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On the foundations of the endochronic theory of viscoplasticity (*)

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PREVIOUSLY we proposed the ENDOCHRONIC theory of viscoplasticity by introducing the concept of intrinsic time. In this paper we broaden the foundations of the theory. This we do by discussing the geometric motivation for the intrinsic time measure; by giving this measure a stress-time base as an alternative to the original strain-time base; by devising dual forms of the constitutive equations in terms of the Gibbs free energy function ϕ. We use a simple model of such a ϕ-form to show that *linear unloading* can be a constitutive consequence and not an additional assumption.

Poprzednio zaproponowano endochroniczną teorię lepkoplastyczności wprowadzając koncepcję czasu wewnętrznego. W niniejszej pracy rozszerzono podstawy teorii. Dokonano tego przez dyskusję geometrycznej motywacji dla miary czasu wewnętrznego, nadanie tej mierze bazy odkształcenie-czas oraz ustanowienie dualnych postaci równań konstytutywnych, wyrażonych przez funkcję energii swobodnej Gibbsa ϕ. Wykorzystano prosty model takiej formy ϕ celem wykazania, że *liniowe odciążenie* może być konstytutywną konsekwencją a nie dodatkowym założeniem.

Раньше мы предложили эндохроническую теорию вязкопластичности, вводя концепцию внутреннего времени. В настоящей работе расширяем основы теории. Производим это путем обсуждения геометрического истолкования для меры внутреннего времени, путем придания этой мере базиса деформация-время и путем установления дуальных видов определяющих уравнений, выраженных через функцию свободной энергии Гиббса ϕ. Используем простую модель такой формы ϕ с целью показания, что *линейная разгрузка* может быть следствием определяющих уравнений, а не дополнительным предположением.

Introduction

IN A PREVIOUS paper [1] we proposed the ENDOCHRONIC theory of viscoplasticity by introducing the notion of intrinsic time. The use of the internal variable formalism of irreversible thermodynamics then enabled us to derive constitutive equations which predict the stress and entropy response of materials to general thermomechanical histories. The derivation was in terms of the Helmholtz free energy Ψ.

Since that time, we have applied with considerable success the theory to the response of metals to various types of deformation. For details, we refer the reader to the appropriate references which are discussed at greater length in a later section of this paper.

In this paper, we broaden the foundations of the theory by (i) introducing a geometric motivation for the intrinsic time measure, (ii) giving this measure a stress-time base as

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an alternative to original strain-time base and (iii) deriving *dual* forms of the constitutive equations in terms of the *Gibbs* free energy function ϕ.

Our recent work has shown that the Gibbs free energy formulation gives rise to constitutive equations which are better suited for the representation of the unloading behavior of metals and their response to compression following extension "in the plastic range".

1. Helmholtz free energy formulation

In the above formulation, which we shall call the Ψ-formulation, the independent variables are the Cartesian components c_{ij} of the Right Cauchy Green tensor $\mathbf{C}$, the absolute temperature θ and the Cartesian components q^r_{ij} of n tensor-valued internal variables $\mathbf{q}^r$.

Concisely stated, the constitutive equations then are, in standard notation:

$$\boldsymbol{\tau} = 2\frac{\varrho}{\varrho_0}\frac{\partial \Psi}{\partial \mathbf{C}}, \tag{1.1}$$

$$\eta = -\frac{\partial \Psi}{\partial \theta}, \tag{1.2}$$

$$\frac{d\mathbf{q}^r}{dz} = \mathbf{f}^r(\mathbf{C}, \theta, \mathbf{q}^s), \tag{1.3}$$

$$\mathbf{h} = \mathbf{h}(\mathbf{C}, \mathbf{q}^r, \mathrm{Grad}\theta), \tag{1.4}$$

where $\vec{h}$ is the heat flux per unit undeformed area, Grad is a material operator and $\vec{h} = 0$ whenever Grad $\theta = 0$.

