with a given state of total deformation, and relative to that deformation of the body which coincides with the plastic part of the total deformation or with the sum of the plastic part and the thermal part ([15]). By $\overset{*}{\mathbf{C}}$ and $\bar{\mathbf{C}}$ we shall indicate the Green deformation tensors relative to these states, and from now on, $\overset{*}{\mathbf{g}}$ and $\bar{\mathbf{g}}$ will stand for the corresponding values undertaken by the metric tensor of the system of convected coordinates $\mathbf{X}$. Naturally we shall have:

$$(5.5)\qquad \overset{*}{\mathbf{C}} = 2\mathbf{E}''+\mathbf{1}$$

and

$$(5.6)\qquad \bar{\mathbf{C}} = 2\mathbf{E}''+2\mathbf{E}'''+\mathbf{1}.$$

The conditions of compatibility for these states of partial deformation will therefore be:

$$(5.7)\qquad \mathbf{R}^{(\overset{*}{\mathbf{C}})} = \mathbf{0}$$

and

$$(5.8)\qquad \mathbf{R}^{(\bar{\mathbf{C}})} = \mathbf{0}.$$

We can obtain an expression in component form of these relations directly from (5.4) by replacing $\mathbf{C}$ with $\overset{*}{\mathbf{C}}$ or $\bar{\mathbf{C}}$, and by setting $\mathbf{E}'$ and $\mathbf{E}''$ or only $\mathbf{E}'$ equal to zero, depending on whether we are referring to (5.7) or to (5.8). We can thus obtain, for example, this expression of (5.7) in component form:

$$(5.9)\qquad E''_{KM;PQ}+E''_{PQ;KM}-E''_{KP;MQ}-E''_{MQ;KP}+\overset{-1}{\overset{*}{C}}{}^{RS}(\overset{2E''}{\Gamma}_{KMR}\overset{2E''}{\Gamma}_{PSQ}-\overset{2E''}{\Gamma}_{MQR}\overset{2E''}{\Gamma}_{PKS}) = 0.$$

([15]) For simplicity, we shall not consider explicitly only the thermally deformed state. This will be considered as a particular case of a thermo-plastically deformed state, when the plastic deformations vanish.

6. Universal deformations for elastic-plastic materials under given thermo-plastic deformation

As stated in the introduction, by universal deformations of elastic-plastic continua we mean those states of deformation which can be attained by any elastic-plastic material by means of a process of deformation which is due to the application of surface forces only, and which must not produce any changes in the thermal and in the plastic strain tensors $\mathbf{E}'$ and $\mathbf{E}''$. ERICKSEN [1, 2] dealt with this problem for hyperelastic, isotropic, incompressible or compressible materials and showed that for compressible materials the only universal deformations are the homogeneous ones. In this section we shall consider the same problem for the family of thermo-elastic-plastic materials, and we shall show that if we allow them to undergo suitable thermo-plastic deformations, their set of universal deformations will not be reduced only to the homogeneous deformations.

The conditions that have to be met by the variables describing the configuration of the body, so that the deformation they represent is universal, are: (i) those obtained by imposing that, for any material belonging to the family considered, the equilibrium equations which are related to the state of deformation that the body attains owing to the application of surface forces only, are satisfied; (ii) those obtained by imposing that the deformed configuration of the body is compatible. For isotropic, hyperelastic and compressible materials ERICKSEN [2] found that the condition (i) can be expressed by the relations:

(6.1) $$c^{ij}{}_{;j} = 0$$

and

(6.2) $$c^{j}{}_{j;i} = 0.$$

Let us note that since these relations are obtained by using only the equilibrium equations, they are valid also if referred to incompatible deformations, and moreover that, as follows from (4.16), the materials considered in this paper can be interpreted as hyperelastic materials, provided that the plastic and thermal deformations are kept constant. We can, therefore, deduce that, if we suppose the constitutive relation (4.16) to be isotropic, the condition (i) relative to the thermo-elastic-plastic materials can be directly obtained from (6.1) and (6.2), by replacing $\mathbf{c}$ with the tensor $\mathbf{c}'$ which is relative to the elastic partial deformation, generally non-compatible, occurring between the thermo-plastically deformed state and the final state. Since to describe the state of deformation of a thermo-elastic-plastic material we have so far mostly used Green-St. Venant strain tensors, we had better express the conditions (i) in terms of $\mathbf{E}'$ instead of $\mathbf{c}'$. To this purpose let us note that by means of (2.11) and (3.8) or their analogous expressions in controvariant and mixed components, we can represent the components of $\mathbf{c}'$ in terms of those of $\mathbf{E}'$, in any one of the following terms:

(6.3) $$c'_{ij} = g_{ij} - E'_{LM}\,{}^{\wedge}X^{L}{}_{,i}\,{}^{\wedge}X^{M}{}_{,j},$$

(6.4) $$c'_{ij} = g_{ij} - E'^{L}{}_{M}\,{}^{\wedge}X_{L,i}\,{}^{\wedge}X^{M}{}_{,j},$$

