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On uniqueness and stability of elastic-plastic deformation (*)

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THE CONTINUUM theory for uniqueness and stability laid down by HILL is reviewed, and its application discussed in relation to bars, columns, plates and shells. A brief account is given of the role of geometric imperfections and miscellaneous effects with respect to failure mechanisms pertinent to structural and metal working problems.

Dokonano przeglądu kontynualnej teorii jednoznaczności i stateczności ustanowionej przez HILLA i przedyskutowano jej zastosowanie w odniesieniu do prętów, słupów, płyt i powłok. Zwrócono uwagę na rolę geometrycznych niedoskonałości i różnorodnych efektów na mechanizmy zniszczenia, występujące w zagadnieniach konstrukcyjnych i obróbki metali.

Проведено обозрение континуальной теории однозначности и устойчивости установленной Хиллом и обсуждено ее применение по отношению к стержням, столбам, плитам и оболочкам. Обращено внимание на роль геометрических неидеальностей и разнородных дефектов на механизмы разрушения, выступающие в конструкционных вопросах и в вопросах обработки металлов.

1. Introduction

THE PROGRESS has been halting and at different levels as regards derivation and application of criteria for uniqueness and stability of deformation of elastic-plastic solids. One might contemplate only the classical confusion regarding the Engesser and Considère theories for buckling of columns which lasted for more than half a century until SHANLEY [1, 2], by the aid of a simple mechanical model, demonstrated the important distinction between uniqueness and stability of elastic-plastic deformation. It took another decade until a continuum theory for related matters was laid down by HILL [3]. Since then Hill's basic theorems have been sparsely utilized in the solution of problems of practical interest and it seems that not until recently has a larger group of writers in the field rightly appreciated the merits of this approach.

In a recent, highly recommendable, survey article (containing 639 references) SEWELL [4] has given a detailed exposé of the existing basic principles for uniqueness and stability, including the case of solids having singular yield functions, and furthermore has discussed in particular column buckling problems at length. The state of affairs regarding a diversity of structural buckling problems of engineering importance has been reviewed exemplarily by HUTCHINSON [5] with emphasis on postbifurcation behaviour and imperfection-sensitivity aspects.

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If the discipline as regards application may be considered as fairly well established for problems common to structural mechanics, this state of the art does not seem to apply to the so-called "tensile instabilities" which, e.g., frequently cause failure in metal working processes. A related and familiar example which contains some main features of this class of problems is the necking phenomenon, which frequently occurs in ordinary uniaxial tensile tests of ductile materials. A discussion of this problem might serve as a feasible introduction to a general three-dimensional theory.

The necking phenomenon is commonly treated in text-books in the following fashion. At a generic instant during deformation the axial force in the test specimen

$$P = \sigma A, \tag{1.1}$$

where σ is true stress and A the current area of the specimen cross-section. Then a logarithmic strain measure ε is introduced through

$$d\varepsilon = \frac{dL}{L}, \tag{1.2}$$

L being the current length of the specimen. After neglecting elasticity effects and assuming incompressibility of plastic flow in the form

$$dLA + LdA = 0, \tag{1.3}$$

the basic claim is that at load maximum, i.e. when

$$dP = 0, \tag{1.4}$$

instability intervenes with local necking as a consequence. The combination of the Eqs. (1.1)–(1.4) then yields the final condition

$$\frac{d\sigma}{d\varepsilon} = \sigma. \tag{1.5}$$

From this approach it is evident that (1.5) defines a critical point on the tensile stress-strain curve of the material corresponding to the maximum load a bar may withstand under continued homogenous deformation. As it stands, however, this criterion is neither a necessary nor sufficient condition for the initiation of a local neck in a tensile test.

Firstly, in this kind of test it is in general the relative velocity of the heads of the testing machine which is controlled and not the load itself (the machine being then assumed infinitely stiff). Empirically the neck forms smoothly, indicating a stable process. Secondly, the shape of the neck is not dealt with in an analysis as above. As a matter of fact, at the conjectured critical instant it is tacitly assumed, as is evident from (1.2) and (1.3) that continued deformation is homogeneous in contrast to that which is to be proved.

In order to improve on the second point it is necessary, when still neglecting elasticity effects, to analyse the form of admissible particle velocity fields in a rigid-plastic body under uniaxial stress. Introducing the rate of deformation tensor d_{ij} referring to Cartesian coordinates x_i at a generic instant when only $\sigma_{33} \neq 0$, any isotropic non-singular flow potential yields for the deformation components

$$d_{11} = d_{22} = -(1/2)d_{33}, \quad d_{12} = d_{23} = d_{31} = 0, \tag{1.6}$$

or expressed in associated particle velocities v_i

$$\frac{\partial v_1}{\partial x_1} = \frac{\partial v_2}{\partial x_2} = -\frac{1}{2}\frac{\partial v_3}{\partial x_3}, \quad \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} = 0,$$
(1.7)
$$\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} = 0, \quad \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} = 0.$$

The solution to this, and similar systems, has been studied by PRAGER [6] and also by SEWELL [7, 8]. In the present case PRAGER gave the general solution

$$v_1 = -\frac{a}{4}(x_1^2 - x_2^2 + 2x_3^2) - \frac{b}{2}x_1x_2 - 2cx_1x_3 - \frac{d}{2}x_1,$$
(1.8)
$$v_2 = -\frac{a}{2}x_1x_2 - \frac{b}{4}(x_2^2 - x_1^2 + 2x_3^2) - 2cx_2x_3 - \frac{d}{2}x_2,$$
$$v_3 = ax_1x_3 + bx_2x_3 + c(x_1^2 + x_2^2 + 2x_3^2) + dx_3,$$

in which a, b, c and d are arbitrary mode amplitudes restricted only by the loading condition $d_{33} \geqslant 0$.