The Clausius-Duhem inequality dictates that $\mathbf{f}^r$ cannot be chosen at will but must satisfy the constraints

$$\frac{\partial \Psi}{\partial \mathbf{q}^r}\cdot \mathbf{f}^r \leqslant 0 \ (r \text{ not summed}). \tag{1.5}$$

A particularly suitable form of the Eq. (1.3), which we have used extensively in the past, is:

$$\frac{\partial \Psi}{\partial \mathbf{q}^r} + \mathbf{b}^r \cdot \frac{d\mathbf{q}^r}{dz} = 0 \ (r \text{ not summed}), \tag{1.6}$$

where $\mathbf{b}^r$ is a positive definite viscosity tensor. The Eq. (1.6) satisfies the constraint imposed by the Clausius-Duhem inequality and is in accord with the Onsager notions of linearity between "forces" and "fluxes". This equation is likely to be valid in situations in which the internal variables have a small norm (in a Euclidean sense) even though the overall deformation of the material is large.

The intrinsic time scale merits discussion. It is well known that in dissipative media the stress response is a function of the "strain path" or "strain history". The latter is the strain state expressed as a function of the Newtonian time t. A formal mathematical statement of the above ideas is:

$$\tau = \underset{s=-\infty}{\overset{t}{\mathfrak{F}}}[\mathbf{E}(s)], \tag{1.7}$$

where $\mathfrak{F}$ is a function of the function $E(s)$, $-\infty < s \leqslant t$. This representation is found to be adequate in the case of materials having fading memory with respect to t. It certainly does not apply to non-ageing materials which possess permanent memory (with respect to t) of their thermomechanical history [21].

The endochronic theory, which is based on the notion of intrinsic time, is aimed at the analytical representation of the thermomechanical behavior of materials with permanent memory.

The statement that the stress is a function of the strain "path" gives rise to the question: "which path"? In particular, what is the appropriate path for materials that are "history dependent" but "strain-rate independent"?

We define a path relative to a Riemannian space. Consider a six-dimensional Riemannian space R with metric G_{ij}. Of the six independent components E_i of $\mathbf{E}$ let each be measured along one of the coordinates of R, in a sense of "one-to-one and on-to". Evidently, a state of strain is a point and a strain history is a path in this space. The distance $d\xi$ between two adjacent deformation states is given by the relation

$$d\xi^2 = G_{ij} dE_i dE_j, \tag{1.8}$$

or

$$d\xi = +\sqrt{G_{ij} dE_i dE_j}. \tag{1.9}$$

The intrinsic time ξ is arrived at by the following considerations. If G_{ij} is a material property, then $d\xi$ is a material "time-like interval" between two adjacent strain states: *As such, $d\xi$ is a measure of an intrinsic time scale.* In tensor form the Eq. (1.8) becomes

$$d\xi^2 = d\mathbf{E} \cdot \mathbf{P} \cdot d\mathbf{E}, \tag{1.10}$$

where $\mathbf{P}$ is a material tensor, which may conceivably depend on $\mathbf{E}$.

We showed in a previous paper [2] that the ξ-scale is not adequate for predicting crosshardening when constitutive equations of the convolution type such as the Eqs. (4.2, 3) of Ref. [1]) are used to represent material response. For this purpose the scale ([1]) $\zeta(\xi)$ was introduced, which is such that

$$d\zeta = \frac{d\xi}{f(\xi)}, \tag{1.11}$$

where $f(\xi) > 0$.

If the elapsed time between two adjacent strain states is dt, then:

$$d\xi = \sqrt{G_{ij} \frac{dE_i}{dt} \frac{dE_j}{dt}}\, dt. \tag{1.12}$$

Note that $d\xi$ remains invariant with the transformation

$$dt' = a\,dt \qquad (a > 0) \tag{1.13}$$

which implies that $d\xi$ is independent of the strain rate. If now we set:

$$\boldsymbol{\tau} = \mathop{\mathfrak{F}}_{\zeta'=\zeta_0}^{\zeta}[E(\zeta')], \tag{1.14}$$

([1]) This is a new notation which we introduce for reasons that are discussed in this section.

we have achieved a representation whereby the stress response is history dependent but strain rate independent. The Eq. (1.14) applies, therefore, to plastic materials. We obtain specific forms of the Eq. (1.14) by using the Eqs. (1.1), (1.2) and (1.3) or (1.6).

