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BRIEF NOTES

Additional comments on Suliciu, Malvern and Cristescu's paper. "Remarks concerning the "plateau" in dynamic plasticity(*)"

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IT IS proved that under certain loading conditions, a constitutive equation of type (1) with ψ of the form (2), where $\alpha \in (0,1)$, may admit an absolute plateau. This result corrects the false statement contained in the previous paper, for $\alpha = 1/2$.

THE assertion that the semilinear rate type constitutive equation

$$\dot{\sigma} = E\dot{\varepsilon} + \psi(\varepsilon, \sigma) \tag{1}$$

where

$$\psi(\varepsilon, \sigma) = \begin{cases} -k(\sigma - f(\varepsilon))^{\alpha}, & \sigma > f(\varepsilon), \\ 0 & \sigma \leqslant f(\varepsilon), \end{cases} \tag{2}$$

for $\sigma \geqslant 0$ and $\varepsilon \geqslant 0$, with $\alpha = 1/2$, can not admit an absolute plateau, is false. We arrived to this conclusion by applying formula (3.5)(1). It was proved by SULICIU [15, 19] that formula (2.11) can be applied to a semi-linear constitutive equation (1), in the case when ψ is continuous on some domain $\mathscr{D}$ and possesses bounded partial derivatives on $\mathscr{D}$. The function ψ given by (2) has no bounded partial derivatives on the domain $\mathscr{D} = \{(\varepsilon, \sigma); \sigma > f(\varepsilon), \varepsilon > 0, \sigma > 0\}$, the domain of interest for the instantaneous impacts discussed here. Thus, the formula (3.5) does not hold.

We shall prove now, for a special class of histories of strain (i.e. for such histories of strain that appear close to the impacted end, when a bar is impacted with constant velocity), that the formula (2.11) can be applied and

$$0 \leqslant \int_{t_0}^{t} \left[\exp - \int_{t_0}^{s} \frac{\partial \mu}{\partial \tau} (\varepsilon(s_1), \tau(s_1)) dn_1 \right] \frac{\partial \mu}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial X} ds \leqslant \text{const} \tag{3}$$

(*) Archives of Mechanics, **24**, 5-6, 999-1011, 1972.

(1) Relations with two groups of numbers refer to the paper under discussion.

for any $\geqslant t_0$, while

$$\exp\left[\int_{t_0}^{t} \frac{\partial \mu}{\partial \tau}(\varepsilon(s), \tau(s))\,ds\right] \to 0 \quad \text{when} \quad t \to \tilde{t}_* < \infty. \tag{4}$$

This will lead us to the conclusion that an absolute plateau for ε, v and σ is possible.

Since, close to the impacted end, a jump in strain $\varepsilon_i > \varepsilon_Y$ is found and, as the corresponding stress lies on the instantaneous curve (in this case on Hooke's line), one has $\sigma_i = E\varepsilon_i > f(\varepsilon_i)$. Therefore, in the formula (2.9), ψ is computed for $(\varepsilon_i, \sigma_i) \in \mathscr{D}$ at $t = t_0$, with $\tau(t_0) = 0$ (and obviously $\partial \mathscr{F}/\partial \tau = 1$); thus (2.11) is valid as long as $(\varepsilon(t), \sigma(t) = E\varepsilon(t) + \tau(t)) \in \mathscr{D}$.

Now, one will prove that if ε reaches a plateau (in time and space) above a curve Γ from the characteristic plane, then there exists a curve Γ^* above Γ, where v and σ also reach a plateau. The proof of all these facts is based on the following

PROPOSITION. *The initial value problem*

$$\dot{\tau} = -k(\tau + \Omega)^\alpha, \quad \tau(t_0) = 0 \tag{5}$$

with $\alpha \in (0, 1)$ *and* $\Omega: [t_0, \infty) \to R$, $\Omega \in C^1$, $\Omega(t_0) = \Omega_0 > 0$, $\Omega(t) > 0$ *for all* $t \in [t_0, \infty)$, $\dot{\Omega}(t) > 0$ *for all* $t \in [t_0, t_*)$ *and* $\dot{\Omega}(t) = 0$ *for all* $t \in [t_*, \infty)$, *admits a unique solution of class* C^1 $[t_0, \infty)$ (see for instance HARTMAN Ch. III, §6 [18]). *Moreover, the solution of problem* (5) *has the following properties*:

a) *if*

$$0 < t_* + \frac{\Omega_*^{1-\alpha}}{k(1-\alpha)} + t_0, \tag{6}$$

where $\Omega_* = \Omega(t_*)$, *then there exists* $\tilde{t}_* > t_*$ *such that*

$$\tau(t) + \Omega(t) > 0 \quad \textit{for all } t \in [t_0, \tilde{t}_*); \tag{7}$$

b) *in the interval* $[t_*, \infty)$, *the solution of the Eq.* (5) *has the following expression:*

$$\tau(t) = \begin{cases} -\Omega_* + [(\tau_* + \Omega_*)^{1-\alpha} - k(1-\alpha)(t - t_*)]^{\frac{1}{1-\alpha}}, & t \in [t_*, \tilde{t}_*], \\ -\Omega_*, & \tilde{t}_* < t, \end{cases} \tag{8}$$

which is obtained by direct integration of the Eq. (5) *over the interval* $[t_*, \infty)$, *with* $\tau(t_*) = \tau_*$.

