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On the flow of magnetic fluids

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THE PAPER deals with the behaviour of a ferromagnetic suspension flowing in the presence of an external magnetic field under a condition of weak rotary Brownian motion. Only stationary flows in a uniform field with flow variables depending upon a coordinate normal to streamlines are considered. This proves to be quite sufficient to allow certain important qualitative features of ferrofluids (and of suspensions of dipolar spheres in general), which are displayed in these flows, to be elucidated and properly understood.

Praca dotyczy zachowania się ferromagnetycznej zawiesiny płynącej w zewnętrznym polu magnetycznym przy warunku słabo obrotowego ruchu Browna. Rozważane są jedynie przepływy stacjonarne w polu jednorodnym ze zmiennymi przepływu, zależnymi od współrzędnej prostopadłej do linii prądu. Okazuje się, że jest to wystarczające do wyjaśnienia i właściwego zrozumienia niektórych jakościowych własności ferrocieczy (w ogólności zawiesin kul dipolarnych) występujących w tych przepływach.

Работа касается поведения ферромагнитной взвеси текущей во внешнем магнитном поле при условии слабого вращательного броуновского движения. Рассматриваются только стационарные течения в однородном поле с переменными, зависящими от координаты перпендикулярной к линии тока. Оказывается, что это достаточно для выяснения и соответственного понимания некоторых качественных свойств феррожидкостей (в общем взвеси дипольных сфер), выступающих в этих течениях.

1. Introduction

AN EXHAUSTIVE study of the bulk rheological properties of dilute suspensions of dipolar spherical particles in an external field was first put forward in [1, 2]. Later this treatment was extended to include the effects of rotary Brownian motion [3, 4] and generalized to suspensions of moderate concentration [5]. There are two serious difficulties of the theory in [1–4]. First, the back influence of the oriented dipoles on the external field is completely left out of account [5]. Second, the Maxwellian electromagnetic stresses which should appear in the governing momentum equation are ignored without any explanation and necessary evaluation [6].

One of the purposes pursued below is to draw attention to these non-linear effects and to point out under what conditions the neglect of these complications is actually justified. Another intended purpose lies in exposing some essential qualitative properties of a ferrofluid which might look *a priori* rather inexplicable or difficult to foresee. To make the treatment more comprehensible, other problems irrelevant to the main aims in view are here disregarded, whether they be of considerable interest on their own account or not.

For this reason only stationary flows in a comparatively simple geometry are investigated and, for the sake of simplicity, certain other assumptions are made when possible. The subsequent analysis is based on results in [1, 2] as well as in [5, 7, 8] which are briefly outlined in Sec. 2.

2. Governing equations

Consider a moderately concentrated suspension of rigid spheres under an assumption that the Reynolds number characterizing the relative flow of the ambient fluid around the individual particles is small compared with unity. The sphere radius a is, however, taken sufficiently large in order that the flow of the suspension is not significantly affected by the rotary Brownian motion of the particles (the appropriate inequalities imposed on a are discussed in full detail in [2–5, 8]). Permanent magnetic dipoles of moment $\mathbf{D} = D\mathbf{T}$, $\mathbf{T}$ being a unit vector, are embedded within the suspended spheres(1). The dipoles interact in an applied magnetic field and give rise to external couples acting on the spheres which inhibit their free rotation with the equilibrium angular velocity, defined by the local vorticity of a suspension flow and by the fraction of particles by volume [5].

Below we confine ourselves to flows which are uni-directional, steady or almost stationary when the distribution of the orientation of the particle dipole vector $\mathbf{T}$ is nearly the equilibrium one (see [2, 5]), and both this distribution and the flow variables are dependent on a sole coordinate $r_1 = x$ normal to the flow direction along the axis $r_2 = y$. Further, we suppose that the fluid and particles are incompressible and there are no external body forces.

2.1. Flow equations

The equations of mass, momentum and angular momentum conservation of the continua describing both phases of a concentrated suspension of spherical dipolar particles were discussed in [5]. With no reason to dwell upon these equations in further detail, we consider them in the form they reduce to in the particular type of plane flow under study.

First of all, the equations of mass conservation are satisfied identically for both phases. The equation of momentum conservation for the continuum modelling the disperse phase is

$$\mathbf{f} = \frac{3}{4}\varrho\mu_0\left(\frac{6}{a^2}F^{(1)}\varepsilon\mathbf{u}+F^{(2)}\frac{d^2\mathbf{c}}{dx^2}\right) = 0, \quad \varepsilon = 1-\varrho. \tag{2.1}$$

Here ϱ and ε are the fraction of the particles and ambient fluid by volume, respectively, μ_0 is the fluid viscosity, $\mathbf{c}$ and $\mathbf{u}$ are the mean suspension velocity and the fluid slip velocity, that is

$$\mathbf{c} = \varepsilon\mathbf{v}+\varrho\mathbf{w}, \quad \mathbf{u} = \mathbf{v}-\mathbf{w}, \tag{2.2}$$

$\mathbf{v}$ and $\mathbf{w}$ being the mean velocities of the fluid and disperse phases. The vector $\mathbf{f}$ in (2.1) represents the interphase interaction force per unit volume of the suspension.