Materials which are history as well as strain rate dependent must, a *fortiori*, possess an intrinsic time scale z which must be related, in some sense, to the Newtonian time scale t. Such a scale is constructed from the relation given below:

$$dz^2 = \varkappa^2\{d\zeta^2+g^2dt^2\}, \tag{1.15}$$

where g is a material constant and $\varkappa$ depends on $\dot{\zeta}$, as was found from constant strain rate tests on superpure aluminum. The above equation reduces further to the simple form:

$$dz = \varkappa(\dot{\zeta})dt. \tag{1.16}$$

In view of the nature of the dependence of $\dot{\zeta}$ on $\dot{\varepsilon}_{ij}$, under monotonically increasing proportional loading, $\varkappa$ is a special sort of function of $\dot{\varepsilon}_{ij}$ as well as $\dot{\varepsilon}_{ij}$ though its dependence on the latter may be rather weak.

Note the differences between the present form of dz and the one given in our previous paper. There,

$$d\zeta^2 = d\xi^2+g^2dt^2 \tag{1.17}$$

and

$$dz = \frac{d\zeta}{f(\zeta)}. \tag{1.18}$$

This definition gives rise to conceptual difficulties. Under conditions of zero strain (i.e., $d\xi = 0$), $d\zeta = gdt$. This last result will lead to material ageing as a consequence of the Eq. (1.18). This can be averted if g is a function of $\dot{\varepsilon}$ and furthermore if $g(0) = 0$. However, this constraint eliminates also stress relaxation under constant strain, an essential characteristic of viscoplastic materials. Another possibility is to set $g(0) \neq 0$, but $f(\zeta) = 1$. This set of constraints eliminates ageing, but unfortunately strain hardening as well. Thus, in the old definition, there is no way of circumventing these difficulties.

The advantage of the new definition for dz, as given by the Eq. (1.16), is that it accommodates the above effects in a self-consistent fashion.

Explicit forms of constitutive equations which are potentially applicable to large deformation histories and isothermal conditions may be constructed by introducing a scalar function $\Psi_0(\mathbf{C})$ and a generalized strain tensor $\mathbf{\Psi}_1(\mathbf{C})$ with respect to which the free energy Ψ retains its quadratic form in the internal variables. Thus, under these conditions:

$$\Psi = \Psi_0(\mathbf{C})+\sum_r \mathbf{\Psi}_1\cdot\mathbf{B}^r\cdot\mathbf{q}^r+1/2\sum_r \mathbf{q}^r\cdot\mathbf{C}^r\cdot\mathbf{q}^r, \tag{1.19}$$

where $\mathbf{B}^r$ and $\mathbf{C}^r$ are constant fourth the order material tensors. In conjunction with the Eq. (1.6), Eq. (1.19) then leads, without essential difficulty, to the constitutive equation (1.20)

$$\tau = 2\frac{\varrho}{\varrho_0}\left\{\frac{\partial\Omega_0}{\partial\mathbf{C}}+\frac{\partial\Psi_1}{\partial\mathbf{C}}\int_0^z \lambda(z-z')\frac{\partial\Psi_1}{\partial z'}dz'+2\frac{\partial\mathbf{\Psi}_1}{\partial\mathbf{C}}\cdot\int_0^z \mu(z-z')\frac{\partial\mathbf{\Psi}_1}{\partial z'}dz'\right\}, \tag{1.20}$$

where $\Psi_1 = \text{trace}\, \mathbf{\Psi}_1$,

$$\Omega_0 = \Psi_0 - 1/2\lambda(0)\Psi_1^2 - \mu(0)\mathbf{\Psi}_1 \cdot \mathbf{\Psi}_1. \tag{1.21}$$

