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Material connections and induced metrics on inhomogeneous materially uniform two-solids

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WE HERE focus on the derivation of a result obtained in the course of a research program outlined in a preceeding Brief Note [1]. In a theory intended to describe the mechanical response of *Inhomogeneous Materially Uniform Higher Gross Order Bodies* **[2]** the r-reference maps in the charts of a material atlas $\mathfrak{A}^r$ for an r-th order body $\mathscr{B}$ induce a *material* r-metric field $\mathbf{g}_{\mathfrak{A}^r}$ on $\mathscr{B}$, $r = 1, 2, \ldots$. In turn the material-geometric structure is described by a *material r-connection field* $\mathbf{\Gamma}_{\mathfrak{A}^r} = (\Gamma^i_{i_1 i_2}, \ldots, \Gamma^i_{i_1 \cdots i_{r+1}})$ on $\mathscr{B}$. The requirement that there vanish the (generalized) covariant derivative, with respect to the material connection, of the induced metric field establishes relations between the components of $\mathbf{g}_{\mathfrak{A}^r}$ and $\mathbf{\Gamma}_{\mathfrak{A}^r}$. Because of its potential import we here explore the case $r = 2$. We show that, once assigned, the material 2-metric field $\mathbf{g}_{\mathfrak{A}^2}$ on $\mathscr{B}$ uniquely determines the material 2-connection field $\mathbf{\Gamma}_{\mathfrak{A}^2}$ on $\mathscr{B}$. This appears to be a *new* result. We close this Note with a brief discussion of its significance.

1. Basic notions

THE CONCEPT of a higher order body has been outlined in(1) [1]. To each point p in the (abstract) body C^r-differentiable manifold there may be assigned r-vector [resp., r-covector] spaces $\mathscr{T}^r(\mathscr{B}, p)$ [resp., $\mathscr{T}_r(\mathscr{B}, p)$]. That is, set-theoretically:

$$(1.1)\qquad \mathscr{T}^r(\mathscr{B}, p) = \left\{\mathbf{U}_p \,\middle|\, \mathbf{U}_p = U_p^{i_1} \frac{\partial}{\partial x^{i_1}}\bigg|_p + \cdots + U_p^{i_1 \ldots i_r} \frac{\partial^r}{\partial x^{i_1} \ldots \partial x^{i_r}}\bigg|_p \right\},$$

and

$$(1.2)\qquad \mathscr{T}_r(\mathscr{B}, p) = \{\mathbf{U}_p^* | \mathbf{U}_p^* = U^p_{i_1} d_p x^{i_1} + \cdots + U^p_{i_1 \ldots i_r} d_p x^{i_1} \cdots x^{i_r}\},$$

where relative to a given *base space chart* $(U_\alpha(\ni p), \psi_\alpha = (x^i, n))$, $n = \dim\mathscr{B}$, in an atlas $\mathfrak{A}^r$ of $\mathscr{B}$, the $\dfrac{\partial^r}{\partial x^{i_1} \ldots \partial x^{i_k}}\bigg|_p$ and $d_p x^{i_1} \cdots x^{i_k}$, $k = 1, 2, \cdots, r$, are the basis-vectors and -covectors respectively. It is readily verified that the *form* of the elements in (1.1) and (1.2) is unaltered under change of chart. In fact, upon invoking r-jet notation [1], [2],

(1) There called *r-gauge*. Cf., [2] for a modified and improved definition.

if an element $\mathbf{U}_p \in \mathscr{T}^r(\mathscr{B}, p)$ has, relative to a chart $(U_\alpha(\ni p), \psi_\alpha = (x^i, n))$, the representation $\mathbf{U}_{p,\alpha} = \mathbf{e}_{p,\alpha}\mathbf{u}_{p,\alpha}$, where $\mathbf{e}_{p,\alpha} \equiv \left(\left.\frac{\partial}{\partial x^{i_1}}\right|_p, \cdots, \left.\frac{\partial^r}{\partial x^{i_1}\cdots\partial x^{i_r}}\right|_p\right)$ and $\mathbf{u}_{p,\alpha} \equiv (U^{i_1}_{p,\alpha}, \cdots, U^{i_1\cdots i_r}_{p,\alpha})$, then, relative to a chart $(U_\beta(\ni p), \psi_\beta = (y^i, n))$,

$$\mathbf{U}_{p,\beta} = \mathbf{e}_{p,\beta} j^r_{\psi_\alpha(p), <\psi_\beta(p)>}(\psi_\beta \circ \psi_\alpha^{-1})\mathbf{u}_{p,\alpha}, \tag{1.3}$$

where the juxtapositions are to be interpreted via the r-jet composition rule [1], [2], and the operator $\langle \cdot \rangle$ deletes the target from the indicated r-jet.