The analogous equation for the continuum modelling the fluid phase gives [5]

$$0 = -\frac{\partial p}{\partial y}+\frac{d}{dx}\left(\mu\frac{dc}{dx}\right)+\frac{d\sigma_{21}}{dx}, \quad 0 = -\frac{\partial p}{\partial z}+\frac{d\sigma_{31}}{dx}, \tag{2.3}$$

(1) For definiteness ferromagnetic particles are considered but all the results obtained below are also valid, in their original or a slightly modified form, for dipoles of other physical origin.

where p is the mean fluid pressure, μ is the effective viscosity of the suspension of the same particles without dipoles and σ_{ij} are components of an antisymmetric stress tensor evaluated in [5]. For these quantities there are the following equations:

$$\mu = \mu_0\left(1+\frac{5}{2}\varrho S\right), \quad \sigma_{ij} = \frac{1}{2}n\varepsilon_{ijk}q_k, \quad n = \left(\frac{4}{3}\pi a^3\right)^{-1}\varrho, \tag{2.4}$$
$$\mathbf{q} = DH(\boldsymbol{\tau}\times\mathbf{H}^\circ) = 8\pi a^3\mu_0\nu\beta M(\boldsymbol{\tau}\times\mathbf{H}^\circ), \ \boldsymbol{\tau} = \langle\mathbf{T}\rangle, \quad 2\boldsymbol{\nu} = \operatorname{rot}\mathbf{c}.$$

Here τ is the vector $\mathbf{T}$ averaged over the orientational distribution function, $\mathbf{H}^\circ$ denotes a unit vector in the direction of the local magnetic field $\mathbf{H}$ as it would be at the centre of the particle if the latter were absent, ε_{ijk} is the unit isotropic alternating triadic, and the vector $\mathbf{q}$ is the external torque acting upon a single particle and due to the dipole being in the field $\mathbf{H}$. The Eqs. (2.1) and (2.4) contain functions $F^{(1)}$, $F^{(2)}$, S and M of the volume concentration increasing as ϱ grows and turning to unity as ϱ tends to zero. These functions were found for a moderately concentrated suspension in [7]. A parameter β is introduced into (2.4)

$$\beta = \frac{DH}{8\pi a^3\mu_0\nu M}, \tag{2.5}$$

which is the ratio of the magnetic couples to the hydrodynamic ones.

The equation for the conservation of the angular momentum of the disperse phase determines the angular velocity of particle rotation and has been accounted for while formulating (2.4). The similar equation for the fluid phase yields in the case under consideration

$$\frac{d}{dx}\left(\varrho X\frac{dv_3}{dx}\right)\sim\frac{d}{dx}\left(\varrho X\frac{d^2c}{dx^2}\right) = 0, \tag{2.6}$$

where X is an increasing function of ϱ equalling unity at $\varrho = 0$ and evaluated for moderate values of ϱ in [7].

The flow is supposed to be bounded by planes $x = \pm h$, where certain boundary conditions are imposed.

2.2. Field equations

Equations describing a stationary magnetic field are

$$\operatorname{rot}\mathbf{H} = 0, \quad \operatorname{div}\mathbf{B} = 0, \tag{2.7}$$

where $\mathbf{B}$ is the magnetic induction connected to the field strength by

$$\mathbf{B} = \lambda\mathbf{H}+4\pi nD\boldsymbol{\tau}, \tag{2.8}$$

λ being the effective magnetic permeability of the suspension of spheres with no embedded dipoles depending upon the permeabilites of the phase materials and ϱ. This dependence is presumed known, in the particular case of a dilute suspension it is given by the Maxwellian formula. The second term on the right-hand side of (2.8) is an additional intrinsic induction of the suspension caused by the polarization of the dipolar spheres in the same way as the usual intrinsic induction is due to the polarization of elementary currents on the molecular level.

The orientation within a flowing suspension due to the external field thus influences the field itself. This "back" influence makes such flows non-linear. Not only the applied field affects the behaviour of a ferromegnetic suspension but also any alignment of the dipolar particles results in a change of the field which may in certain circumstances be important. In this respect there is a drastic distinction between systems of particles with dipoles of electro-magnetic origin and those with gravitational dipoles (due to the centre of mass not coinciding with the centre of friction). This distinction allowing a "screening action" of aligned dipoles on the field was pointed out in [5] but, unfortunately, has not been taken into account in the majority of other papers on the topic. As a consequence of this distinction $\mathbf{H}$ and $\mathbf{H}^\circ$ involved in the above hydrodynamic equations cannot be regarded as independent quantities but are related to the variables $\mathbf{c}$ and ϱ.