Now, if (1.8) is to represent the initiation of a cylindrically symmetric neck, then necessarily v_1 and v_2 must be odd functions of x_1 and x_2 implying that $a = b = 0$. Furthermore, if the existence of symmetry with respect to a transverse plane is required, then $c = 0$ also. Consequently, under these restrictions continued homogeneous deformation is the only surviving possibility. Incidentally it has been shown by MILES [9] from continuity arguments that the velocity amplitudes may not be given different values in different regions of the bar. Whether this shortcoming of rigid-plastic theory may be remedied by the aid of introduction of elasticity effects, geometric imperfections or perhaps more realistic boundary conditions has not until recently been answered in a satisfactory manner.

The criterion (1.5), which is well known to be applicable in practice, may be tentatively explained, though from an approximate imperfection approach (cf. e. g. CAMPBELL [10]). The unfortunate drawback of the formal success of this simple theory, however, seems to be that its philosophy has been extrapolated to apply to more complicated problems which are of interest when dealing for instance with sheet metal or shells in a membrane state under combined loads. As a consequence a confused literature has developed in which frequently the boundary value problem is not satisfactorily set, a uniqueness condition is lacking or a stability condition vaguely stated, and in particular no general form of plausible bifurcation modes are dealt with (as is the case e. g. in a contribution of my own [11]).

So there seems to be a strong case for a proper formulation of the boundary value problem and associated definition and rigorous derivation of criteria for uniqueness and stability.

2. Uniqueness and stability criteria

The basic results in this section are due to HILL, who has given an account in compact form in [12].

The occurrence of bifurcations and instabilities at deformation of continua is sensitive to the particular form of boundary conditions prevailing at a generic instant. For the sake of simplicity a simple (though practically common) form of conditions for a mixed boundary value problem is chosen here, i.e. nominal traction rates $\dot{\mathbf{F}}$ (corresponding to the *load* vector) are prescribed on part of the body surface S_F and particles velocities $\mathbf{v}$ on the remainder S_v. No body forces are assumed to act in the volume V. It is a routine matter to include these.

A proper form of a constitutive equation describing the mechanical behaviour of elastic-plastic solids at finite strain is a topic of great current interest. Several contributors to this Colloquium have discussed for instance appropriate decompositions of total deformation in its elastic and plastic parts. In the present treatment it is assumed however that whatever the general form of the constitutive equation, it may be put in the classical

$$d_{ij} = \left(M_{ijkl}+\frac{\beta}{h}\mu_{ij}\mu_{kl}\right)\dot{\tau}^{kl} \tag{2.1}$$

rate-form to describe continued deformation at a generic instant.

In (2.1), natural time does not enter and all tensorial quantities are referred to an embedded coordinate system. The convected derivative (with respect to any parameter which increases monotonically with deformation) of Green's strain tensor, choosing the current configuration as reference, is then

$$d_{ij} = \frac{1}{2}(v_{i,j}+v_{j,i}), \tag{2.2}$$

a comma denoting the covariant derivative, and $\dot{\tau}^{ij}$ is the convected derivative of the contravariant form of the symmetric Kirchhoff (second Piola-Kirchhoff) stress. The elastic compliance tensor M_{ijkl} is assumed to have $ij \leftrightarrow kl$ symmetry (apart from the ordinary symmetries) and β is different from zero and equals one only when plastic flow occurs, i.e. when a yield condition is fulfilled and the normal μ_{ij} to the smooth yield surface has a positive scalar product with $\dot{\tau}^{ij}$. The current hardening state of the solid is then prescribed through the scalar measure h, being a functional of the deformation history.

In general, the particular symmetries prevailing in the complete constitutive matrix as in (2.1) allow constitutive potentials (symmetrized functions of degree two of the strain-rates) to be introduced whatever conjugate strain and stress measures are chosen. Concerning the convenience of application when geometry effects are significant, the particular choice of an objective stress-rate tensor, such as the one in (2.1), is somewhat crucial, as some measures admit a potential for the nominal (first Piola-Kirchhoff) stress rate $\dot{s}_{ij}$ to be introduced but some do not, as for instance diverse rates of Cauchy-stress. The Kirchhoff-stress as employed here differs from that of Cauchy only in the case of non-isochoric motion and the distinction is slight in many practical cases. When Cauchy-stress is derivable as a gradient of some energy function per unit mass, which is a common result of thermo dynamic considerations, the Kirchhoff-stress rate appears quite naturally in the rate form of the constitutive equation. This is the case for instance in the state variable approach to elastic-plastic behaviour recently presented by FARDSHISHEH and ONAT [13].

Now by utilizing the relation

$$\dot{s}^{ij} = \dot{\tau}^{ij}+\sigma^{ik}v^{j}{}_{,k} \tag{2.3}$$

between the two stress-rates referred to the current configuration, a potential

(2.4) $$U = \frac{1}{2}(\dot{\tau}^{ij}d_{ij}+\sigma^{ij}v^{k}{}_{,i}v_{k,j})$$

generating

(2.5) $$\dot{s}^{ij} = \frac{\partial U}{\partial v_{j,i}}$$

may be introduced. The potential U should here be understood as a function of velocity gradients only through (2.1).

In this setting the condition for continued equilibrium (rotational balance being secured through the symmetry of τ_{ij}) may be expressed in the form

(2.6) $$\frac{\partial}{\partial x_i}\left(\frac{\partial U}{\partial v_j/\partial x_i}\right) = 0,$$

which has far-reaching consequences as any solution to the boundary value problem, unique or not, may then be sought from the variational equation

(2.7) $$\delta\left(\int U dV - \int \dot{F}^i v_i dS_F\right) = 0.$$

As regards the properties of a specific material the magnitude of the constitutive parameters in (2.1) do of course depend on the particular choice of variables. A detailed account of this matter and related transformation formulas has been given by HILL [14]. This matter has also been discussed at this Colloquium in connection with non-hardening solids by Dr. DUSZEK, who emphasized that the very definition of a perfectly plastic material [when nominally $h \to 0$ in (2.1)] is closely associated with the particular stress-rate chosen. In the following it is assumed that, except when stated otherwise, the potential U is positive definite in a stress free state implying uniqueness and stability of incipient deformation.