2. Inner products. Induced metrics

Abstractly [3] *an inner product* on $\mathscr{V} \times \mathscr{V}$, where $\mathscr{V}$ is a real linear vector space, is a real ($\mathscr{R}$)-valued bilinear map:

$$\{\cdot, \cdot\}: \mathscr{V} \times \mathscr{V} \to \mathscr{R}: (\mathbf{U}, \mathbf{V}) \to \{\mathbf{U}, \mathbf{V}\} \tag{2.1}$$

which is: (1) symmetric in $\mathbf{U}$ and $\mathbf{V}$, and (2) positive definite on the diagonal of $\mathscr{V} \times \mathscr{V}$. In conformance with these requirements introduce:

DEFINITION 2.1. *A k-vector inner product* is a bilinear map on $\mathscr{T}^r(\mathscr{B}, p) \times \mathscr{T}^r(\mathscr{B}, p) \to \mathscr{R}$, which relative to a given chart $(\mathscr{U}_\alpha(\ni p), \psi_\alpha = (x^i, n))$ in an atlas for $\mathscr{B}$ *is* defined by:

$$\{\mathbf{U}_p, \mathbf{V}_p\}^{(k,r)} \equiv \sum_{i_1} U^{i_1}_{p,\alpha} V^{i_1}_{p,\alpha} + \sum_{i_1, i_2} U^{i_1 i_2}_{p,\alpha} V^{i_1 i_2}_{p,\alpha} + \ldots + \sum_{i_1, \cdots, i_k} U^{i_1\cdots i_k}_{p,\alpha} V^{i_1\cdots i_k}_{p,\alpha}. \tag{2.2}$$

A *k-covector inner product*, $\{\cdot, \cdot\}_{(k,r)}$, *on* $\mathscr{T}_r(\mathscr{B}, p) \times \mathscr{T}_r(\mathscr{B}, p)$ and a *k-cross-inner product*, $\{\cdot, \cdot\}^{(k,r)}_{(k,r)}$, *on* $\mathscr{T}^r(\mathscr{B}, p) \times \mathscr{T}_r(\mathscr{B}, p)$ are similarly defined.

In the above only the k-cross-inner product is form-invariant under change of base-space chart.

A chart in a material (r-reference) atlas for $\mathscr{B}$ has the form $(\mathscr{U}_\alpha, \mathbf{j}^r_\alpha)$, where $\mathbf{j}^r_\alpha$ is a target-deleted r-jet field on $\mathscr{U}_\alpha$ whose coordinate representation relative to the corresponding base-chart $(\mathscr{U}_\alpha, \psi_\alpha)$ may be expressed as:

$$\mathbf{x}(\mathbf{j}^r_\alpha) = (J^i_{i_1}, J^i_{i_1 i_2}, \cdots, J^i_{i_1 \cdots i_r}), \tag{2.3}$$

where $\mathbf{x}$ is the coordinate map on the space of the indicated r-jets. This serves to motivate:

DEFINITION 2.2. With $1 \leqslant k \leqslant r$, an *induced k-metric on* $\mathscr{B}$ [relative to a material atlas $\mathfrak{A}^r$ *of* $\mathscr{B}$] is a real-valued symmetric bilinear map $[g_{\mathfrak{A}^r}(p; k)](\cdot, \cdot)$ on $\mathscr{T}^r(\mathscr{B}, p) \times \mathscr{T}^r(\mathscr{B}, p)$ such that for each $p \in \mathscr{U}_\alpha$:

$$[g_{\mathfrak{A}^r}(p; k)](\mathbf{U}_p, \mathbf{V}_p) \equiv \{\mathbf{j}^r_\alpha(p)\mathbf{u}_{p,\alpha}, \mathbf{j}^r_\alpha(p)\mathbf{v}_{p,\alpha}\}^{(k,r)} \tag{2.4}$$

for all $(\mathbf{U}_p = \mathbf{e}_{p,\alpha}\mathbf{u}_{p,\alpha}, \mathbf{V}_p = \mathbf{e}_{p,\alpha}\mathbf{v}_{p,\alpha}) \in \mathscr{T}^r(\mathscr{B}, p) \times \mathscr{T}^r(\mathscr{B}, p)$, where the inner product is that of (2.2).