A quasi-linear form of the Eq. (1.9) which applies to isothermal conditions and small strains may be obtained by setting

$$\mathbf{\Psi}_1 = \mathbf{E} \sim \boldsymbol{\epsilon}, \tag{1.22}$$

where $\boldsymbol{\epsilon}$ is the small strain tensor. In this event, within terms of order $||\varepsilon||^2$

$$\boldsymbol{\tau} = \frac{\partial \Omega}{\partial \boldsymbol{\epsilon}} + \int_0^z \lambda(z-z') \frac{\partial (\mathrm{tr}\boldsymbol{\epsilon})}{\partial z'} dz' + 2\int_0^z \mu(z-z') \frac{\partial \boldsymbol{\epsilon}}{\partial z'} dz'. \tag{1.23}$$

The Eq. (1.23) may be fully linearized by setting

$$\Omega = 1/2\lambda_\infty(\mathrm{tr}\,\boldsymbol{\epsilon})^2 + \mu_\infty(\mathrm{tr}\,\boldsymbol{\epsilon}^2). \tag{1.24}$$

The one-dimensional counterpart of the Eq. (1.20) is

$$\tau = \frac{d\Omega}{d\lambda} + \frac{d\omega}{d\lambda} \int_0^z E(z-z') \frac{d\omega}{dz'} dz', \tag{1.25}$$

where ω is a function of λ and $E(z)$ is a uniaxial "heredity modulus". A quasi-linear form of the above equation may be obtained by setting $\omega = \lambda - 1 \equiv \varepsilon$. In this case

$$\tau = f(\varepsilon) + \int_0^z E(z-z') \frac{\partial \varepsilon}{\partial z'} dz', \tag{1.26}$$

where $f(\varepsilon) = d\Omega/d\lambda$. A fully linear form is obtained by setting $f(\varepsilon) = f_0\varepsilon$. In this event

$$\tau = \int_0^z E_L(z-z') \frac{\partial \varepsilon}{\partial z'} dz', \tag{1.27}$$

where

$$E_L = f_0 H(z) + E(z), \tag{1.28}$$

i.e. E_L is the heredity modulus of the fully linear theory.

2. Gibbs free energy formulation

In this formulation, which we shall call the Φ-formulation, the *stress components* in the material system ([2]) (calculated per unit undeformed area), the absolute temperature θ as well as the n internal variables $\mathbf{q}^r$ are regarded as the independent variables. One then seeks to derive the strain and entropy response of a material system to a thermomechanical excitation.

With this in mind, we write the first law in the form:

$$\dot{\varepsilon} = \Pi^{\alpha\beta} \dot{E}_{\alpha\beta} - h^{\alpha}_{,\alpha}, \tag{2.1}$$

([2]) These are the components of the Piola stress tensor $\mathbf{\Pi}$.

where $\mathbf{E}$ is the Green deformation tensor defined by the relation:

(2.2) $$\mathbf{E} = \frac{1}{2}(\mathbf{C}-\boldsymbol{\delta})$$

and

(2,3) $$\Pi^{\alpha\beta} = \frac{\varrho_0}{\varrho}\frac{\partial x^\alpha}{\partial y_i}\frac{\partial x^\beta}{\partial y_j}T_{ij},$$

where $\partial y_i/\partial x$ are the deformation gradient components and T_{ij} the components of the Cauchy stress tensor defined in the spatial frame of reference. The Eqs. (1.1) and (1.2) may be written in the compact form:

(2.4) $$\dot{\Psi}|_{\mathbf{q}} - \Pi^{\alpha\beta}\dot{E}_{\ \beta} + \eta\dot{\theta} = 0,$$

where the first term on the left-hand side of the Eq. (2.4) is the time rate change of Ψ keeping $\mathbf{q}^r$ constant for all r.