In the present formulation an obviously sufficient condition for unique continued motion at a generic instant is

(2.8) $$I = \int \Delta\dot{F}^j \Delta v_j dS \neq 0,$$

where Δ denotes the difference between two admissible solutions to the field variable following. Clearly, this condition also covers cases when boundary conditions of mixed traction rate/particle velocity type are prescribed.

From stability aspects, to be dealt with below, only positive values of I are of practical interest. As $\dot{F}^j = n_i\dot{s}^{ij}$ (n_i being the outward unit body surface normal) application of Gauss theorem yields

(2.9) $$I = \int \Delta\dot{s}^{ij}\Delta v_{j,i}dV > 0$$

for continued quasi-static motion and by introduction of (2.5)

(2.10) $$I = \int \Delta\left(\frac{\partial U}{\partial v_{j,i}}\right)\Delta v_{j,i}dV > 0.$$

In many practical situations rigid constraints prevail on the part S_v of the boundary surface. In such a situation the zero velocity field is admissible in (2.10) which consequently implies

$$\int \frac{\partial U}{\partial v_{j,i}} v_{j,i} dV > 0, \tag{2.11}$$

or by aid of the definition of U and Euler's theorem for homogeneous functions

$$2\int U dV > 0. \tag{2.12}$$

It should be emphasized however that, in general (2.11) or alternatively (2.12), does not imply (2.10) as the constitutive equation (2.1) is non-linear due to ist two different branches (dependent on whether plastic flow occurs or not) and, consequently, the integrand of I is multivalued with respect to Δv_i for material elements at yield, while in (2.11) and (2.12) the functionals are single-valued with respect to v_i.

Turning now to the question of stability, this concept is not given by any law of nature. A classical definition of stability in dynamic sense claims that if a body, in equilibrium under dead loading on part of the boundary surface and rigid constraints on the remainder, is perturbed by any kind of agency, then the state is stable if the resulting motion, compatible with the constraints, stays vanishingly small (in some sense) when the perturbation itself does.

A sufficient condition for stability in this sense may then be derived, starting from the requirement that during the initiated motion the internal change of energy should exceed the work done by the external loads on the displacements i.e.

$$J = \int_0 \left(\int s^{ij} v_{j,i} dV - \int F^j v_j dS_F \right) dt > 0. \tag{2.13}$$

The value of J is strongly path-dependent but, through an estimation of J to the second-order for most likely paths (cf. [3]), (2.13) yields the sufficient condition

$$J = \frac{1}{2} t^2 \int \dot{s}^{ij} v_{j,i} dV > 0. \tag{2.14}$$

Thus (2.12) and (2.14) are equivalent conditions but neither implies (2.10) as already mentioned. For quasi-static motion (2.14) is equivalent to (in the case of rigid constraints)

$$\int \dot{F}^j v_j dS_F > 0 \tag{2.15}$$

and thus there always exists the possibility of stable bifurcations under changing external loads (when (2.14) is fulfilled but (2.10) fails) as once heuristically argued by SHANLEY. In this sense uniqueness implies stability but the converse may not necessarily be true. In practical situations then the uniqueness condition (2.10) is of most importance, as the occurrence of a bifurcation usually marks the end-point of the satisfactory service life of a member.

In general this condition does not lend itself easily to application. For common constitutive properties of elastic-plastic solids, however, it may be shown, [15], that this condition may be expressed in a weakened but more useful form by introducing a potential U_L

constructed from the single loading branch ($\beta = 1$ always) of the constitutive equation for material elements stressed to their yield points. In a situation when there exists an overall plastic flow solution for a particular problem (which incidentally is the case in all the illustrations discussed in detail below), utilizing the concept of this so-called "linear comparison solid", bifurcations may be sought as eigen-states from (2.7) under homogeneous boundary conditions, as for linear solids the question of uniqueness does not depend on the specific boundary values prescribed on S_F and S_v. When superposing eigen-solutions found in this manner on the fundamental solution, the amplitudes of the eigen-modes are then bounded from above through the loading condition in the constitutive equation.

The principles discussed above have a great potential when solving practical problems. There exist, however, other kinds of boundary conditions which are of interest in practice. HILL [16] has studied configuration-dependent loading such that the associated problem is still self-adjoint. The main ideas are the same but when for instance hydrostatic pressure p acts on part of the boundary S_p (which is probably the most practically important condition in the family dealt with) the relevant functionals must be supplemented so that, for instance, (2.8) takes the form

$$\int \Delta \dot{F}^j \Delta v_j dS - \int p(n_j \Delta v^j{}_{,k} - n_k \Delta v^j{}_{,j}) \Delta v^k dS_p \neq 0 \tag{2.16}$$

the basic arguments being unchanged.

Some general aspects of uniqueness under loading by uniform fluid pressure have been discussed in detail by MILES [17] with a particular illustration of the influence of lateral pressure loading in a tensile test.

3. Illustrations; uniaxial homogeneous steady stress state

In the case of a purely uniaxial stress state, with $\sigma_{33} = \sigma \neq 0$ (say), the relevant functional for the search of eigen-states in the "linear comparison solid" becomes

$$F = \int (\dot{\tau}^{ij} d_{ij} + \sigma v^k{}_{,3} v_{k,3}) dV, \tag{3.1}$$

where $\dot{\tau}^{ij}$ is related to the velocity gradients through (2.1), the loading branch being adopted for elements at yield.

Any effort to solve the equation

$$\delta F = 0 \tag{3.2}$$

for a three-dimensional case will mostly lead to prohibitive numerical procedures even for homogeneous steady stress states. In analogy with the classical theory, e.g. for the bending of beams, plates and shells, it is in some situations feasible to introduce approximations of (3.1) through simple kinematical assumptions. As the stationary value of F is zero and an analytic minimum, then, from the associated Rayleigh principle, close upper bounds for critical load parameters may be expected in many cases.

