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On the invariance of Hamilton's function

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Two theorems, for mechanical systems with non-autonomous Hamiltonians, are proved: a) to every constant of motion there corresponds an infinitesimal transformation that leaves the Hamiltonian invariant; b) every constant of motion is invariant under an infinitesimal transformation. The Emden-Fauller equation is analysed as an example.

1. It is well known (see for example [1], p. 95, [2], p. 224, [3 and 4]) that a constant of motion of a dynamical system generates an infinitesimal canonical transformation in phase space with respect to which the Hamiltonian of the problem is invariant. The theorem is valid for holonomic conservative dynamical systems whose Hamiltonian and the constant of motion are autonomous, that is, these quantities do not depend explicitly on time. Recently, applying Noetherian theory a number of non-autonomous first integrals for mechanical systems with dissipative forces and with non-autonomous Hamiltonians have been obtained [5–7].

The objective of this short note is to establish the connection between the conserved quantities of motion and the invariance properties of a Hamiltonian in the case where the constant of motion and the Hamiltonian depend explicitly on time.

2. Let us consider a holonomic mechanical system with n-degrees of freedom, where the q^i are regarded as generalized coordinates, the p_i are the generalized momenta and t is the time ([1]). The motion of the system is such that it is possible to construct a Hamiltonian function in the form $H = H(t, q^i, p_i)$. Now, the time variation of any dynamical variable $F = F(t, q^i, p_i)$ is given by

$$\frac{dF}{dt} = [F, H] + \frac{\partial F}{\partial t}, \tag{1}$$

where the equations of motion $\dot{q}^i = \partial H/\partial p_i$; $\dot{p}_i = -\partial H/\partial q^i$ are used and $[F, H]$ denotes the Poisson bracket of F and H. If Δt, Δq^i and Δp_i are infinitesimal transformation as of time, generalized coordinates and momenta than these produce a change ΔD in the value of any dynamical variable $D(t, q^i, p_i)$ in the form

$$\Delta D = \frac{\partial D}{\partial t}\Delta t + \frac{\partial D}{\partial q^i}\Delta q^i + \frac{\partial D}{\partial p_i}\Delta p_i. \tag{2}$$

([1]) Unless stated otherwise, Latin indices take the values $1, 2, \ldots, n$ and the summation convention is adopted.

Assuming that

$$\Delta q^i = \varepsilon\frac{\partial G}{\partial p_i}, \qquad \Delta p_i = -\varepsilon\frac{\partial G}{\partial q^i}, \tag{3}$$

where ε is a small parameter of the transformation and $G = G(t, q^i, p_i)$ is an arbitrary function, (2) reduces to

$$\Delta D = \frac{\partial D}{\partial t}\Delta t + \varepsilon[D, G]. \tag{4}$$

3. Applying (1) to the constant of motion G, i.e. when $dG/dt = 0$, and applying (4) to the Hamiltonian ($D \equiv H$) and combining these results we obtain the change in the value of the Hamiltonian

$$\Delta H = \frac{\partial H}{\partial t}\Delta t + \varepsilon\frac{\partial G}{\partial t}. \tag{5}$$

When

$$\Delta t = -\varepsilon\frac{\partial G/\partial t}{\partial H/\partial t} \tag{6}$$

it is obvious that $\Delta H = 0$ and we have the following theorem:

THEOREM 1. *To every constant of motion $G(t, q^i, p_i)$, for the dynamical system described by the Hamiltonian $H(t, q^i, p_i)$, there corresponds an infinitesimal transformation* (3) *and* (6) *that leaves the Hamiltonian invariant.*

Remark. The transformation (3) and (6) is not a canonical transformation as in the case of an autonomous system [1] p. 94.

4. Applying (1) and (4) to the constant of motion, D, and for $G \equiv H$, we obtain the change in the value of D

$$\Delta D = \frac{\partial D}{\partial t}(\Delta t - \varepsilon). \tag{7}$$

Hence for

$$\Delta t = \varepsilon \tag{8}$$

we have immediately that $\Delta D = 0$, and the following theorem is established:

THEOREM 2. *Every constant of motion $D(t, q^i, p_i)$, for a dynamical system with Hamiltonian $H = H(t, q^i, p_i)$, is invariant under the infinitesimal transformation* (3), *where $G \equiv H$, and* (8).

5. Let us consider the Emden-Fauller equation

$$t\ddot{x} + 2\dot{x} + at^{\nu}x^{2\nu+3} = 0, \tag{9}$$

where a and ν are arbitrary constants.

The Hamiltonian of the problem is [6]

$$H = \frac{1}{2}\left(p^2t^{-2} + \frac{a}{\nu+2}t^{\nu+1}x^{2\nu+4}\right), \tag{10}$$

where $p = t^2\dot{x}$ is the generalized momentum. The Eq. (9) possesses a first integral [6]

$$G = p^2t^{-1}+xp+\frac{a}{\nu+2}t^{\nu+2}x^{2(\nu+2)} = \text{const.} \tag{11}$$

According to Theorem 1 the Hamiltonian (10) is invariant under the infinitesimal transformation

$$\begin{aligned} &\Delta x = \varepsilon(x+2pt^{-1}), \quad \Delta p = -\varepsilon(p+at^{\nu+2}2x^{2\nu+3}),\\ &\Delta t = -\varepsilon\frac{at^{\nu+1}x^{2(\nu+2)}-p^2t^{-2}}{\dfrac{a(\nu+1)}{2(\nu+2)}t^{\nu}x^{2(\nu+2)}-p^2t^{-3}}, \end{aligned} \tag{12}$$

while, according to Theorem 2, the quantity G, given by (11), is invariant with respect to the transformation

$$\Delta t = \varepsilon, \quad \Delta x = \varepsilon pt^{-2}, \quad \Delta p = -\varepsilon at^{\nu+1}x^{2\nu+3}. \tag{13}$$

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