(6.5) $$c'_{ij} = g_{ij} - E'^{LM}\,{}^{\wedge}X_{L,i}\,{}^{\wedge}X_{M,j}.$$

Once again let us point out that (6.3)–(6.5) are three different relations between components, that cannot be deduced from only one tensorial equation ([16]) and also that, as far as the lower-case indices are concerned, each of these relations can be considered as the expression in component form of a tensorial relation (these indices, in fact, can be raised or lowered with the help of the metric tensor **g**). From (6.3)–(6.5) by means of covariant derivation ([17]) in the spatial system of reference and by means of an inessential change in the position of the lower-case indices, we have:

$$c'_{il;j} = -2(E'_{AL}\,{}^{\wedge}X^{A}{}_{,i}\,{}^{\wedge}X^{L}{}_{,l})_{;j}, \tag{6.6}$$

$$c'^{i}{}_{l;j} = -2(E'^{A}{}_{L}\,{}^{\wedge}X_{A}{}^{,i}\,{}^{\wedge}X^{L}{}_{,l})_{;j}, \tag{6.7}$$

$$c'^{il}{}_{;j} = -2(E'^{AL}\,{}^{\wedge}X_{A}{}^{,i}\,{}^{\wedge}X_{L}{}^{,l})_{;j}. \tag{6.8}$$

Thus the analogous expressions of (6.1) and (6.2), relative to the elastic partial deformation and expressed in terms of **E**′, are respectively:

$$(E'^{AL}\,{}^{\wedge}X_{A}{}^{,i}\,{}^{\wedge}X_{L}{}^{,j})_{;j} = 0 \tag{6.9}$$

$$(E'^{A}{}_{L}\,{}^{\wedge}X_{A}{}^{,j}\,{}^{\wedge}X^{L}{}_{,j})_{,i} = 0. \tag{6.10}$$

If we choose the system of convected coordinates in such a way that its final deformed configuration with base vectors $\hat{\mathbf{b}}_L$ coincides with the spatial system of reference and if the latter is orthogonal, then (6.9) and (6.10) assume the simpler form:

$$\delta^{j}{}_{L}\,E'^{AL}{}_{;j} = 0 \tag{6.11}$$

and

$$E'^{A}{}_{A;i} = 0, \tag{6.12}$$

respectively. The condition (i) is thus represented by (6.9) and (6.10), or by the equivalent (6.11) and (6.12). If **E**′ satisfies these relations, then the stresses generated in the material by the deformation can be sustained by applying only surface forces to the body. Of course the total deformation will be compatible only if we can verify the Eqs. (5.4) which, therefore, constitute the condition (ii) for our family of materials. In addition to the conditions (i) and (ii) thus established, we shall have to introduce a third condition which ensures that the state of deformation of the body will be compatible with its constitutive characteristics. This condition (iii) can explicitly be obtained by imposing that the strain **E**′ or, if we prefer, the state of stress that follows this strain according to (4.16), will not cause any plastic deformation in addition to that represented by **E**″ which is supposed assigned. In view of the constitutive relations for **E**″, laid down in Sec. 4, the condition (iii) is met if we impose that in every point of the body we shall have:

$$f(\mathbf{S}, \mathbf{E}'', \vartheta) \leqslant k. \tag{6.13}$$

For assigned thermo-plastic deformations, (6.9) and (6.10) (or equivalents) together with (5.4) and (6.13) help us to choose, out of the possible elastic deformations, those that render the total deformation universal. The conditions for the elastic strain tensor, which can thus be obtained, can be immediately expressed by means of (3.1), in terms of

([16]) In fact the position of the capital indices of **E**′ can be varied by means of the metric tensor $\mathring{\mathbf{g}}$, while the position of the capital indices of ${}^{\wedge}X^{M}{}_{,l}$ by means of the metric tensor $\hat{\mathbf{g}}$.

([17]) In Sec. 2 we have already discussed the possibility of deriving covariantly this kind of relations between components.

the total deformation tensor **E**. As we can easily see by inspection of (5.4), the difficulties that are actually met with in determining the universal deformations, depend crucially on the form of the strain tensors $\mathbf{E}''$ and $\mathbf{E}'''$. Let us note incidentally, that if the state of deformation defined by $\mathbf{E}''+\mathbf{E}'''$ is homogeneous or, in particular, if $\mathbf{E}''+\mathbf{E}'''$ vanishes everywhere, our problem will coincide with that of ERICKSEN [2] [apart from the condition (6.13) which, however, we can suppose to be satisfied in practice, so long as the elastic deformations are not too large]. Since the non-homogeneous thermo-plastic deformations cause changes in the form of the equations of compatibility, with respect to the case in which such deformations are homogeneous or null, it is clear that they cause changes also in the family of universal deformations. For some practical solutions of this problem (cf. [9]). It should be observed, moreover, that we only have to make slight and obvious changes to our approach if we want to restrict it to the thermo-elastic or the elastic-plastic deformations, instead of to the more general thermo-elastic-plastic deformations. Finally, let us note that the relations established in this section can be used in the study of many analogous problems, among which that of determining the plastic or the thermal strains (and therefore also the field of temperatures in the body) that are to be added to a given state of elastic deformation in order to make the total deformation universal or, on the other hand, that of determining the plastic or the thermal strain necessary to render universal a given state of deformation ([18]). A remarkable simplification which will come in handy in the latter case is that we can avoid using the conditions of compatibility (5.4), which in fact are identically satisfied because the assigned total deformation is supposed to be compatible.

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ISTITUTO DI SCIENZA DELLE COSTRUZIONI
UNIVERSITÁ DI CAGLIARI, ITALY.

([18]) Of course, if we introduce the hypothesis that the plastic deformations are isochoric, in order to determine $\mathbf{E}''$ we must add a relation that ensures that this hypothesis is fulfilled.