3.1. Columns

In an investigation of buckling of an inelastic column built in at one end, HILL and SEWELL [18] utilized a velocity field of the type

$$v_1 = w+\frac{1}{2}\nu(x_1^2-x_2^2)\frac{\partial^2 w}{\partial x_3^2},$$

(3.3)

$$v_2 = \nu x_1 x_2\frac{\partial^2 w}{\partial x_3^2}, \quad v_3 = -x_1\frac{\partial w}{\partial x_3},$$

in a Cartesian coordinate system.

In (3.3) $w(x_3)$ denotes the transverse velocity of particles on the line of centroids in an assumed (non-uniform) bending mode. Except for covering the common assumption of line elements originally perpendicular to the neutral axis remaining perpendicular to the deformed axis (frequently attributed to EULER-BERNOULLI), this field allows for shearing effects to be taken into account if $\partial^3 w/\partial x_3^3 \neq 0$. The instantaneous material state was assumed orthotropic and of such symmetry that the contraction ratio ν for the longitudinal direction was independent of the direction of transverse continued loading. In particular then for the field (3.3), $\dot{\tau}_{11}, \dot{\tau}_{12}, \dot{\tau}_{22} \equiv 0$ and the boundary conditions for the lateral surface of the column may be fulfilled at least in the transverse directions.

The resulting critical load when uniqueness is lost in a slender column (lower Shanley load) was then

$$P_{cr} = \frac{\pi^2}{4}E_t\frac{I}{L^2} \tag{3.4}$$

in customary notation, E_t being apart from a practically negligible factor, the tangent modulus to the true stress-strain curve in compression. The associated buckling mode was then of sinusoidal shape.

To arrive at the approximate result (3.4), however, it is essential that the instantaneous moduli of transverse shear and longitudinal compression are of the same order of magnitude. If these circumstances do not prevail, shear stiffening will not be negligible and in the extreme case when the response to a shear stress increment is rigid (as is the case in a rigid-plastic situation), the critical velocity field must necessarily be of the form (1.8) given by PRAGER. The resulting estimate for the buckling load is for this case

$$P_{cr} = 3E_t\frac{I}{L^2} \tag{3.5}$$

and consequently the shear stiffening effect is then of the order of 20 per cent of the critical load.

The column problem was subsequently studied by HILL and SEWELL [19] from a stability point of view in the same three-dimensional formulation and the accuracy of the classical "reduced modulus" load verified (when shearing effects are negligible). The incipient bifurcation paths possible below this critical load, with regions of the column undergoing elastic unloading, were also dealt with, [20], with the aid of an especially devised variational approach bearing resemblance to the Galerkin and Kantorovich methods.

3.2. Plates

In-plane shearing effects become a major issue when it comes to buckling of plates. It is well known that for this kind of problem there exists an appreciable discrepancy between experimental results and theoretical predictions based on classical flow theory. It is interesting to notice in this connexion that critical loads predicted by analyses based on a deformation theory of plasticity in Hencky's meaning usually agree well with experimental findings. This fact may hardly serve as a motivation for an unreflecting use of such constitutive theories in plastic buckling problems. It is evident though that the choice of constitutive model is crucial.

This feature was discussed early in relation to buckling of plates by BATDORF [21], who pointed out that the creation of a corner on the yield surface would remarkably lower the theoretical buckling load. The development of such corners is characteristic of slip theories of plasticity and in a recent extensive analysis of a rectangular plate under compression, SEWELL [22] has discussed the issue at length supplementing an earlier study [7, 23], starting from classical flow theory.

SEWELL adopted, in his final approximation, a velocity field of the form

$$
\begin{aligned}
v_1 &= u - x_3\frac{\partial w}{\partial x_1}, \qquad v_2 = v - x_3\frac{\partial w}{\partial x_2},\\
v_3 &= w + x_3 R(u, v) + x_3^2 S(w),
\end{aligned} \tag{3.6}
$$

where u, v, w are the middle surface particle velocities, being functions of Cartesian coordinates x_1 and x_2, and R and S are differential operators (dependent on the constitutive properties) implying $\dot{\tau}_{33} = 0$. This field represents, apart from locally homogeneous deformation, bending based on Kirchhoff's classical assumption that material elements perpendicular to the original middle surface remain perpendicular to the deformed middle surface.

An alternative method which accounts in (3.1) for continued zero transverse normal stress, when transverse shear effects are negligible, is to delete the last two terms in $(3.6)_3$ but before inserting $\dot{\tau}_{ij}$ in (3.1) solve the rate equations for a transverse plane stress state.

The particular form of constitutive equation adopted by SEWELL, originally proposed by HILL [24] for slip deformation of single crystals, reads for plastic loading

$$
d_{ij} = (M_{ijkl} + h_{\alpha\beta}^{-1}\nu_{ij\alpha}\nu_{kl\beta})\dot{\tau}^{kl}, \tag{3.7}
$$

where $\nu_{ij\alpha}$ ($\alpha = 1, 2, \ldots, N$) are second-order tensors interpretable as different flow branches at a corner of the yield surface and $h_{\alpha\beta}$ is a hardening matrix allowing for coupling between the different branches when the matrix is not purely diagonal. In the case $N = 1$, (3.7) reduces to (2.1) implying a smooth yield surface.

In particular SEWELL discussed the state of affairs in the case of a local Tresca-type criterion when N was assumed to be two and correspondingly the two ν_{ij} tensors being parallel to the normals of the facets of the yield surface intersecting at the current stress point. The hardening matrix is then

$$
h_{\alpha\beta} = h\begin{pmatrix} 1 & \Phi \\ \Phi & 1 \end{pmatrix},
$$

where Φ accounts for the amount of coupling at a generic instant.

Sewell's constitutive assumptions imply that the instantaneous shear modulus retains its elastic value during uniaxial stressing. Yet a substantial reduction of the buckling load was obtained when adopting a Tresca criterion instead of a von Mises criterion. For instance for the value 0.5 of the coupling parameter Φ, the relative reduction was approximately 30% in a particular situation when adopting a Ramberg-Osgood uniaxial stress-strain relation. This value of Φ prevails in the slip theory proposed by BUDIANSKY and WU [25], being based on "kinematically" hardening slipping, and incidentally it was found by SEWELL that the constitutive rate-equations are then isotropic and any property which does not depend on the shear modulus (e.g. the buckled shape of the plate) might directly be found from any elastic theory in which the constitutive parameters have been replaced by those of Hencky's deformation theory.