Another important feature of ferrofluid flows should be emphasized here, namely, one can use equations of the type of (2.7) only when the electrical conductivity of the ambient fluid is zero, even if the field under study is stationary. When the effective conductivity of the suspension does not vanish, and the motion of the suspension results in the intersecting of magnetic strength lines, macroscopic electric currents are generated and influence in their turn the applied field. In particular, rot $\mathbf{H}$ is not necessarily equal to zero, so that there is an additional body force proportional to $\operatorname{rot}\mathbf{H}\times\mathbf{H}$, which must be included in (2.3). Such a force can formally be taken into account by introducing the electro-magnetic stress tensor into the expression for the tensor of total momentum flux density, as is done in magneto-hydrodynamics. The above equations are therefore valid under the condition of negligible electrical conductivity of the suspension. Note that this important complication is generally overlooked although it may sometimes be of primary significance.

To simplify the problem, the external field strength $\mathbf{H}_e$ is made uniform everywhere outside the flow region $-h \leqslant x \leqslant h$. Inside this region the quantity $\mathbf{H}$ must satisfy the Eqs. (2.7) and (2.8).

2.3. Closure of the equations

The set of hydrodynamic and field equations would be closed if a suitable expression for $\boldsymbol{\tau}$ were known. Here we make use of such expressions derived in [2, 5].

As described in [2, 5] two principally different situations occur. The first corresponds to $\beta \geqslant 1$ or $\beta < 1$ and $\gamma \neq \pi/2$, γ being an angle between the directions of the vorticity vector $\boldsymbol{\nu}$ and $\mathbf{H}$. If $\mathbf{x}^\circ$, $\mathbf{y}^\circ$ and $\mathbf{z}^\circ$ are the unit vectors of a right-handed coordinate system, $\mathbf{c}$ and $\boldsymbol{\nu}$ being proportional to $\mathbf{y}^\circ$ and $\mathbf{z}^\circ$, respectively, then each particle achieves a unique and stable terminal orientation $\boldsymbol{\tau}$ which is independent of its initial orientation and is given by a relation

$$\begin{aligned} \boldsymbol{\tau} &= \sin\psi_1\cos(\psi_2+\theta)\mathbf{x}^\circ+\sin\psi_1\sin(\psi_2+\theta)\mathbf{y}^\circ+\cos\psi_1\mathbf{z}^\circ, \\ \sin\psi_1 &= \left\{\frac{1}{2}(1+\beta^2)-\left[\frac{1}{4}(1+\beta^2)^2-\beta^2\sin^2\gamma\right]^{1/2}\right\}^{1/2}, \\ \sin\psi_2 &= (\beta\sin\gamma)^{-1}\sin\psi_1, \quad \psi_1\in(0,\pi), \quad \psi_2\in(0,\pi/2), \end{aligned} \tag{2.9}$$

the parameter β being defined in (2.5) and the angles γ and θ determining the direction of the vector $\mathbf{H}^\circ$

$$\mathbf{H}^\circ = \sin\gamma\cos\theta\mathbf{x}^\circ+\sin\gamma\sin\theta\mathbf{y}^\circ+\cos\gamma\mathbf{z}^\circ. \tag{2.10}$$

This "regular" situation, in which particle rotation around any axis normal to $\boldsymbol{\tau}$ is completely hindered, is favoured by comparatively strong fields.

In a special case $\beta < 1, \gamma = \pi/2$ an alternative "singular" situation is realized, in which the hydrodynamic couples prevail and ensure the rotation of the embedded dipole vectors even in the therminal state $t \to \infty$. The periodic orbits performed by the ends of these vectors do not depend upon the initial orientation of the particles: the initial orientations are eventually forgotten through disorientating Brownian rotations whatever the relative magnitude of the random Brownian and deterministic hydrodynamic or magnetic couples. These orbits in a dilute suspension were closely investigated in [8]; the generalization to a concentrated suspension happended to be fairly easy and can be found in [5]. The vector $\mathbf{H}$ lies in this case in the flow plane (x, y) and the vector $\boldsymbol{\tau}$ is described as follows:

$$\boldsymbol{\tau} = -F(\beta)\sin\theta\mathbf{x}^\circ + F(\beta)\cos\theta\mathbf{y}^\circ,$$

(2.11)
$$F(\beta) = \frac{1}{\beta}\left\{1 - \frac{1-\beta^2}{\sqrt{3}} \frac{\ln[(\sqrt{2+\beta^2}+\beta\sqrt{3})(\sqrt{2+\beta^2}-\beta\sqrt{3})^{-1}]}{\ln[(\sqrt{2+\beta^2}+\beta)(\sqrt{2+\beta^2}-\beta)^{-1}]}\right\}.$$

Asymptotic evaluation of $F(\beta)$ at $\beta \to 0$ and $\beta \to 1-0$ is given in [8]. Note that the modulus of $\boldsymbol{\tau}$ in (2.11) is smaller than unity, since the alignment of the particle dipoles in one preferable direction is not complete.

