Archives of Mechanics • Archiwum Mechaniki Stosowanej • 27, 5-6, pp. 883–887, Warszawa 1975

BRIEF NOTES

Effective stress in mixture theory(*)

L. W. MORLAND (NORWICH)

DESCRIBING the gross response of a fluid-saturated porous solid or composite medium by a mixture theory introduces partial stresses and partial densities through the concept of overlapping particles. An effective stress-effective deformation theory is used to retain the known properties of the individual constituents, but the partial stress-effective stress relation becomes a constitutive postulate. An interpretation of effective stress is introduced which rationalises and extends the earlier direct postulates, and in particular equates the cross-section area and volume fractions of a constituent within a mixture element.

1. Mixture theory

A MIXTURE or interacting continua theory is a convenient framework in which to describe the gross response of a continuum composed of two or more constituents [e.g., TRUESDELL, 1965]. That is, when a representative element of mixture, small compared with the length scale of interest, contains sufficient of each constituent $\overset{(\alpha)}{s}$ $(\alpha = 1, \ldots, r)$ to assume that every point $\mathbf{x}$ in the mixture is occupied by a particle of each $\overset{(\alpha)}{s}$ at all times t. Then for each $\overset{(\alpha)}{s}$ there is a velocity field $\overset{(\alpha)}{\mathbf{v}}(\mathbf{x}, t)$ which represents in some mean sense the velocity of $\overset{(\alpha)}{s}$ particles in a neighbourhood of $\mathbf{x}$ at time t, and an associated particle path

$$\mathbf{x} = \overset{(\alpha)}{\boldsymbol{\chi}}(\overset{(\alpha)}{\mathbf{X}}, t), \quad \overset{(\alpha)}{\mathbf{v}} = \partial \overset{(\alpha)}{\boldsymbol{\chi}} / \partial t, \tag{1.1}$$

where $\overset{(\alpha)}{\mathbf{X}}$ is a reference position of the $\overset{(\alpha)}{s}$ particle. The deformation gradient of the $\overset{(\alpha)}{s}$ motion is

$$\overset{(\alpha)}{\boldsymbol{F}} = \nabla \overset{(\alpha)}{\boldsymbol{\chi}}, \quad \overset{(\alpha)}{F}_{ij} = \partial \overset{(\alpha)}{\chi}_i / \partial \overset{(\alpha)}{X}_j, \tag{1.2}$$

where the indices $(i, j = 1, 2, 3)$ refer to rectangular Cartesian coordinates.

Mass conservation implies

$$\overset{(\alpha)}{J} = |\det \overset{(\alpha)}{\boldsymbol{F}}| = \overset{(\alpha)}{\varrho}_0 / \overset{(\alpha)}{\varrho}, \tag{1.3}$$

where $\overset{(\alpha)}{\varrho}$ is the partial density of the mass $\overset{(\alpha)}{s}$ per unit volume of mixture, and $\overset{(\alpha)}{\varrho}_c$ is the value in the reference state. The traction on unit cross-section of mixture with unit out-

(*) The paper has been presented at the *EUROMECH* 54 *COLLOQUIUM* on "*FINITE DEFORMATIONS IN PLASTICITY*", Jabłonna, September 30–October 3, 1974.

ward normal $\mathbf{n}$ is the sum of the partial tractions $\overset{(\alpha)}{\mathbf{t}}_{(\mathbf{n})}$ — the traction supported by $\overset{(\alpha)}{s}$ in the unit cross-section of mixture. Linear momentum balance implies the existence of a partial stress tensor $\overset{(\alpha)}{\boldsymbol{\sigma}}$ such that

$$\overset{(\alpha)}{\mathbf{t}}_{(\mathbf{n})} = \overset{(\alpha)}{\boldsymbol{\sigma}}\mathbf{n}, \tag{1.4}$$

and then

$$\overset{(\alpha)}{\varrho}\,\frac{\overset{(\alpha)}{D}\overset{(\alpha)}{v}_i}{Dt} = \frac{\partial \overset{(\alpha)}{\sigma}_{ij}}{\partial x_j} + \overset{(\alpha)}{\varrho}\overset{(\alpha)}{b}_i + \varrho\overset{(\alpha)}{\beta}_i, \quad i = 1, 2, 3. \tag{1.5}$$