The role of the shear modulus is more explicitly displayed when analysing elastic-plastic buckling of a compressed cruciform column. With this kind of column geometry [consisting in principle of four joined plates, of width b and thickness t (say)], the dominant buckling mode is one of purely longitudinal torsion. In a recent study, HUTCHINSON and BUDIANSKY [26] have recapitulated the fact that from any non-singular plastic flow theory a stress at bifurcation

$$\sigma_{cr} = G(t/b)^2 \tag{3.8}$$

is predicted where G is the elastic shear modulus. For any deformation theory (3.8) still holds if G is replaced by the appropriately reduced effective shear modulus, and experimental findings seem to favour this result.

Such experiments, however, may not be understood to be conclusive regarding the choice of constitutive theories, as the results of Hutchinson and Budiansky's analysis strengthen the earlier conception of the extreme sensitivity of the load-carrying capacity to the presence of practically unavoidable imperfections in the column shape. In an early study by ONAT and DRUCKER [27] it was shown that the favourable deformation theory prediction might as well be achieved from the classical flow theories taking imperfections into account.

3.3. Shells

When dealing with buckling of shells additional difficulties appear, in particular, regarding the relevance of the common kinematical *ad hoc* assumptions. In elasticity there exists in essence no thick-shell theory and not until the last decade has it been shown, especially through the basic work by KOITER, that the classical Kirchhoff-Love assumptions generally lead to a correct first-order theory in the thin-shell limit. This result has been demonstrated only for a Hookean shell material and in a strict sense the applicability in the case of anisotropic constitutive rate equations, as is presently dealt with, still remains to be proved. In any case, when introducing a restricted velocity field into Hill's uniqueness functional in order to derive an estimate of the critical load from a bifurcation point of view, it must be demonstrated that a consistent two-dimensional theory results. I have recently discussed this matter elsewhere [28] and in a particular illustration analysed the case of a thin-walled, circular cylinder under axial compression.

In a cylindrical coordinate system, the kinematical restriction that during continued motion material elements perpendicular to the middle surface of the shell will remain perpendicular and not change their length, yields approximately for the particle velocities in an arbitrary point

$$v_1 = u - z\frac{\partial w}{\partial x}, \quad v_2 = v - z\left(\frac{\partial w}{a d\varphi} - \frac{v}{a}\right), \quad v_3 = w, \tag{3.9}$$

where u, v, w are the velocities of particles on the middle surface and z denotes the distance from the middle surface having radius of curvature a.

Introduction of the associated velocity gradients into the relevant functional yields

(3.10)

$$\int U_L dV = \int_{-\frac{t}{2}}^{\frac{t}{2}} \int_0^{2\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} \left\{ m_{11}\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right)^2 + 2m_{12}\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial v}{a\partial\phi} + \frac{w}{a} - \frac{z}{a}\frac{\partial^2 w}{a\partial\phi^2}\right)\right.$$

$$+ m_{22}\left(\frac{\partial v}{a\partial\phi} + \frac{w}{a} - \frac{z}{a}\frac{\partial^2 w}{a\partial\phi^2}\right)^2 + \mu\left(\frac{\partial u}{a\partial\phi} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w}{a\partial x\partial\phi}\right)^2$$

$$\left. - \sigma\left[\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial v}{\partial x} - z\frac{\partial^2 w}{a\partial x\partial\phi}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]\right\} a\,dx\,d\phi\,dz,$$

t and L being the local thickness and length of the cylinder and σ the current compressive stress. In (3.10) an orthotropic plane stress form of the constitutive equation has been adopted, namely

$$\begin{bmatrix} \dot{\tau}_{11} \\ \dot{\tau}_{22} \\ \dot{\tau}_{12} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{12} & m_{22} & 0 \\ 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} d_{11} \\ d_{22} \\ d_{12} \end{bmatrix}.$$

Furthermore, when constructing (3.10), terms of the order $|z|/a$ have been deleted whenever comparable to unity. In the present approximation there is no evident reason to refine the kinematics further than this.

Carrying out the integration in the thickness direction in (3.10), the variational equation (3.2) yields the Euler equations

$$(m_{11}-\sigma)\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + (m_{12}+\mu)\frac{\partial^2 v}{\partial x\partial y} + m_{12}\frac{1}{a}\frac{\partial w}{\partial x} = 0, \tag{3.11}$$

$$(m_{12}+\mu)\frac{\partial^2 u}{\partial x\partial y} + (\mu-\sigma)\frac{\partial^2 v}{\partial x^2} + m_{22}\frac{\partial^2 v}{\partial y^2} + m_{22}\frac{1}{a}\frac{\partial w}{\partial y} = 0,$$

$$m_{12}\frac{1}{a}\frac{\partial u}{\partial x} + m_{22}\frac{1}{a}\frac{\partial v}{\partial y} + \alpha a^2\left[(m_{11}-\sigma)\frac{\partial^4 w}{\partial x^4} + (2m_{12}+4\mu-\sigma)\frac{\partial^4 w}{\partial x^2 dy^2} + m_{22}\frac{\partial^4 w}{\partial y^4}\right]$$

$$+ m_{22}\frac{w}{a^2} + \sigma\frac{\partial^2 w}{\partial x^2} = 0,$$

and the associated boundary condition

$$(3.12)\quad \oint\Bigg\{\left[(m_{11}-\sigma)\frac{\partial u}{\partial x}+m_{12}\left(\frac{\partial v}{\partial y}+\frac{w}{a}\right)\right]\delta u+\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-\sigma\frac{\partial v}{\partial x}\right]\delta v$$
$$+\alpha a^2\left[(m_{11}-\sigma)\frac{\partial^2 w}{\partial x^2}+m_{12}\frac{\partial^2 w}{\partial y^2}\right]\delta\left(\frac{\partial w}{\partial x}\right)-\left[\alpha a^2(m_{11}-\sigma)\frac{\partial^3 w}{\partial x^3}\right.$$
$$\left.+\alpha a^2(m_{12}+4\mu-\sigma)\frac{\partial^3 w}{\partial x\,\partial y^2}+\sigma\frac{\partial w}{\partial x}\right]\delta w\Bigg\}dy=0,$$

where $\alpha = t^2/(12a^2)$ and dy denotes $ad\phi$.