The map $\mathbf{g}_{\mathfrak{A}^r}(p; k)$, by (2.3) and (2.4), may be explicitly exhibited, relative to a base-space chart, as:

$$\mathbf{g}_{\mathfrak{A}^r}(p; k) = \sum_{s,t=1}^{k} g_{i_1\cdots i_s; j_1\cdots j_t}(p; k) d_p x^{i_1} \ldots x^{i_s} \otimes d_p x^{j_1} \ldots x^{j_t}, \tag{2.5}$$

where the tensor product is denoted by $\otimes$, the usual summation convention holds, and the *components* of the metric (2.5) may be expressed in terms of the J's of (2.3). For the case $r = k = 2$, to which we shall subsequently confine our attention, relative to a given base space chart, the juxtaposition to the right in (2.4) has the explicit component form:

$$\mathbf{x}(\mathbf{j}^2_\alpha(p)\mathbf{u}_{p,\alpha}) = (J^i_j U^j_{p,\alpha} + J^i_{jk} U^{jk}_{p,\alpha}, J^i_k J^j_l U^{kl}_{p,\alpha}). \tag{2.6}$$

3. Metrics and cometrics

The dual of the k-metric (2.5) is the *k-cometric*, which relative to a base-space chart has the form:

$$\mathbf{g}^*_{\mathfrak{A}^r}(p;k) = \sum_{s,t=1}^{k} g^{i_1\cdots i_s;j_1\cdots j_t}(p;k) \left.\frac{\partial^s}{\partial x^{i_1}\ldots\partial x^{i_s}}\right|_p \otimes \left.\frac{\partial^t}{\partial x^{j_1}\ldots\partial x^{j_t}}\right|_p, \tag{3.1}$$

where, with $\langle\cdot,\cdot\rangle$ denoting the action of the first on the second entry,

$$\left\langle d_p x^{i_1}\ldots x^{i_s}, \left.\frac{\partial^t}{\partial x^{j_1}\ldots\partial x^{j_t}}\right|_p \right\rangle \equiv \begin{cases} \delta^{i_1}_{j_1}\ldots\delta^{i_s}_{j_s} & \text{if} \quad s=t, \\ 0 & \text{if} \quad s\neq t. \end{cases} \tag{3.2}$$

The components of the k-cometric are determined by requiring that the k-cross-inner product be preserved; that is, by imposing an orthogonality condition of the form $(1 \leqslant k \leqslant r)$:

$$\langle [\mathbf{g}_{\mathfrak{A}^r}(p;k)](\mathbf{U}_p), [\mathbf{g}^*_{\mathfrak{A}^r}(p;k)](\mathbf{V}^*_p)\rangle = U^{i_1}V_{i_1} + U^{i_1 i_2}V_{i_1 i_2} + \cdots + U^{i_1\cdots i_k}V_{i_1\cdots i_k}. \tag{3.3}$$

When $k = r = 2$, (3.3) yields the following simultaneous equations for the components of $\mathbf{g}^*_{\mathfrak{A}^r}(p;2)$:

$$\begin{aligned} &g^{ia}g_{aj} + 4g^{i;ab}g_{ab;j} = \delta^i_j, \qquad && g^{ij;ab}g_{ab;kl} + 4g^{ij;a}g_{a;kl} = \delta^i_k\delta^j_l, \\ &g^{ij;a}g_{ak} + g^{ij;ab}g_{ab;k} = 0, \quad \text{and} && g^{ia}g_{a;jk} + g^{i;ab}g_{ab,jk} = 0, \end{aligned} \tag{3.4}$$

where we have invoked the symmetry requirements on the 2-metric and 2-cometric as reflected in the component-field symmetries:

$$\text{and} \quad \begin{aligned} &g_{ij} = g_{ji}, \quad g_{i;jk} = g_{jk;i}, \quad g_{ij;kl} = g_{kl;ij}, \\ &g^{ij} = g^{ji}, \quad g^{i;jk} = g^{jk;i}, \quad g^{ij;kl} = g^{kl;ij}. \end{aligned} \tag{3.5}$$