3. Flows without singularities

We begin by considering flows in the regular situation, when either $\beta \geqslant 1$ or $\beta < 1$ but simultaneously $\gamma \neq \pi/2$, so that the equations in (2.9) are valid. Here, and in the remainder of the paper the external magnetic field $\mathbf{H}_e$ outside the flow region $-h \leqslant x \leqslant h$ is proposed uniform and defined by a relationship similar to (2.10)

(3.1)
$$\mathbf{H}_e^\circ = \sin\gamma_e\cos\theta_e\mathbf{x}^\circ + \sin\gamma_e\sin\theta_e\mathbf{y}^\circ + \cos\gamma_e z^\circ,$$

where the polar and azimuthal angles γ_e and θ_e are in general not equal to those determining the field strength $\mathbf{H}$ within the flow region.

The Eqs. (2.7) permit the conclusion that H_y, H_z and B_x are constant throughout the flow. Thus these quantities are determined by the boundary conditions of continuity for the tangential components of the field strength and for the normal component of the magnetic induction. This gives three equations,

(3.2)
$$\begin{aligned}\lambda_e H_e \sin\gamma_e\cos\theta_e &= \lambda H \sin\gamma\cos\theta + 4\pi n D \sin\psi_1\cos(\psi_2+\theta),\\ H_e\sin\gamma_e\sin\theta_e &= H\sin\gamma\sin\theta,\\ H_e\cos\gamma_e &= H\cos\gamma,\end{aligned}$$

λ_e being the external magnetic permeability, ψ_1 and ψ_2 depending on β from (2.5) in accordance with (2.9) and n being the number concentration of particles defined in(2.4). The Eqs. (2.8) to (2.10) and (3.1) have been taken into account while deriving (3.2).

Two different problems may be of interest. A direct one lies in expressing H, γ and θ

in terms of the presumed known quantities H_e, γ_e and θ_e and the flow variables ϱ and c', a prime designating differentiation with respect to x. An inverse problems is to find the external field necessary for the realization of a given field inside the flow region. Both problems can be readily solved with the help of (3.2). In particular, one is able to regard hereafter

$$H = H(\varrho, c'), \quad \gamma = \gamma(\varrho, c'), \quad \theta = \theta(\varrho, c'), \tag{3.3}$$

as known functions depending in addition upon the external field parameters.

By integrating the Eq. (2.6) one obtains

$$\varrho X(\varrho)\frac{d^2c}{dx^2} = \text{const}. \tag{3.4}$$

This allows ϱ to be expressed as a function of c''.

Thus only c remains to be determined. It is possible to show with the help of the results in [2, 5] that there is an alternative relation for σ_{ij} to that in (2.4)

$$\sigma_{ij} = 3\varrho M\mu_0\nu\varepsilon_{ijk}(\tau_k\cos\psi_1 - \mathring{\tau}_k), \tag{3.5}$$

$\mathbf{v}^\circ$ being a unit vector in the vorticity direction coinciding with $\mathbf{z}^\circ$ for the plane flow under study. By making use of (2.9) and (3.5) one derives from the first equation (2.3) when the pressure gradient is constant and directed along $-\mathbf{y}^\circ$

$$\begin{gathered}\frac{d}{dx}\left(\hat{\mu}\frac{dc}{dx}\right) = \frac{dp}{dy} = -P, \quad \hat{\mu} = \mu_0\left(1+\frac{5}{2}\varrho S+\frac{3}{2}\varrho M\phi(\beta,\gamma)\right),\\ \phi(\beta,\gamma) = \frac{1+\beta^2}{2}\left\{1-\left[1-\frac{4\beta^2\sin^2\gamma}{(1+\beta^2)^2}\right]^{1/2}\right\}.\end{gathered} \tag{3.6}$$

The quantity $\hat{\mu}$ here plays the role of an apparent suspension viscosity depending, in view of (2,5), (3.3) and (3.4), upon c' and c''. A necessary condition for the existence of a plane flow follows from the second equation (2.3) after calculating σ_{31} on the basis of (2.9) and (3.5); this becomes

$$\frac{d}{dx}\left(\varrho M\sin\psi_1\cos\psi_1\sin(\psi_2+\theta)\frac{dc}{dx}\right) = 0. \tag{3.7}$$

Hence plane flow of the type considered is only possible under rather special conditions imposed on the flow and field variables. For instance, such a flow can be brought into existence when $\gamma = 0$ or $\gamma = \pi/2$ at any values of ϱ and c' and when $\varrho = \text{const}$, $c' = \text{const}$, γ and θ being arbitrary.

Alternatively if (3.7) does not hold, there is a velocity component normal to the pressure gradient. If even the velocity is collinear to the latter in some plane $x = \text{const}$, this is not the case for neighbouring parallel planes. Note that the secondary circulation in flows of magnetic suspensions was previously discussed in [9].