At this point we introduce the Gibbs free energy Φ by the relation

(2.5) $$\Phi = \Psi - \Pi^{\alpha\beta}E_{\alpha\beta}.$$

The Eq. (1.1) implies that $\mathbf{E}$ is a function of $\mathbf{\Pi}$, θ, and $\mathbf{q}^r$, assuming the requisite condition of invertibility. In that event Φ is also a function of $\mathbf{\Pi}$, θ, and $\mathbf{q}^r$. Furthermore, as a result of the Eq. (2.5).

(7.6) $$\dot{\Psi}|_q = \dot{\Phi}|_q + \Pi^{\alpha\beta}\dot{E}_{\alpha\beta}|_q + \dot{\Pi}^{\alpha\beta}E_{\alpha\beta}.$$

Substitution of the Eq. (2.6) in Eq. (2.4) and recall of the independence of the appropriate new variable, yields the equation:

(2.7) $$E_{\alpha\beta} = -\left.\frac{\partial\Phi}{\partial\Pi^{\ \beta}}\right|_{\theta,q},$$

(2.8) $$\eta = -\left.\frac{\partial\Phi}{\partial\theta}\right|_{\Pi,q}.$$

Furthermore the Eqs, (2.5), (2.7) and (2.8) lead immediately to the relation

(2.9) $$\frac{\partial\Psi}{\partial\mathbf{q}^r}\cdot\dot{\mathbf{q}}^r = \frac{\partial\Phi}{\partial\mathbf{q}^r}\cdot\dot{\mathbf{q}}^r \quad (r \text{ not summed})$$

in which event the Clausius-Duhem inequality now is:

(2.10) $$\frac{\partial\Phi}{\partial\mathbf{q}^r}\cdot\mathbf{q}^r \leqslant 0,$$

or

(2.11) $$\frac{\partial\Phi}{\partial\mathbf{q}^r}\cdot\frac{d\mathbf{q}^r}{dz} \leqslant 0$$

assuming, of course, that $dz/dt > 0$.

Since we are dealing with an endochronic theory, the concomitant evolution equations for the $\mathbf{q}$'s will be of the type

(2.12) $$\frac{d\mathbf{q}^r}{dz} = \mathbf{g}^r(\mathbf{\Pi}, \theta, \mathbf{q}^r),$$

where $\mathbf{g}^r$ must satisfy the constraint (2.10).

Needless to say that $\mathbf{g}^r$ is related to $\mathbf{f}^r$ by the equation

$$\mathbf{g}^r = \mathbf{f}^r(\mathbf{E}(\mathbf{\Pi}, \mathbf{q}^r, \theta), \mathbf{q}_r). \tag{2.13}$$

In deriving explicit forms of constitutive equations we take the view that in the vicinity of small strains, at least the Eq. (2.11) does in fact have the form [3]

$$\frac{\partial \Phi}{\partial \mathbf{q}^r} + \mathbf{b}^r \cdot \dot{\mathbf{q}}^r = 0 \qquad (r \text{ not summed}), \tag{2.14}$$

where $\hat{\mathbf{q}}^r \equiv d\mathbf{q}^r/dz$. The constraint (2.12) is satisfied if $\mathbf{b}^r$ are positive definite fourth-order tensors for all r.

Other, more general, forms of the Eq. (2.14) are of course possible. We pointed out in earlier papers that the inequality (2.11) implies, in fact, a functional relationship between $d\mathbf{g}^r/dz$ and $\partial\Phi/\partial\mathbf{q}^r$. Thus, the Eq. (2.12) will a fortiori have the more specific form,

$$\frac{d\mathbf{q}^r}{dz} = g^r\left(\frac{\partial \Phi}{\partial \mathbf{q}^s}\right), \tag{2.15}$$

i.e. $\mathbf{g}^r$ will in general be a function of the Gibbs free energy gradients $\partial\Phi/\partial\mathbf{q}^s$.

The heat conduction equation will also be of the form:

$$\mathbf{h} = \mathbf{h}(\mathbf{\Pi}, \theta, \mathbf{q}^r, \mathrm{Grad}\,\theta). \tag{2.16}$$

The Φ-formulation is now complete.