Equations (3.4) have two sets of solutions, differing by a constant factor, for each of the components of $\mathbf{g}^*_{\mathfrak{A}^2}(\cdot;2)$. The value of the factor is determined by requiring the coalescence of the two sets of solutions. The orthogonality conditions (3.4) then reduce to:

$$\begin{aligned} &g^{ia}g_{aj} = \frac{1}{2}\delta^i_j, \quad g^{ij;ab}g_{ab;kl} = \frac{1}{2}\delta^i_k\delta^j_l, \\ &g^{i;ab}g_{ab;j} = \frac{1}{8}\delta^i_j, \quad g^{ij;a}g_{a;kl} = \frac{1}{8}\delta^i_k\delta^j_l, \end{aligned} \tag{3.6}$$

and

$$g^{i;jk} = -2g^{ia}g_{a;bc}g^{bc;jk}. \tag{3.7}$$

The result $(3.6)_1$ for 2-metrics may be compared with that obtained in classical (1-) tensor analysis: $g^{ia}g_{aj} = \delta^i_j$; of course, the remainder of the relations (3.6), (3.7) have no counterparts in classical theory. Relations (3.6) are frequently invoked in the course of the subsequent development.

4. Two-connections and two-metrics

A description of the methods whereby it can be shown that (smooth) material inhomogeneities in higher order bodies may be characterized by higher order material connections is beyond the scope of this Brief Note. So is an outline of the theory for determining covariant derivatives relative to such connections. These two topics are covered in [2].

A material two-connection field on $\mathscr{B}$ has, relative to a material- and corresponding base-space-chart, the representation $\Gamma_{\mathfrak{A}^2} = (\Gamma^i_{jk}, \Gamma^i_{jkl})$. The body $\mathscr{B}$ is said to be a *2-solid* if and only if the first entry in the coordinate representation of its relative isotropy group $\mathscr{G}^2_n(\mathfrak{A}^2) = \{(a^i_j, a^i_{jk})\}$ is such that the set $\{a^i_j | \det a^i_j \neq 0\} \subseteq \mathscr{Q}(n)$, the full orthogonal group of transformations of $\mathscr{R}^n$. As should be the case, it is readily shown that this definition of a 2-solid does not depend upon the choice of base-space chart. An *intrinsic tensor*(²) at $p \in \mathscr{B}$ is one which is invariant under all transformations of the intrinsic isotropy group at p. *Intrinsic tensor fields on* $\mathscr{B}$ are similarly defined. It follows that the induced 2-metric field on a 2-solid is intrinsic. Thus, its covariant differential must vanish on $\mathscr{B}$. Coordinatewise this condition is reflected by the three equations [2, § 12]:

$$\frac{\partial g_{ij}}{\partial x^w} + \Gamma^u_{iw} g_{uj} + \Gamma^u_{jw} g_{iu} = 0,$$

(4.1)
$$\frac{\partial g_{ij;k}}{\partial x^w} + \Gamma^u_{iw} g_{uj;k} + \Gamma^u_{jw} g_{i;uk} + \Gamma^u_{kw} g_{ij;u} + \Gamma^u_{ikw} g_{uj} + \Gamma^u_{jkw} g_{iu} = 0,$$

and

$$\frac{\partial g_{ij;kl}}{\partial x^w} + \Gamma^u_{iw} g_{uj;kl} + \Gamma^u_{jw} g_{iu;kl} + \Gamma^u_{kw} g_{ij;ul} + \Gamma^u_{lw} g_{ij;ku} + \Gamma^u_{ijw} g_{uk;l} = 0.$$

We seek to invert these equations so as to obtain the Γ's in terms of the g's. Clearly, by $(3.6)_3$, $(4.1)_3$ yields an expression for Γ^u_{ijw} in terms of the Γ^i_{jk}'s and the $g_{ij,kl}$'s. Let $\Gamma^i_{(jk)}$ and $\Gamma^i_{[jk]}$ denote the symmetric and skew-symmetric parts of Γ^i_{jk} respectively. Then, invoking the classical solution-method, $(4.1)_1$ may be solved for $\Gamma^i_{(jk)}$. The substitution of these two results in $(4.1)_2$ yields an equation for $\Gamma^i_{[jk]}$ whose solution is:

$$\Gamma^u_{[kw]} + \Gamma^n_{(kw)} = 32 g^{u;ij} \left\{ \frac{\partial g_{[ij]:k}}{\partial x^w} + g_{[jv]:k} \Gamma^v_{(iw)} + g_{[vi]:k} \Gamma^v_{(jw)} + g_{ji;v} \Gamma^v_{(kw)} \right\} - 3\Gamma^u_{(kw)}. \tag{4.2}$$