To gain an approximate idea of the character of regular ferrofluid flows it is worth analysing two particular cases, namely, Couette flow and Poiseuille flow with $\gamma = \pi/2$ and ϱ constant. The Eq. (3.7) then holds true. In the first case, $\mathbf{H}$ and β happen to be uniform so that σ_{ij} and $\hat{\mu}$ do not depend on x. After some calculation we obtain non-zero antisymmetric stresses

$$
\begin{aligned}
-\sigma_{12} = \sigma_{21} &= \frac{3}{2}\varrho M\mu_0 c' \sin^2\psi_1, \\
-\sigma_{13} = \sigma_{31} &= \frac{3}{2}\varrho M\mu_0 c' \sin\psi_1 \cos\psi_1 \sin(\psi_2+\theta), \qquad (3.8)\\
-\sigma_{23} = \sigma_{32} &= -\frac{3}{2}\varrho M\mu_0 c' \sin\psi_1 \cos\psi_1 \cos(\psi_2+\theta),
\end{aligned}
$$

and from (3.6) an expression for $\hat{\mu}$ in terms of β and γ. A representative dependence of $\hat{\mu}$ on β is plotted in Fig. 1 in which the magnetic suspension appears to be a shear-thinning medium. The relations in (3.8) resemble those derived earlier in [2, 5]. There is, however,

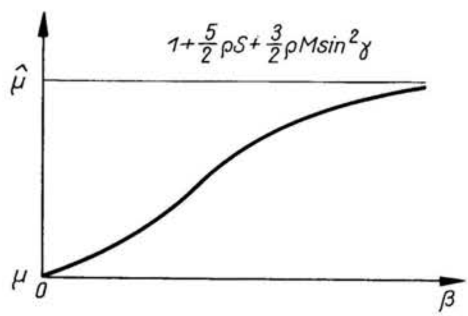

FIG. 1.

an essential distinction since the Eqs. (3.8) involve the local field which is different from the external field. The former is related to the latter one by the Eqs. (3.2) and it is easy to see that in general neither the magnitude nor the direction of **H** coincide with those of $\mathbf{H}_e$. This enables us to conclude magnetic suspensions to be in every way an anisotropic media, that is in both hydrodynamic and electro-magnetic respects. On the one hand the shear stress is no longer parallel to the shear velocity and on the other the total effective magnetic permeability, including the macroscopic part due to the particle dipole alignment, is a tensor quantity. Note in this connection that magnetic suspensions do not represent quasi-Newtonian media in which there exists a linear tensor relation between the stresses and the rates of strain, in contrast to a suggestion in [10]. This is by no means strange because the reasoning in [10] applies apparently to suspensions with the same instantaneous microstructure. That this is not the case in the present study can readily be seen from the above consideration: the suspension microstructure is assumed to be essentially dependent on the particle alignment related to the local magnetic field magnitude and direction. As evidenced by the curve in Fig. 1 the apparent viscosity grows as β increases or $c' = 2\nu$ decreases and the suspension behaves like a shear-thinning pseudoplastic fluid.

We turn further to the Poiseuille flow supposing ϱ is constant and H_e lies in the flow plane (x, y). Then equations in (3.6) yield (the parameter β has to be remembered to exceed unity for a regular flow with $\gamma = \pi/2$)

$$\phi(\beta, \gamma) = 1, \quad \hat{\mu} = \mu_0\left(1+\frac{5}{2}\varrho S+\frac{3}{2}\varrho M\right), \tag{3.9}$$

so that the suspension is indistinguishable under the present circumstances from a Newtonian fluid of constant viscosity. The same conclusion, valid for a dilute suspension was previously made in [2] where a discussion of this phenomenon was presented.

We bring this section to an end with a brief analysis of the internal field $\mathbf{H}$ when $\mathbf{H}_e$ is given. It follows from the third equation (3.2) that $\mathbf{H}$ lies in the flow plane as well as $\mathbf{H}_e$. The other equations in (3.2) permit H and θ to be found. Two simple cases may be of special interest: $\sin\theta_e = 0$ and $\sin\theta_e = 1$. In the first case, when $\mathbf{H}_e$ is either parallel or antiparallel to $\mathbf{x}^\circ$, $\theta = \theta_e$ and by introducing the quantities

$$\varkappa = \frac{\lambda}{\lambda_e}, \quad m = \frac{H}{H_e}, \quad \beta_e = \frac{\beta}{m}, \quad \alpha_e = \frac{4\pi n D}{\lambda_e H_e}, \tag{3.10}$$

one obtains the following equation form:

$$1-\varkappa m = \alpha_e[1-(m\beta_e)^{-2}]^{1/2}. \tag{3.11}$$

If $\beta_e \geqslant \varkappa$ this equation has a unique root satisfying the inequality $\varkappa^{-1} \leqslant m \leqslant \beta_e^{-1}$. In the opposite case there is no root of (3.11). The latter is quite natural since β then appears to be smaller than unity and the flow is singular in the sense that (2.11) must be used instead of (2.9).