Explicit constitutive equations may again be obtained by expanding Φ in terms of $\phi_0(\Pi)$, $\phi_1(\mathbf{\Pi})$ and $\mathbf{q}^r$, in a form analogous to the Eq. (1.19), i.e.

$$\Phi = \phi_0(\mathbf{\Pi}) + \sum_r \boldsymbol{\phi}_1 \cdot \mathbf{D}^r \cdot \mathbf{q}_1^r + 1/2 \sum_r \mathbf{q}^r \cdot \mathbf{F}^r \cdot \mathbf{q}^r. \tag{2.17}$$

Use of this equation in conjunction with the Eqs. (2.7) and (2.14) yields a constitutive equation very much like Eq. (1.20), in form, of the type:

$$\mathbf{E} = \frac{\partial \chi}{\partial \mathbf{\Pi}} + \frac{\partial \phi}{\partial \mathbf{\Pi}} \int_0^z J_1(z-z') \frac{\partial \phi}{\partial z'} dz' + \frac{\partial \boldsymbol{\phi}_1}{\partial \mathbf{\Pi}} \cdot \int_0^z J_2(z-z') \frac{\partial \boldsymbol{\phi}_1}{\partial z'} dz', \tag{2.18}$$

where

$$\phi = \mathrm{tr}\,\boldsymbol{\phi}_1 \tag{2.19}$$

and

$$\chi = -\phi_0 + 1/2 J_1(0)\phi^2 + 1/2 J_2(0) \boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_1. \tag{2.20}$$

A quasi-linear form of the Eq. (2.20) which applies to small strains is obtained by setting $\mathbf{E} \sim \boldsymbol{\epsilon}$, $\boldsymbol{\phi}_1 = \boldsymbol{\Pi} \equiv \boldsymbol{\sigma}$ in which event

$$\boldsymbol{\epsilon} = \frac{\partial \chi}{\partial \boldsymbol{\sigma}} + \boldsymbol{\delta} \int_0^z J_1(z-z') \frac{\partial \sigma_{kk}}{\partial z'} dz' + \int_0^z J_2(z-z') \frac{\partial \boldsymbol{\sigma}}{\partial z'} dz'. \tag{2.21}$$

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[3] The Eq. (2.14) is, in fact, the counterpart of the Eq. (1.6).

The Eq. (2.21) may be fully linearized by setting

$$\chi = 1/2J_{10}(\mathrm{tr}\,\boldsymbol{\sigma})^2 + 1/2J_{20}(\mathrm{tr}\,\boldsymbol{\sigma}^2). \tag{2.22}$$

The uniaxial counterpart of the Eq. (2.21) is

$$\varepsilon = g(\sigma) + \int_0^z J(z-z')\frac{\partial\sigma}{\partial z'}dz', \tag{2.23}$$

where

$$g(\sigma) = \left.\frac{\partial\chi}{\partial\sigma_1}\right|_{\sigma_2=\sigma_3=0}, \quad (\sigma_{ij} = 0, \quad i \neq j)$$

and $\sigma \equiv \sigma_1$.

In its fully linear form the Eq. (2.23) becomes

$$\varepsilon = \int_0^z J_L(z-z')\frac{\partial\sigma}{\partial z'}dz', \tag{2.24}$$

where

$$J_L = \chi_0 H(z) + J_1(z), \tag{2.25}$$

$H(z)$ being the Heaviside step function of z.