P r o o f. Let us prove assertions (7) and (8). Denote by $\Omega_* = \Omega(t_*) \geqslant \Omega(t)$ for all $t \in [t_0, \infty)$; then we have

$$-k(\tau + \Omega)^\alpha \geqslant -k(\tau + \Omega_*)^\alpha \quad \text{for all} \quad t \in [t_0, \infty) \tag{9}$$

and therefore (see for instance HARTMAN Ch. III, §4 [18])

$$\tau(t) \geqslant \tau_0(t) \quad \text{for all} \quad t \in [t_0, \infty), \tag{10}$$

where $\tau(t)$ is the solution of problem (5) and $\tau_0(t)$ is the solution of the following problem:

$$\dot{\tau}_0 = -k(\tau_0 + \Omega_*)^\alpha, \quad \tau_0(t_0) = 0. \tag{11}$$

$\tau_0(t)$ has the following form:

$$(12)\quad \tau_0(t) = \begin{cases} -\Omega_* + [\Omega_*^{1-\alpha} - k(1-\alpha)(t-t_0)]^{\frac{1}{1-\alpha}} & \text{for} \quad t \in \left[t_0, \dfrac{\Omega_*^{1-\alpha}}{k(1-\alpha)}\right], \\ -\Omega_* & \text{for} \quad t > \dfrac{\Omega_*^{1-\alpha}}{k(1-\alpha)} + t_0 . \end{cases}$$

From (10) and (12) we find that

$$(13)\qquad \tau(t) > -\Omega_* \quad \text{for} \quad t \in [t_*, \tilde{t}_*),$$

where

$$(14)\qquad \tilde{t}_* = t_* + \frac{(\tau_* + \Omega_*)^{1-\alpha}}{k(1-\alpha)}$$

and thus (7) and (8) follow.

We now choose $\Omega = E\varepsilon - f(\varepsilon)$; since $\mathscr{F} = E\varepsilon + \tau$, with $\tau(t_0) = 0$, we have

$$\mu(\varepsilon, \tau) = \psi(\varepsilon, \tau + E\varepsilon)$$

and from (2.12) and (2) we obtain

$$(15)\qquad \frac{\partial\mu}{\partial\varepsilon} = \frac{\partial\psi}{\partial\varepsilon} = -k\frac{E - f'(\varepsilon)}{(E\varepsilon - f(\varepsilon) + \tau)^{1-\alpha}}, \qquad \frac{\partial\mu}{\partial\tau} = -\frac{k\alpha}{(E\varepsilon - f(\varepsilon) + \tau)^{1-\alpha}}.$$

Now, since we assumed that ε has reached a plateau (in time and space) above a curve Γ, i.e. for $t > t_* = g(X_*)$, $(X_*, t_*) \in \Gamma$, $\dfrac{\partial\varepsilon}{\partial X}(X_*, t) = 0$, $\dfrac{\partial\varepsilon}{\partial t}(X_*, t) = 0$, then from (7) it follows that $(\varepsilon(t_*), \sigma(t_*)) \in \mathscr{D}$ (i.e. this point is not on the relaxation curve), hence $\partial\mu/\partial\varepsilon$ and $\partial\mu/\partial\tau$ are finite and therefore (3) follows, since its left-hand side remains constant for $t \geqslant t_*$.

From (8) and (15) we obtain

$$\frac{\partial\mu}{\partial\tau} = -\frac{k\alpha}{(\tau_* + \Omega_*)^{1-\alpha} - k(1-\alpha)(t-t_*)}, \qquad t \in [t_*, \tilde{t}_*)$$

so we can write

$$\exp\left(\int_{t_0}^{t} \frac{\partial\mu}{\partial\tau}\, ds\right) = \left[\exp\left(\int_{t_0}^{t_*} \frac{\partial\mu}{\partial\tau}\, ds\right)\right] \frac{1}{(\tau_* + \Omega_*)} \left[(\tau_* + \Omega_*)^{1-\alpha} - k(1-\alpha)(t-t_*)\right]^{\frac{\alpha}{1-\alpha}}$$

for all $t \in [t_*, \tilde{t}_*)$. From this equality and (14) the assertion (4) follows and thus a plateau in velocity and stress will appear for $t \geqslant t_*$.

In his numerical analysis on rate effect, KUKUDJANOV [9] did use examples of type (2) for $\alpha = 1/2$, $\alpha = 1$, $\alpha = 3$ and $\alpha = 5$. A constitutive equation of type (1), with ψ given by (2), cannot possess an absolute plateau for $\alpha \geqslant 1$, as can easily be seen by applying formula (3.5); indeed, in these cases, formula (3.5) holds as the conditions of the theorem cited above ([15, 19]) are satisfied.

References

15. The correct address of reference [15] of the paper under discussion is: vol. **25**, No. 1, pp. 53–170, 1973, in the same Journal.
18. P. HARTMAN, *Ordinary Differential Equations*, John Wiley & Sons Inc., New York-London-Sydney 1964.
19. I. SULICIU, *Classes of discontinuous motions in elastic and rate type materials. One-dimensional case*, Archives of Mechanics, **26**, 687–711, 1974.

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