If $\theta_e = \pi/2$, the θ and m are calculated from

$$\varkappa m\left(1-\frac{1}{m^2}\right)^{1/2} = \alpha_e\left[\frac{1}{\beta_e m^2}-\left(1-\frac{1}{m^2}\right)^{1/2}\left(1-\frac{1}{\beta_e^2 m^2}\right)^{1/2}\right], \tag{3.12}$$
$$\theta = \arcsin(m^{-1}),$$

where the parameters (3.10) are used again. There is a unique root of the first equation in (3.12) with $m > 1$ and $m > \beta_e^{-1}$. The y-component of the internal field strength is evidently unchanged and coincides with $\mathbf{H}_e$ whereas the z-component differs from zero through the flow region and depends on β_e and, consequently, on the shear rate.

Obviously, it is not difficult to consider regular flows of more complex geometry in a similar manner, taking into account the restrictions imposed by the Eqs. (3.4) and (3.7).

4. Singular flows

We now proceed to investigate flows of the singular type when the relations $\beta < 1$ and $\gamma = \pi/2$ hold simultaneously. It is easy to see that γ_e is also equal to $\pi/2$ in this case, i.e. the applied external field is normal to the vorticity vector. As before, the quantities H and θ can be looked upon as certain functions of c' and ϱ defined by the Eqs. (3.2) in a form similar to (3.3). Analogously the Eq. (3.4) enables us to express ϱ as a function of c'' so that the problem is again reduced to determination of the suspension velocity $c(x)$.

The Eqs. (2.4) and (2.11) show that only the components

$$-\sigma_{12} = \sigma_{21} = \frac{3}{2}\varrho M\mu_0 \Phi(\beta)\frac{dc}{dx}, \quad \Phi(\beta) = \beta F(\beta), \tag{4.1}$$

of the antisymmetric stress tensor do not vanish in the case under study. Then the second equation in (2.3) is identically satisfied and the first one takes the form (cf. (3.6))

$$\begin{gathered} \frac{dG}{dx} = \frac{dp}{dy} = -\mathbf{P}, \quad G = G_1 + G_2, \\ G_1 = \mu\frac{dc}{dx} = \mu_0\left(1 + \frac{5}{2}\varrho S\right)\frac{dc}{dx}, \quad G_2 = \frac{3}{2}\mu_0\varrho M\Phi(\beta)\frac{dc}{dx}. \end{gathered} \tag{4.2}$$

This equation governs flows of a non-Newtonian fluid whose viscosity is dependent on both c' and c''.

Accounting for this dependence leads to a rather laborious and cumbersome calculation. In order to simplify the problem as much as possible while demonstrating the main flow properties, it is natural to treat by way of example a pure Poiseuille flow symmetric about the central plane $x = 0$ and to assume the pressure gradient to be sufficiently large. Then c'' is also large and its dependence on x can approximately be neglected as compared with a similar dependence of c'. In other words, ϱ can be regarded in the first approximation as a constant quantity. Moreover, we assume an inequality

$$4\pi nD \ll H_e \tag{4.3}$$

to hold. This gives an opportunity to set H approximately equal to H_e and to neglect the dependence of H upon both c' and c'' while solving the Eq. (4.2). Thus, the fact that the parameter from (2.5) is inversely proportional to $\nu = c'/2$ is only significant in (4.2).

4.1. General consideration

In the vicinity of the symmetry plane $x = 0$ the shear rate is always small and tends to zero with x. The parameter β is therefore high in a flow core region and goes to infinity as x approaches zero. For c' smaller than a critical value c'_- corresponding to $\beta = 1$, the results obtained in the end of Sec. 3 are valid. Particularly, the suspension within the core displays Newtonian properties and its apparent viscosity $\hat{\mu}$ is determined by (3.9). The

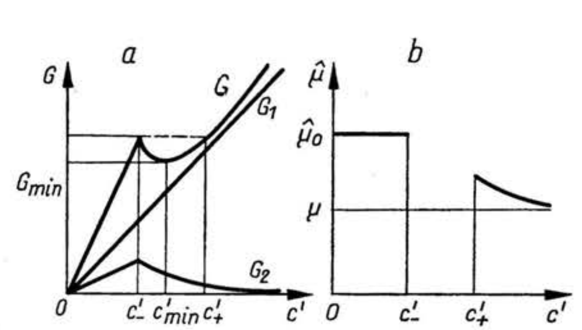

FIG. 2

second derivative c'' and the volume concentration of particles are constant and equal to c_0'' and ϱ_0, respectively.