As the final rate equations were generated from a potential, it is gratifying to establish the symmetry in the appearance of the operators for the middle surface particle velocities in (3.11). From the view-point of consistency it is, however, the interpretation of the natural complement to the rigid boundary conditions in (3.12) which is of more fundamental interest.

To this end it is feasible to follow Sewell's approach in a corresponding plate problem and introduce nominal stress-rate resultants and stress couples as

$$(3.13)\quad \dot{n}^{\alpha\beta} = \int_{-t/2}^{t/2} \dot{s}^{\alpha\beta}\,dz$$

and

$$(3.14)\quad \dot{m}^{\alpha\beta} = \int_{-t/2}^{t/2} \varepsilon^{\beta\delta}(\dot{s}^{\alpha}{}_{\delta}z+s^{\alpha}{}_{\delta}w)\,dz,$$

respectively, where $\varepsilon_{\alpha\beta}$ is the two-dimensional permutation tensor.

It turns out then that within the present approximation, $t/a \ll 1$, (3.12) is equivalent to

$$(3.15)\quad \oint\left[\dot{n}_{xx}\,\delta u+\dot{n}_{xy}\,\delta v-(\dot{m}_{xy}-n_{xx}w)\,\delta\left(\frac{\partial w}{\partial x}\right)+\left(\dot{q}_z-\frac{\partial\dot{m}_{xx}}{\partial y}\right)\delta w\right]dy=0,$$

where naturally the transverse shear stress resultant $\dot{q}_z$ must be obtained from the local rotational balance condition, i.e.

$$\dot{q}_z = \frac{\partial\dot{m}_{xy}}{\partial x}+\frac{\partial\dot{m}_{yy}}{\partial y}$$

as transverse shear effects were neglected in the chosen restricted class of particle velocities and consequently only local in-plane resultants may be found in the manner outlined above. Furthermore (3.11) may now be shown to be, still within the thin-shell approximation, the conditions for local balance of momentum.

To solve the non-linear eigenvalue equation associated with (3.11) and (3.12) generally, accounting for the influence of all geometric and constitutive parameters seems to be a formidable task and of dubious practical value. Suffice it to say in this context that by a linearization of (3.11), which corresponds to a Donnell approximation in elasticity and plausibly retains the main features of the buckling problem in the present context, it was shown that there exists a remarkable sensitivity to the boundary condition similar to that in elasticity.

Thus by relaxing the boundary condition from rigid to in-plane shear free ends of the cylinder, the buckling load is theoretically reduced by a factor two (approximately), e.g. in the case of a linearly strain-hardening material.

Naturally a comparison with experimentally found buckling loads has to be performed with the utmost precaution. All the constitutive aspects discussed above in relation to the plate problem apply also to the present cylinder problem and, furthermore, it is well known from several investigations that the buckling load of an elastic cylinder is highly sensitive to geometric imperfections. A corresponding satisfactory analysis of the elastic-plastic case seems to be lacking so far.

3.4. Tensile bars

Returning now to the necking problem discussed above it was recently shown by MILES [9] through Hill's sufficient condition for uniqueness that the maximum load point provides a lower bound for the true stress at which necking will occur in a thought tensile test in which the axial velocity is controlled at shear-free ends of the specimen. The proof was carried out for quite general material properties.

When dealing with these kinds of problems it must be emphasized that the requirement of positive definiteness of the constitutive matrix as in (2.1) has to be relaxed. Usually the amount of straining preceding loss of uniqueness is high and the role of deformation increments in bifurcation modes is of the same order as rotations in contrast to the circumstances ordinarily prevailing in buckling problems. This state of affairs calls for a careful interpretation of the constitutive parameters when modeling real material characteristics through a mathematical expression.

By the aid of a velocity field of the form

$$v_r = -mF\cos mx, \quad v_\phi = 0, \quad v_x = \frac{1}{r}\left(F + r\frac{dF}{dr}\right)\sin mx, \tag{3.16}$$

(in a cylinder system) satisfying incompressibility, Miles explicitly demonstrated the existence of an exact necking solution in a cylindrical bar. In (3.16) m is proportional to the axial wave number and in the final solution, $F(r)$ turned out to be a Bessel function of the first order. The critical stress (furnishing an upper bound for a primary eigensolution) was found to be unrealistically high from a practical view-point. Anyhow Miles' bifurcation solution provided some insight into the specimen geometry dependence of the phenomenon.

More realistic bifurcation stresses were found by HUTCHINSON and MILES [29] through asymptotic and numerical methods still thought for an incompressible material. One of their interesting results, which exhibits the influence of elasticity effects, showed that for slender specimens the true stress at the initiation of necking does only exceed the true stress at maximum load by a few per cent, even if the ratio of the tangent modulus and Young's modulus is of the order 0.001 for the material in question. Naturally, the exact values depend on the change of strain-hardening properties after the maximum load point has been passed, but this is a remarkable result when seen in the light of the absence of bifurcations within rigid-plastic theory.

The necking phenomenon in the case of isotropic compressible elastic-plastic materials having parabolic strain-hardening properties was dealt with rigorously and at length in a numerical study by NEEDLEMAN [30]. In a bifurcation solution with boundary conditions as in the earlier discussed analyses, NEEDLEMAN calculated, besides the stress at the initiation of necking, the magnitude of relevant variables during the necking down procedure by solving Hill's variational equation (2.7) through a finite element technique. Thus, among other interesting features, NEEDLEMAN was able to offer a comparison with Bridgman's well-known semi-empirical formulae for the stress distribution in the neck. Good agreement was found particularly in the earlier stages of necking, though in the latter stages Bridgman's results were shown to underestimate the maximum hydrostatic tension (acting on the axis of the specimen).