$\overset{(\alpha)}{D}/Dt$ is the material time derivative following $\overset{(\alpha)}{s}$, ϱ is the mass of unit volume of mixture, the sum of the $\overset{(\alpha)}{\varrho}$, $\overset{(\alpha)}{\mathbf{b}}$ is the external body force on $\overset{(\alpha)}{s}$ per unit mass of $\overset{(\alpha)}{s}$, and $\overset{(\alpha)}{\beta}$ is the interaction body force per unit mass of mixture on $\overset{(\alpha)}{s}$ due to other constituents. Finally, angular momentum balance in the absence of external body couple shows that the skew part of $\overset{(\alpha)}{\boldsymbol{\sigma}}$ is given by

$$\overset{(\alpha)}{\boldsymbol{\sigma}}_{[\,]} = \varrho\overset{(\alpha)}{\boldsymbol{\lambda}}, \tag{1.6}$$

where $\overset{(\alpha)}{\boldsymbol{\lambda}}$ is a skew tensor defining the interaction couple per unit mass of mixture on $\overset{(\alpha)}{s}$.

Here thermomechanical coupling is excluded for brevity, since it does not influence the concept of effective stress introduced shortly. Energy balance is needed only to compute energy storage in a given motion. This mechanical theory is completed by prescribing constitutive laws for the interaction terms $\overset{(\alpha)}{\boldsymbol{\beta}}$, $\overset{(\alpha)}{\boldsymbol{\lambda}}$ subject to the restrictions

$$\sum_{\alpha}\overset{(\alpha)}{\boldsymbol{\beta}} = \mathbf{0}, \quad \sum_{\alpha}\overset{(\alpha)}{\boldsymbol{\lambda}} = \boldsymbol{0}, \tag{1.7}$$

and for the symmetric parts of the partial stresses, $\overset{(\alpha)}{\boldsymbol{\sigma}}_{(\,)}$.

2. Effective stress and deformations

Here attention is focussed on constitutive laws for the symmetric partial stresses $\overset{(\alpha)}{\sigma}_{(\,)}$, following the effective stress, effective deformation theory proposed by MORLAND [1972]. Introducing the volume fraction $\overset{(\alpha)}{n}$ of each constituent $\overset{(\alpha)}{s}$, the effective density $\overset{E(\alpha)}{\varrho}$ of $\overset{(\alpha)}{s}$ — the mass per unit volume of $\overset{(\alpha)}{s}$ — is given by

$$\overset{(\alpha)}{\varrho} = \overset{(\alpha)}{n}\overset{E(\alpha)}{\varrho}, \tag{2.1}$$

and

$$\sum_{\alpha}\overset{(\alpha)}{n} = 1, \quad 0 \leqslant \overset{(\alpha)}{n} \leqslant 1. \tag{2.2}$$

Then an effective deformation gradient $\overset{E(\alpha)}{F}$ was defined by

$$\overset{E(\alpha)}{F} = (\overset{(\alpha)}{n}/\overset{(\alpha)}{n}_0)^{\frac{1}{3}}\overset{(\alpha)}{F}, \tag{2.3}$$

where $\overset{(\alpha)}{n}_0$ is the initial volume fraction of $\overset{(\alpha)}{s}$.