3. Prediction of unloading with the linear theory

The Eqs. (1.21) and (2.24) are completely equivalent; in fact, J_L and E_L are related by the integral equation

$$\int_0^z E_L(z-z')\frac{dJ_L}{dz'}dz' = H(z). \tag{3.1}$$

To examine how the linear theory predicts unloading behavior we need only study the Eq. (1.27). For the sake of clarity we drop the suffix L and write (1.27) as:

$$\tau = \int_0^z E(z-z')\frac{d\varepsilon}{dz'}dz'. \tag{3.2}$$

Consider a deformation history, in which the strain increases monotonically from zero to some value ε_1 and is then decreased. The process is assumed to take place at a substantially constant absolute value of the strain rate. Let the slope of the unloading stress strain curve be denoted by E_-. Then it can be shown that

$$E_- = 2E_0 - E_+, \tag{3.3}$$

where E_0 is the initial modulus $d\sigma/d\varepsilon|_{\varepsilon=0}$ and E_+ is the tangent modulus at the point of unloading. Since in the plastic range E_+ is substantially less than E_0, the Eq. (1.7) fails to predict the experimentally observed unloading slope which is essentially equal to E_0, for metals at room temperature.

It is worth emphasizing that Eq. (1.2) is true, irrespective of the form of $E(z)$ and $z(\zeta)$, provided that the inequalities in Eq. (3.4) hold, i.e.,

$$\frac{d\zeta}{dz} > 0, \quad k > 0 \tag{3.4a,b}$$

where

$$d\zeta = k|d\varepsilon|. \tag{3.5}$$

4. Prediction of unloading with quasi-linear theories

The quasi-linear Ψ-form of the theory [Eq. (1.26)] can predict the observed unloading slope by suitably choosing the form of the function $f(\varepsilon)$. The pertinent relation is:

$$\left.\frac{d\tau}{d\varepsilon}\right|_{+} + \left.\frac{d\tau}{d\varepsilon}\right|_{-} = 2E(0)+2f'(\varepsilon). \tag{4.1}$$

In particular, the condition

$$\left.\frac{d\tau}{d\varepsilon}\right|_{-} = E_0 \tag{4.2}$$

leads readily to the relation

$$f'(\varepsilon) = 1/2\left.\frac{d\tau}{d\varepsilon}\right|_{+}. \tag{4.3}$$

The quasi-linear Φ-form of the theory [Eq. (2.23)] will also predict the observed unloading slope by a suitable choice of the function $g(\sigma)$. Evidently, from Eq. (2.23) $g(0) = 0$. Also it can be shown that without loss of generality one can set $g'(0) = 0$. It then transpires that $J(0) = 1/E_0$, where E_0 is initial tangent modulus $d\sigma/d\varepsilon|_{\varepsilon=0}$. Equation (2.23) may then be re-written in the form:

$$E_0\,\varepsilon = \int\limits_0^z J(z-z')\frac{d\sigma}{dz'}\,dz'+F(\sigma), \tag{4.4}$$

where $F(\sigma) = E_0 g(\sigma)$, and $J(0) = 1$. The following relation is then found to hold

$$2E_0 = \left(1+\frac{dF}{d\sigma}\right)\left(\left.\frac{d\sigma}{d\varepsilon}\right|_{+}+\left.\frac{d\sigma}{d\varepsilon}\right|_{-}\right). \tag{4.5}$$

In the event that the unloading slope is equal to E_0 it follows that

$$\frac{dF}{d\sigma} = \frac{E_0-E_t}{E_0+E_t}, \tag{4.6}$$

where $E_t = d\sigma/d\varepsilon|_{+}$.

Thus, the quasi-linear Φ-theory can also predict unloading behavior correctly. However, we would like to point out that there is evidence that the Φ-form possesses other attractive features and may prove more suitable for the representation of the mechanical behavior of metals.

5. A simple model of the quasi-linear Φ-form

The Eq. (4.4) may be expressed in the alternative form

$$E_0 \frac{d\varepsilon}{dz} = \frac{d\sigma}{dz} + \int_0^z \dot{J}(z-z') \frac{d\sigma}{dz'} dz' + \frac{dF}{dz}. \tag{5.1}$$

In the event that $\dot{J} = \alpha$, where α is a constant (a "Maxwell model"), then the Eq. (5.1) simplifies to the expression

$$E_0 \frac{d\varepsilon}{dz} = \frac{2E_0}{E_0 + E_t(|\sigma|)} \frac{d\sigma}{dz} + \alpha\sigma. \tag{5.2}$$