In a second, perhaps more realistic, approach NEEDLEMAN assumed in the setting of the boundary value problem that the ends of the cylinder bar were cemented to rigid grips. The axial homogeneity of the problem is then immediately lost at the onset of deformation but the numerical results show that on the whole the circumstances in the final necking procedure are very similar to those obtained from the bifurcation approach. In the two cases the maximum load was essentially the same although it was reached at a smaller strain by the cemented specimen which also necked down faster.

4. The role of geometric imperfections and miscellaneous effects

It is evident from the arguments pertinent to the relations (2.10), (2.14) and (2.15) above that bifurcations in elastic-plastic solids may mostly be expected to occur under increasing external loading (exceptions being for instance the compressed cruciform column and the tensile specimen as discussed above). In such situations the system to be analysed is not imperfection-sensitive in the sense generally accepted in the theory of elasticity. In elasticity imperfection-sensitivity is an important feature as it is well known that small imperfections, which may be due to various reasons, may have a strong influence on the total load-carrying capacity of a structure. In practice it is particularly at elastic buckling of shells that initial geometric imperfections constitute the latent cause of failure at loads smaller than those predicted by classical bifurcation theories.

Despite the formal insensitivity when plastic effects are significant it seems rewarding to study the influence of imperfections in such situations too as imperfections in geometry may rapidly grow when non-linear material effects are momentous. This conclusion may be drawn especially from some recent results given by HUTCHINSON [31]. For some simple systems, a compressed idealized column with a non-linear spring support and an externally pressurized spherical shell which are both imperfection-sensitive in the elastic range, HUTCHINSON has shown that the increase in load-carrying capacity after a bifurcation has occurred in deformation of the geometrically perfect system, is not significant in the plastic range. When initial imperfections are present the load-carrying capacity is substantially reduced as for instance in the shell problem the maximum load is decreased by a factor two when the amplitude of an initial imperfection in the middle surface geometry is around 0.4 times the shell thickness. This result was achieved for specific but realistic shell parameters.

The theory for post-buckling behaviour and imperfection-sensitivity of elastic structures has during the last decades undergone a rapid development in Koiter's spirit. The main features of a similar approach to elastic-plastic systems have been discussed at length by HUTCHINSON [32, 33]. One of Hutchinson's findings shows that for some systems containing imperfections, the maximum of a single load parameter P^{Max} is very sensitive to the load $\hat{P}$ at which strain reversal begins. For a qualitative description of the main buckling features and with no claim of generality Hutchinson offers the approximate formula

$$\frac{\hat{P}}{P_c} \approx \frac{P^{\mathrm{Max}}}{P_0^{\mathrm{Max}}}$$

for the maximum load in a system for which, in case of perfect geometry, P_c and P_0^{Max} are the bifurcation load and the maximum load, respectively. As concluded by HUTCHINSON [31], in practice imperfection-sensitivity might not be such an important problem when plastic effects are present as in elasticity. This is due to the fact that mostly "thick" structures are likely to fail in the plastic range and such bodies are more easily manufactured with "perfect" geometry than "thin" members.

When discussing initial strain-path deviations, other effects, which have consequences similar to those of geometric imperfections, are such as excentricity of loading and the presence of residual stresses. An extensive literature has been devoted to such aspects (cf. e. g. [34]).

Geometry or material imperfections are also of interest when predicting failure under predominantly tensile stresses. A notoriously difficult problem which has withstood many efforts of a rigorous explanation is the local thinning effect which frequently occurs in the stretching of sheet metal. In this field it seems like a large quantity of experimental findings has been collected without a corresponding theoretical understanding. An improvement in this state of affairs e.g. regarding failure mechanisms in press-shop operations would no doubt be of a great practical importance.

Within a von Mises rigid-plastic theory the velocity equations for deformation of a sheet under plane stress are hyperbolic when the ratio of the in-plane principal stresses exceeds two. It may then be conjectured that local necking will occur along the characteristic surfaces containing the directions of zero stretch. For instance in a rectangular isotropic strip subjected to a uniaxial stress the neck will then form a 54.7° angle with the tension axis (cf. e.g. [35]). In the case of elliptic compatibility equations, MARCINIAK [36] has attempted an imperfection approach in order to provide a failure criterion. This line of attack to simulate so-called diffuse necking seems to have had a varying degree of success depending on the particular circumstances prevailing and will be discussed in more detail by Professor MELLOR at this Colloquium. In essence Marciniak postulates the existence of an initial material or geometric defect but does not analyse in great detail its initial shape and growth. The resulting limiting strains then become functions of some gross measure of the initial inhomogeneity, which has to be arbitrarily assumed in most instances, and are indeed very sensitive to its magnitude. It remains to be shown whether a more affirmative view of the applicability of Marciniak's basic philosophy may be obtained by a more detailed kinematic analysis of the local necking down procedure. The pioneering and to my knowledge still single effort in that direction for a plane problem is the study by COWPER and ONAT [37] of a rectangular bar uniaxially stressed under transverse plane strain.

There exists of course the possibility that local thinning may be predicted from a bifurcation approach when taking elasticity effects into account. In the strict treatment of necking of cylindrical bars under uniaxial stress it has been clearly displayed that elasticity effects are of paramount interest even if elastic and plastic moduli are of quite different orders of magnitude. The technical difficulties seem very severe though, as for instance in many metal working problems, already the achievement of a fundamental rigid-plastic solution is a very complicated task (cf. e. g. [38]). It seems clear though that failure mechanisms must be described by *fields* of relevant variables, as in the necking problems discussed above, and that conditions dealing only with the state in a local material point (where the flow properties are described only through some single strain-hardening exponent and perhaps a simple measure of anisotropy) will not suffice in general.