In a peripheral region of the flow, c' is, however, large enough to make β smaller than unity. The dependence of G_1 on c' does not change in this region from that at $\beta > 1$ but G_2 undergoes an abrupt change and is now given by $\Phi(\beta)$ determined according to (2.11) and (4.1). Typical curves for both G_1 and G_2 as well as for their sum G are plotted in Fig. 2a. It is essential that the slope of the tangent to the curve $G_2(c')$ at the point $c'_- + 0$ is negative and infinitely large. The curve $G(c')$ has therefore a minimum at a certain value of c' exceeding c'_-, as shown in Fig. 2a. Apparently this requires a discontinuity surface within the flow when c' approaches c'_- from below. The condition of stress continuity at such a surface means that c' increases sharply from c'_- up to a value c'_+ which has the same value for G as c'_-. This is also illustrated in Fig. 2a. Note that there is no reason for either c or ϱ to have a discontinuity at this surface.

Within the framework of this paper the region of drastic change in the flow parameters is looked upon as a discontinuity surface of zero thickness. This is undoubtedly due to assuming the effect of the rotary Brownian motion to be negligible. In fact, the thickness of this region is finite and dictated by processes of the rotary Brownian diffusion, no matter how weak the latter, and can be in principle evaluated with the help of the technique proposed in [5, 8].

A dependence of the apparent viscosity on c' corresponding to that of the effective stress G is plotted in Fig. 2b. Clearly, the Newtonian core is surrounded with a flow region where the suspension behaves like a shear-thinning media, the apparent viscosity tending in the limit $c' \to \infty$ to the effective viscosity of a suspension of the same particles without the dipoles.

Within the peripheral region, c'' is larger than c_0'' in the core. Hence it follows that ϱ evaluated in accordance with (3.4) is smaller than ϱ_0 and depends on x outside the core, decreasing to a certain limiting value as x grows to infinity. Thus, there is an accumulation of particles in a central part of the flow. Let us emphasize that this is a new effect having nothing in common with the known segregation of particles of a suspension with no dipole interaction with an external field.

All these conclusions obviously hold even in more complex situations when the Eq. (4.3) is invalid, ϱ cannot be regarded as an approximately constant quantity, and the dependence of $\mathbf{H}$ on the flow parameters is strong and cannot be ignored.

4.2. Examples

Below we consider simple concrete examples in which $\mathbf{H}_e$ is either normal or parallel to the suspension velocity. At the beginning we assume $\mathbf{H}_e$ to be normal to the flow ($\theta_e = 0$ or $\theta_e = \pi$). Then the Eqs. (2.11) and (3.2) lead to a conclusion that the field strength $\mathbf{H}$ is parallel to $\mathbf{H}_e$ and constant throughout the peripheral flow region $|x| > x_*$, where an inequality $\beta < 1$ holds, i.e.

$$\theta = \theta_e, \quad H = \frac{\lambda_e}{\lambda} H_e = \frac{1}{\varkappa} H_e. \tag{4.4}$$

The field strength $\mathbf{H}_0$ within the core region $-x_* \leqslant x \leqslant x_*$ is to be found with the help of equations similar to (3.10) and (3.11), the role of an external field being played in this case by $\mathbf{H}$ defined in (4.4). As a matter of fact, one obtains

$$(4.5) \qquad H_0 = m_0 H = \frac{m_0}{\varkappa} H_e, \quad 1 - m_0 = \alpha_e \left[1 - \left(\frac{m_0 \beta_e}{\varkappa}\right)^2\right]^{1/2},$$

where the second equation defines m_0 as a function of α_e and β_e, the latter being expressed in terms of the external field parameters in accordance with (3.10). Thus H_0 is also parallel to $\mathbf{H}_e$ but its modulus depends on a local value of the vorticity. At $|x| = x_*$ the quantity m_0 equals unity so that the magnetic field strength varies continuously everywhere, including the surfaces $x = \pm x_*$.

Further, by solving (3.6) with $\hat{\mu} = \hat{\mu}_0$ resulting from (3.9) at $\varrho = \varrho_0$, one obtains relations for the fluid velocity and vorticity within the core

$$(4.6) \qquad c = c_0 - \frac{P}{2\hat{\mu}_0} x^2, \quad \nu = \frac{P}{2\hat{\mu}_0} x,$$

where c_0 is constant.

The flow in the peripheral region can be determined by means of solving the nonlinear equation (4.2), ϱ and H being understood as functions of c' and c''. In this particular case H is constant and ϱ is implicitly determined by a relation

$$(4.7) \qquad \varrho X(\varrho) = \varrho_0 X(\varrho_0) \frac{c_0''}{c''}, \quad c_0'' = -\frac{P}{\hat{\mu}_0},$$

following from (3.4) and (4.6). Therefore (4.2) is a differential equation of the third order and three boundary conditions are needed. These conditions result from the continuity of c and ϱ as well as from the condition of continuity of the effective tangential stress at $|x| = x_*$

$$(4.8) \qquad c = c_0 - \frac{P}{2\hat{\mu}_0} x_*^2, \quad \frac{dc}{dx} = c_+', \quad \frac{d^2c}{dx^2} = -\frac{P}{\hat{\mu}_0}, \quad (x = \pm x_*)$$

the quantity c_+' being calculated from a relation

$$(4.9) \qquad \Phi(\beta_+) c_+' = \pm \frac{P}{\hat{\mu}_0} x_*, \quad \beta_+ = \frac{DH_e}{4\pi a^3 \varkappa \mu_0 |c_+'| M(\varrho_0)},$$

in compliance with the analysis in the previous subsection.