Consider the very simple case where $z = \zeta$, and $d\zeta = |d\varepsilon|$ (no strain hardening such as in mild steel at moderate strains). Then the Eq. (5.2) may be integrated to yield, under monotonic loading condition:

$$E_0 \varepsilon = -\sigma_0 \left\{ 2\log\left(1 - \left|\frac{\sigma}{\sigma_0}\right|\right) + \frac{\sigma}{\sigma_0} \right\}, \tag{5.3}$$

where σ_0 is the ultimate stress of the material. In this event

$$E_t(\sigma) = E_0 \left(\frac{\sigma_0 - |\sigma|}{\sigma_0 + |\sigma|} \right) \tag{5.4}$$

and the Eq. (5.2) now simplifies to the form:

$$E_0 \frac{d\varepsilon}{dz} = \left\{ 1 + \left|\frac{\sigma}{\sigma_0}\right| \right\} \frac{d\sigma}{dz} + E_0 \left(\frac{\sigma}{\sigma_0} \right). \tag{5.5}$$

It may readily be shown that the Eq. (5.5) will predict, under loading conditions, as it must, stress strain curve given by the Eq. (5.3), whereas under unloading *it gives the exact result*,

$$\frac{d\sigma}{d\varepsilon} = E_0, \tag{5.6}$$

i.e., the constitutive equation (5.5) predicts that the unloading part of the stress-strain curve is, in fact, an exact straight line with slope E_0.

A truly remarkable result.

6. A stress-based intrinsic time scale

Let me begin with the view that the state of strain is a function of the stress path, in the Riemannian space τ_α ($\alpha = 1, 2, \ldots, 6$) with metric $R_{\cdot\beta}$ such that the square of an element of path length $d\xi$ in this space is:

$$d\xi^2 = R_{\;\beta} d\Pi \; d\Pi_\beta \tag{6.1}$$

or in tensor form:

$$d\xi^2 = d\boldsymbol{\Pi} \cdot \mathbf{R} \cdot d\boldsymbol{\Pi}, \tag{6.2}$$

where $\mathbf{R}$ is a fourth-order positive definite symmetric material tensor.

The Eqs. (1.15) and (1.16) which define the z-scale then apply without change. The constitutive equations (1.20), (1.25), (1.26), (1.27), as well as (2.18), (2.21), (2.23) and (2.24) also remain unchanged *in form.*

The theory is thus formulated in terms of an "intrinsic time measure" which is *stress* based.

7. Substantiation of the theory

A primary characteristic of the mechanical behavior of metals is their permanent memory of their deformation history. Furthermore, they exhibit cross-effects in small strain regions where one would have expected uncoupled linear constitutive equations to apply. Effects such as these are cross-hardening, i.e., hardening in tension due to torsional prestrain (or *vice versa*); cumulative axial extension due to cyclic torsion (cyclic creep) in the presence or *absence* of axial stress; cumulative axial stress relaxation in tension due to cyclic creep; cyclic hardening or softening due to cyclic deformation, depending on the prehistory of the specimen; effect of shear stress on response in tension (or *vice versa*); other effects too numerous to mention.

The effects mentioned specifically above form a sufficiently rigorous set of criteria to be regarded as an essential challenge to any constitutive representation that pertains to the plastic behavior of materials. In the recent past we have been able to predict the above effects with the endochronic theory, sometimes with astonishing accuracy, using very simple forms of the constitutive equations and with minimum analysis.

Effects in the above class have been investigated by MAIR *et Als.* [3], LUBAHN [4], WADSWORTH [5], IVEY [6], BENHAM [7], FREUDENTHAL and RONAY [8, 10], BENDLER and WOOD [9], RONAY [11], UDOGUCHI and ASADA [12], MORROW [13], COFFIN [14], and other people too numerous to mention here.

The representation and prediction of their results by the endochronic theory is given in Refs. [2, 15, 16 and 17].

The application of the theory to situations where the strain rate effects are significant is the object of investigation in Refs. [18, 19, and 20].

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