But $\Gamma^i_{jk} = \Gamma^i_{(jk)} + \Gamma^i_{[jk]}$; hence, (4.2) serves to determine $\Gamma^i_{[jk]} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj})$. In summary, the solution of (4.1) for the Γ's is:

$$\Gamma^i_{jk} = -g^{ia} \left\{ \frac{\partial g_{ja}}{\partial x^k} + \frac{\partial g_{ak}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^a} \right\} + \Gamma^i_{[jk]},$$

(²) We employ "tensor" in a generalized sense; e.g., as an element in $(\overset{R}{\otimes} \mathscr{T}^r(\mathscr{B}, p)) \otimes (\overset{S}{\otimes} \mathscr{T}_r(\mathscr{B}, p))$, $R, S = 1, 2, \ldots$.

$$\Gamma^{i}_{[jk]} = 16 g^{i;ab}\left\{\frac{\partial g_{[ab];[j}}{\partial x^{k]}} + g_{[bw];[j}\Gamma^{w}_{(k]a)} + g_{[wa];[j}\Gamma^{w}_{(k]b)}\right\}, \tag{4.3}$$

and

$$\Gamma^{i}_{jkl} = -\frac{1}{8n} g^{i;ab}\left\{\frac{\partial g_{jk;ab}}{\partial x^{l}} + g_{uk;ab}\Gamma^{u}_{jl} + g_{ju;ab}\Gamma^{u}_{kl} + g_{jk;ub}\Gamma^{u}_{al} + g_{jk;au}\Gamma^{u}_{bl}\right\}.$$

This result may be phrased as:

PROPOSITION 4.1. *Let $\mathscr{B}$ be a 2-solid. Then the assignment of a material 2-metric field on $\mathscr{B}$ uniquely determines the material 2-connection field on $\mathscr{B}$ in accordance with* (4.3).

COROLLARY 4.1. *The material 2-connection $\Gamma_{\mathfrak{A}^2} = (\Gamma^{i}_{jk}, \Gamma^{i}_{jkl})$ field on $\mathscr{B}$ is symmetric in j and k if and only if the induced material 2-metric fields possess the symmetries*:

$$g_{ij;k} = g_{ji;k} \quad \text{and} \quad g_{ij;kl} = g_{jl;kl}. \tag{4.4}$$

5. Discussion

Within the confines of the universe of discourse for simple-(gross) body theory [4] no mechanism is available for determining the *torsion*, $\Gamma^{i}_{[jk]}$, of the connection Γ^{i}_{jk}. As a consequence it must necessarily rest content with focusing its attention at this stage of its development on *Riemannian* (i.e., torsion-free) *material connections*. As shown by (4.3) such a limitation does not arise in the case of 2-solid (gross) bodies. In fact, $(4.3)_2$ indicates that in general the connection symbol Γ^{i}_{jk} *is not* torsion-free. Since the components of the 2-curvature tensor [2], $\mathbf{R}_{\mathfrak{A}^2} = (R^{i}_{jkl}, R^{i}_{jklm})$, are expressed entirely in terms of the Γ's and their first partial derivatives, we conclude from (4.3) that the assignment of a material 2-metric field, $\mathbf{g}_{\mathfrak{A}^2}(\cdot\,;2)$, not only uniquely determines the material 2-connection field, $\Gamma_{\mathfrak{A}^2}(\cdot)$, but also the 2-curvature field $R_{\mathfrak{A}^2}(\cdot)$. However, in general, the converse is not true. That such is the case follows from the fact that, upon assignment of the field $\Gamma_{\mathfrak{A}^2}(\cdot)$ on $\mathscr{B}$, (4.1) may be viewed as a system of three coupled first order linear partial differential equations for the components of $\mathbf{g}_{\mathfrak{A}^2}(\cdot)$. In general, these equations do not possess a unique solution. The study of this inverse problem should, we conjecture, yield new insights into the mathematical properties of inhomogeneous materially uniform 2-solid (gross) bodies.

Finally, another interesting aspect of the result (4.3) is that, in contrast to $\Gamma_{\mathfrak{A}^2}$, the 2-metric $\mathbf{g}_{\mathfrak{A}^2}$, since it is a measure of "distance," possesses greater amenability to physical interpretation and, hence, is easier to assign. Thus, providing still another avenue of ingress to the theoretical study of inhomogeneous 2-solids. Naturally, the equations of motion for such solids would need to be invoked. These have been derived in [2].

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