Conditions on external flow boundaries (e.g., $c = 0$ at $x = \pm h$) enable us to find c_0 involved in the above formulae. At last, the quantity ϱ_0 can be determined from a global balance equation for the suspended particles. For instance, either the total flux of particles or the mean concentration of the flowing suspension may be given in practice and used to the end. This completes the consideration.

Note that the coordinate x_* can be found without the solving of (4.2), namely, it is clearly determined by a condition

$$(4.10) \qquad \frac{DH_0(x_*)}{8\pi a^3 \mu_0 \nu(x_*) M(\varrho_0)} = 1,$$

yielding, after accounting for (3.9), (4.5) and (4.6)

$$x_* = \frac{DH_e}{4\pi a^3 \varkappa P}\left[\frac{1}{M}\left(1+\frac{5}{2}\varrho S+\frac{3}{2}\varrho M\right)\right]_{\varrho=\varrho_0}, \tag{4.11}$$

By comparing x from (4.11) with h one is able to derive a condition for a discontinuity surface to occur in a flow with given h, P, etc.

Within the peripheral region the dipole vector $\mathbf{T}$ of each particle rotates in such a way that only τ_y differs from zero. Within the core region all the particles are stably aligned along the same axis whose slope varies with x so that the axis coincides with the axis x at $x = 0$ and with the axis y at $x = \pm x_*$.

If $\mathbf{H}_e$ is parallel to the flow velocity, then the boundary conditions for the magnetic field and the Eq. (2.11) for $\boldsymbol{\tau}$ give the following relations defining the field parameters within the peripheral region $\beta < 1$:

$$H = \frac{\pm H_e}{\sin\theta}, \quad \theta = \theta_e \pm \arccos[(\eta^2+1)^{1/2}-\eta], \quad \eta = \frac{\varkappa}{2\alpha_e F(\beta)}, \tag{4.12}$$

where upper and lower signs pertain to $\theta_e = \pi/2$ and $\theta_e = 3\pi/2$, respectively. So the field $\mathbf{H}$ changes its slope gradually from that of $\mathbf{H}_e$ to a finite slope achieved at $\beta = 1$. Values of θ and H at $\beta = 1$ determine a field that must be considered as an external field with respect to the field $\mathbf{H}_0$ inside the flow core. The latter field can then easily be found by means of applying (3.2).

The corresponding hydrodynamic problem is to be solved as before, but it is now more complicated because of the dependence on the flow vorticity of $\mathbf{H}$ and $\mathbf{H}_0$.

A similar consideration is possible for a general superposition of the pure Couette and Poiseuille flows with the magnetic field arbitrarily oriented. The principal aspects of ferrofluid flows, manifesting the rather unusual effects, however, remain unchanged.

References

1. W. F. HALL, S. N. BUSENBERG, *Viscosity of magnetic suspensions*, J. Chem. Phys., **57**, 137–144, 1969.
2. H. BRENNER, *Rheology of a dilute suspension of dipolar spherical particles in an external field*, J. Coll. & Interface Sci., **32**, 141–158, 1970.
3. H. BRENNER, *Suspension rheology in the presence of rotary Brownian motion and external couples: elongation flow of dilute suspensions*, Chem. Engng Sci., **27**, 1069–1107, 1972.
4. H. BRENNER, M. H. WEISSMAN, *Rheology of a dilute suspension of dipolar spherical particles in an external field. II. Effects of rotary Brownian motion*, J. Coll. & Interface Sci., **41**, 499–531, 1972.
5. YU. A. BUYEVICH, *Continuum mechanics of monodisperse suspensions. On properties of suspensions of spherical dipoles in an external field* [in Russian], PMM, **38**, 301–311, 1974.
6. M. I. SHLIOMIS, *Effective viscosity of magnetic suspensions* [in Russian], Zh. ETF, **61**, 2411–2418, 1971.
7. YU. A. BUYEVICH, V. G. MARKOV, *Continuum mechanics of monodisperse suspensions. Rheological state equations for moderately concentrated suspensions* [in Russian], PMM, **37**, 1059–1077, 1973.
8. E. J. HINCH, L. G. LEAL, *Note on the rheology of a dilute suspension of dipolar spheres with weak Brownian couples*, J. Fluid Mech., **56**, 803–813, 1972.

9. L. G. LEAL, *On the effect of particle couples on the motion of a dilute suspension of spheroids*, J. Fluid Mech., **46**, 395–416, 1971.

10. G. K. BATCHELOR, *The stress system in a suspension of force-free particles*, J. Fluid Mech., **41**, 545–570, 1970.

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