Similarly, it was proposed that on a cross-section of mixture a constituent $\overset{(\alpha)}{s}$ would occupy only an area fraction $\overset{(\alpha)}{m}$, so that the effective traction $\overset{E(\alpha)}{\mathbf{t}}_{(\mathbf{n})}$ on $\overset{(\alpha)}{s}$ is given by

$$\overset{(\alpha)}{\mathbf{t}}_{(\mathbf{n})} = \overset{(\alpha)}{m}\overset{E(\alpha)}{\mathbf{t}}_{(\mathbf{n})}, \tag{2.4}$$

and hence

$$\overset{E(\alpha)}{t}_{(\mathbf{n})} = \overset{E(\alpha)}{\sigma}\mathbf{n}, \tag{2.5}$$

where

$$\overset{(\alpha)}{\sigma} = \overset{(\alpha)}{m}\overset{E(\alpha)}{\sigma}. \tag{2.6}$$

In the case $\overset{(\alpha)}{\lambda} \neq 0$ only the symmetric part $\overset{(\alpha)}{\sigma}_{(\,)}$ is taken in (2.6). It was then postulated that the constituent $\overset{(\alpha)}{s}$ retains its constitutive description, defined by a symmetric stress functional $\overset{(\alpha)}{\mathscr{F}}$ of deformation history say, within the mixture in terms of the effective stress and deformations; that is

$$\overset{E(\alpha)}{\sigma}(\mathbf{x}, t) = \overset{(\alpha)}{\mathscr{F}}\{\overset{E(\alpha)}{F}(\mathbf{x}, \tau), -\infty < \tau \leqslant t\}. \tag{2.7}$$

The constitutive description of the mixture is then completed by prescribing laws for each $\overset{(\alpha)}{n}, \overset{(\alpha)}{m}$ subject to (2.2) and

$$\sum_{\alpha}\overset{(\alpha)}{m} = 1, \quad 0 \leqslant \overset{(\alpha)}{m} \leqslant 1. \tag{2.8}$$

An implicit assumption in (2.7) is that $\overset{E(\alpha)}{\sigma}$ is independent of the orientation $\mathbf{n}$. This only follows from the assertions (2.4)–(2.6) if $\overset{(\alpha)}{m}$ is independent of $\mathbf{n}$ which is a further assumption of structural isotropy. A simplifying restriction, which incorporates this requirement, is

$$\overset{(\alpha)}{m} = \overset{(\alpha)}{n}, \quad \alpha = 1, \ldots, r, \tag{2.9}$$

which was deduced [MORLAND, 1972] from a mathematical axiom, but without physical motivation. The restriction (2.9) was postulated directly by BIOT [1956] and SCHIFFMAN [1970] for defining effective fluid pressures in linear elastic-fluid mixtures, but effective stress in the elastic motion was not defined in like manner. It is now shown that appropriate interpretation of partial and effective stress leads directly to the result (2.6) with the restriction (2.9).

Let V be the volume of a representative element of mixture, and $\overset{(\alpha)}{B}$ the configuration occupied by $\overset{(\alpha)}{s}$ within the element. Let $\overset{(\alpha)}{\mathbf{\Sigma}}$ be the actual stress tensor (symmetric) within $\overset{(\alpha)}{B}$, and $\overset{(\alpha)}{\mathbf{T}}_{(\mathbf{n})}$ the traction on any section with orientation $\mathbf{n}$ within $\overset{(\alpha)}{B}$. Then for each $\mathbf{n}$,

$$\overset{(\alpha)}{\mathbf{T}}_{(\mathbf{n})} = \overset{(\alpha)}{\sum} \mathbf{n}. \tag{2.10}$$

Now interpret the partial stress as the mixture volume average

$$\overset{(\alpha)}{\boldsymbol{\sigma}} = \frac{1}{V} \int_{\overset{(\alpha)}{B}} \overset{(\alpha)}{\sum} dV + \varrho \overset{(\alpha)}{\boldsymbol{\lambda}}, = \overset{(\alpha)}{\boldsymbol{\sigma}}_{(\,)} + \varrho \overset{(\alpha)}{\boldsymbol{\lambda}}, \tag{2.11}$$

where the skew contribution is the interaction term required for angular momentum balance (1.6) of the element. The partial traction for each $\mathbf{n}$ is

$$\overset{(\alpha)}{\mathbf{t}}_{(\mathbf{n})} = \overset{(\alpha)}{\sigma}_{(\,)}\mathbf{n} + \varrho \overset{(\alpha)}{\boldsymbol{\lambda}} \mathbf{n}. \tag{2.12}$$