Some interesting studies of plausible bifurcation modes in incompressible elastic-plastic solids have been carried out by ARIARATNAM and DUBEY (cf. e. g. [39]). In one illustration dealt with by these writers, they displayed that transcendental necking modes may be initiated in thin sheets under balanced biaxial tension. In the critical states associated with these modes the stress intensity found was however of the same order as the elastic shear modulus. Thus in most practical situations when metals are concerned their results do not bear conviction but may be interpreted as upper bounds for critical states.

In this context it seems appropriate to mention the growing interest for different instability phenomena in sheets or plates perforated by holes or cracks (cf. [40, 41]). When such sheets are subjected nominally to tension, local compressive stresses may cause transverse buckling having severe effects, e.g. from a fracture mechanics point of view as regards both testing and service members. Also crack growth may be assisted by local necking phenomena due to the interaction of cracks and voids. So far these problems seem to have been dealt with theoretically only in a crude manner but more refined results are likely to be obtained by aid of a finite element technique based on the fundamental principles discussed above.

Resuming for a moment the role of elasticity effects, this issue may be further illustrated by the results of some investigations of bifurcations in thick-walled shells by STRIFORS and myself [42, 43, 44]. We found by adopting a three-dimensional rigid-plastic approach that for a closed-end cylinder, in the thin-shell limit a simple tangent modulus formula applied for buckling under external pressure. On the contrary in a corresponding spherical case buckling was ruled out by rigid-plastic theory; a result which has to be viewed with a high degree of suspicion from a practical view-point. This shortcoming is an immediate consequence of the restrictions imposed on particle velocity fields by the flow rule, as in the necking problem discussed above, and fortifies the comprehension that when investigating bifurcation and instability phenomena the success of a rigid-plastic approach depends severely on the character of the velocity field equations in each particular case. Another finding, worth mentioning in this context, is that buckling modes were shown to exist under *increasing internal* pressure for a cylinder of particular material hardening properties. The existence of such a possibility seems to have been overlooked by many writers in the field. Our study did not deal with local "bulging" modes but such phenomena may plausibly be treated in COWPER and ONAT'S manner as I have indicated earlier [45].

When utilizing a full elastic-plastic constitutive equation some phenomena still withstand explanation within classical flow theories as indicated above. Some writers have advocated the uncritical use of deformation theories in efforts to provide a comprehension. In some situations the use of, e.g. a J_2-deformation theory for polycrystals, may be defended by some arguments SANDERS [46], who pointed out that when a yield surface corner develops with infinitely many branches in a flow theory ($N \to \infty$ in (3.7) above) the resulting constitutive equation may be interpreted as the rate form of Hencky's theory for a restricted range of deformation paths. Situations where proportional loading does not prevail but deformation theories still appear physically sound have been analysed also by BUDIANSKY [47].

These arguments rest upon the mathematical assumption of yield surface singularities. In practice this matter might not be crucial, as remarked by PHILLIPS [48]. Many materials develop anisotropic properties with pointed yield surfaces under quite small amounts of straining, and in such situations initial unavoidable imperfections, which cause slight deviations from an idealized deformation path, may have a decisive effect on the change of, for instance, local shear moduli. Thus, as pointed out by some writers, when performing experiments pertinent to e.g. some controversial buckling problems, it is necessary to determine the constitutive material behaviour in parallel in order to achieve a deeper understanding of the phenomena involved.

The discussion so far has been based upon the assumption of time-independence of plastic behaviour which is of course an idealization when it comes to real materials. In many cases though such an assumption may be adopted with success even for problems when several decades of strain-rates are involved. At higher strain-rates though rate effects (in their true sense) may have a pronounced influence on the load-carrying capacity of structures. A study of related questions in connexion with the tensile test has been carried out by CAMPBELL [10] through an imperfection approach (cf. also KLEPACZKO's analysis of adiabatic processes [49]).

On the other hand in slow processes material creep might be essential. On some occasions [50, 45], I have discussed the effects of creep as regards uniqueness and stability of the plastic rate-problem. The essential problem then becomes one of calculating the time during which constitutive and geometry changes occur such that a critical state is induced. The studies mentioned deal solely with visco-plastic solids but a generalization to include also elasticity effects may be carried out in a quite straightforward manner.

When deriving the fundamental theorems above it has been tacitly assumed that deformation is isothermal. Temperature changes in time and space will add further to the difficulties already prevailing. Some aspects of this matter as regards uniqueness of deformation has been recently dealt with by MRÓZ and RANIECKI [51].

5. Concluding remarks

In the current particular branch of non-linear continuum mechanics the second-order effects causing different kinds of failures depend to a great extent on the close interaction between mechanical material properties, body geometry, way of load introduction and different constraints. Hill's definite framework serves as a unified foundation for a proper formulation of boundary value problems and derivation of uniqueness and stability criteria. When dealing with non-trivial systems, however, usually great computational difficulties are encountered. The devised variational methods, have proved to be suitable for the use of numerical methods admitting a choice of accuracy according to what the particular situation demands. Needleman's analysis of the growth of plastic zones in a cylindrical bar under tension serves as a neat illustration how the difficulties may be mastered by the aid of the finite element technique and in particular how realistic boundary conditions may be taken into account. General methods for the application of the finite element technique to problems involving large plastic strain have been outlined by HIBBITT, MARCAL and RICE [52] and also discussed by HUTCHINSON [53].

In many practical cases bifurcation loads found by classical methods might be of questionable practical value and instead there might exist some other effect which may be the latent cause of failure. In some situations for instance primary bifurcations may be expected in a perfect system only in the plastic range, while the real system fails in the elastic range due to some deviation from the mathematical simulation of the behaviour of the real system. Thus the choice of the method of attack when analysing many practical problems requires a certain degree of experience and sound judgment by the analyst.

The practical importance and challenge of the discipline dealt with has attracted many distinguished workers to the field. In this brief survey it has not been possible to deal in detail with many aspects and much important work has had to be left out of the discussion. The treatment has by necessity been cursory and biased towards matters I happen to be somewhat familiar with. Suffice it to say then that outstanding contributions have been given to the field but there is ample room for more efforts in order to achieve a more fundamental understanding of different aspects, some of which have been mentioned above.

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