Similarly, interpret the effective stress as the $\overset{(\alpha)}{s}$ volume average

$$\overset{E(\alpha)}{\boldsymbol{\sigma}} = \frac{1}{\overset{(\alpha)}{n} V} \int_{(\alpha)} \overset{(\alpha)}{\sum} dV, = \frac{1}{\overset{(\alpha)}{n}} \overset{(\alpha)}{\boldsymbol{\sigma}}_{(\,)}, \tag{2.13}$$

and the effective traction for each $\mathbf{n}$ as

$$\overset{E(\alpha)}{\mathbf{t}}_{(\mathbf{n})} = \frac{1}{\overset{(\alpha)}{n} V} \int_{\overset{(\alpha)}{(B)}} \overset{(\alpha)}{\mathbf{T}}_{(\mathbf{n})} dV = \frac{1}{\overset{(\alpha)}{n}} \overset{(\alpha)}{\boldsymbol{\sigma}}_{(\,)} \mathbf{n} = \frac{1}{\overset{(\alpha)}{n}} \overset{(\alpha)}{\mathbf{t}}_{(\mathbf{n})} - \frac{1}{\overset{(\alpha)}{n}} \varrho \overset{(\alpha)}{\boldsymbol{\lambda}} \mathbf{n}. \tag{2.14}$$

The result (2.13) is (2.6) applied to $\overset{(\alpha)}{\boldsymbol{\sigma}}_{(\,)}$ and subject to (2.9), while (2.14) shows that the effective traction is not just a scaled partial traction (2.4) subject to (2.9), but also incorporates a contribution from the interaction couple. This is an extension of the postulates (2.4) consistent with (2.6) when an interaction couple is present. Using the effective stress law (2.7) with the interpretations (2.11), (2.13) to determine the partial stress shows the explicit role of $\operatorname{div} \varrho \overset{(\alpha)}{\boldsymbol{\lambda}}$ in the linear momentum balance. While it is plausible to set $\overset{(\alpha)}{\boldsymbol{\lambda}} \equiv \boldsymbol{0}$ in a fluid-saturated porous solid, it is not clear that there is no interaction couple between bonded solids in a representative element of a composite material.

The above interpretations of effective stress are the volume average definitions of mean stress adopted by HILL [1963] in his linear elastic composite theory, there using the initial volume fractions. The present theory applied to a mixture of elastic solids and linearised in the infinitesimal strain approximation [MORLAND, 1972] is analogous to Hill's theory if the effective strains are also interpreted as the volume average means adopted by HILL, but here the volume fraction $\overset{(\alpha)}{n}$ can be a linear function of the strains.

References

1. M. A. BIOT, *Theory of propagation of elastic waves in a fluid-saturated porous solid*, 1, *Low frequency range*, J. Acoust. Soc. Amer., **28**, 168–178, 1956.
2. R. HILL, *Elastic properties of reinforced solid, some theoretical principles*, J. Mech. Phys. Solids, **11**, 357–372, 1963.
3. L. W. MORLAND, *A simple constitutive theory for a fluid-saturated porous solid*, J. Geophys. Res., **77**, 890–900, 1972.
4. R. L. SCHIFFMAN, *The stress components of a porous medium*, J. Geophys. Res., **75**, 4035–4038, 1970.
5. C. TRUESDELL, *The rational mechanics of materials*, in "On the Foundations of Mechanics and Energetics", pp. 293–304, Gordon and Breach, New York 1965.

SCHOOL OF MATHEMATICS AND PHYSICS
UNIVERSITY OF EAST ANGLIA, NORWICH, ENGLAND.