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Axisymmetric punch problem under condition of consolidation

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In the paper the exact solution of a problem of axisymmetric punch which is pressed into visco-elastic consolidating half-space by known force $P(t)$ is given. On the base of the three-dimensional, linear theory of consolidation with rheological properties of skeleton by different dilatational, shear and due to the fluid pressure creep — the states of strain and stress are given. The formulas for contact stresses and displacement of the punch are presented. Particularly, for media with elastic skeleton the graphs of stresses and punch displacements for different load velocities are given. The result obtained in the paper shows, that the character of singularity of contact stresses in consolidating medium is the same as in the one-phase medium.

W pracy podano ścisłe rozwiązanie osiowo-symetrycznego problemu stempla wciskanego w konsolidującą półprzestrzeń lepkosprężystą znaną siłą $P(t)$. Na gruncie trójwymiarowej, liniowej teorii konsolidacji z uwzględnieniem własności reologicznych szkieletu przy różnym pełzaniu postaciowym, objętościowym oraz wywołanym przez ciśnienie cieczy w porach — wyznaczono stan naprężenia i deformacji ośrodka uzyskując efektywne wzory dla naprężeń kontaktowych oraz przemieszczenia stempla. W szczególności dla ośrodka o szkielecie sprężystym podano wykresy naprężeń i przemieszczeń stempla dla różnych chwil czasu i różnych prędkości obciążenia. Wykazano, że charakter osobliwości naprężeń kontaktowych w ośrodku konsolidującym jest taki sam jak w ośrodku jednofazowym.

В работе дается точное решение осе-симметричной задачи штампа вдавливаемого в консолидующее вязко-упругое полупространство известной силой $P(t)$. На грунте трехмерной, линейной теории консолидации с учетом реологических свойств скелета, при разной ползучести формы и объемной ползучести, а также ползучести, вызванной давлением жидкости в порах, определены напряженное и деформированное состояния среды, получая эффективные формулы для контактных напряжений и перемещения штампа. В частности для среды с упругим скелетом даются диаграммы напряжений и перемещений штампа для разных моментов времени и разных скоростей нагружения. Показано, что характер особых контактных напряжений в консолидующейся среде аналогичен, как в однофазной среде.

1. Introduction

CONSOLIDATION of subsoil under the punch action and the associated evolution of stress concentration is one of the most fundamental problems in mechanics of porous medium saturated with liquid. The knowledge of the contact stress distribution and of the displacement of punch in time is very important from the cognitive as well as from the technical point of view.

Though this problem has already been considered by some authors (DERSKI, ZARETSKY, SZEFER and DOMSKI, BOOKER), its full and exact analysis still remains an open question. DERSKI [3] considered the rigid punch in solid with elastic skeleton and reduced the mixed boundary problem to the dual integral equations only. The discussion of the solution of these equations remains open up to now. ZARETSKY [16] in analogical case has applied the successive approximations method to solve the couple system of equations of the theory of consolidation, and calculated in the first step the liquid pressure from the simplified (without deformation element) filtration equation. Owing to this he avoided the solution of the complicated dual equations but on the other hand, he departed from exactness of the solution. The papers of SZEFER and DOMSKI [13] and BOOKER [2] present the numerical aproach (in the paper [13] the method of finite differences was applied and rich illustrations of the stress state are given, BOOKER on the contrary applied the finite element method but restricted his considerations to show the general procedure in case of mixed boundary conditions only).

On this occasion it is worth to make a note that the problem of a force acting on half-space, which is in strong connection with the discussed problem (the singularity of this solution determined the character of integral equation in the boundary punch problem) and which was considered by FREUDENTHAL and SPILLERS [4], and later by SOBCZYŃSKA [9], is unfortunately not free from defects. These follow from the non-correctly posed initial conditions which ignore the property of non-normality of the system of consolidation theory equations.

In the present paper the problem of action of the rigid, smooth punch with known contact zone, resting on a consolidating viscoelastic half-space is considered. The question under conditions of axisymmetry on base of three-dimensional (coupled) BIOT's theory of consolidation is discussed. The aim of the paper is the effective calculation of contact stresses under the punch and its displacement in time. The problem will be solved by general assumptions on the rheological properties of the skeleton, while for the medium with elastic skeleton the detailed graphs of interesting functions will be given. The solutions are given both for the permeable and impermeable edge.

2. The set of equations of consolidating viscoelastic solid

The subject of our considerations will be a two-phase continuous medium, one phase of which constructed a porous, viscoelastic, homogeneous and isotropic skeleton, while the second one constituted fluid which filtrated through the pores according to the Darcy's law. The rheological properties of the skeleton we shall characterize by different shear, dilatational and, due to the fluid pressure, creep kernels. This last fact, included and discussed by ZARETSKY [16], is a result of observed property of soils, which demonstrate different characteristics of dilatational deformation in process of compression and tension. According to the MESTCHIAN's [7] experiments, we assume the Boltzmann's hereditary principle for material without aging.

For material so characterized, the constitutive equations have the form [16, 11]

$$\sigma_{ij} = 2N\varepsilon_{ij}+M\varepsilon_{kk}\delta_{ij}-Ap\,\delta_{ij}, \tag{2.1}$$

where quantities N, M, A are Volterra's integral operators of the second kind

$$\begin{gathered} N = \mu\Big[1-\int\limits_0^t R(t-\tau)\,\dots\,d\tau\Big], \quad M = \frac{1}{3}(A_v-2N), \quad A = A_v A_p^{-1}, \\ A_v = \alpha_v\Big[1-\int\limits_0^t R_v(t-\tau)\,\dots\,d\tau\Big], \quad A_p^{-1} = \frac{1}{\alpha_p}\Big[1+\int\limits_0^t K_p(t-\tau)\,\dots\,d\tau\Big]. \end{gathered} \tag{2.2}$$

The following notations are used: μ — shear modulus of the skeleton, α_v — dilatational modulus of the skeleton, α_p — dilatational modulus due to the fluid pressure in the pores, $K(t-\tau)$, $K_v(t-\tau)$, $K_p(t-\tau)$ — kernels describing the shear, dilatational and due to the fluid pressure creep, $R(t-\tau)$, $R_v(t-\tau)$ — resolvents of the kernels $K(t-\tau)$, $K_v(t-\tau)$.

Moreover, in formula (2.1) we assume the notations: σ_{ij} — elements of the stress tensor in two-phase medium, ε_{ij} — elements of the strain tensor, p — hydrostatic pressure of the fluid, δ_{ij} — Kronecker's symbol.

Additionally, on the basis of Darcy's law and principle of mass continuity, we consider the filtration equation

$$\frac{k}{\gamma}\Delta p = \frac{3n}{\alpha_w}\dot{p}+\dot{\varepsilon}_{kk} \tag{2.3}$$

in which k denotes filtration coefficient, γ specific weight of the fluid, n porosity, α_w compressibility modulus dilatations of the fluid, Δ Laplace operator, $(\dot{\cdot}) = \partial/\partial t$.

Beside the physical relations (2.1), (2.3) we shall add the known equilibrium and geometrical equations of continuous medium. Including these relations by substitution of (2.1) we come to the known system of displacement equations [16]

$$N\Delta u_i+(N+M)\varepsilon_{kk,i}-Ap_{,i} = 0, \quad i = 1, 2, 3. \tag{2.4}$$

These, with the filtration equation (2.3), will constitute the basis of further considerations.

3. Formulation and general solution of the problem

In cylindrical system of coordinates $0rz$ let us consider a flat, smooth punch pressed by a known force $P(t)$ into half-space $z \geqslant 0$ (Fig. 1).

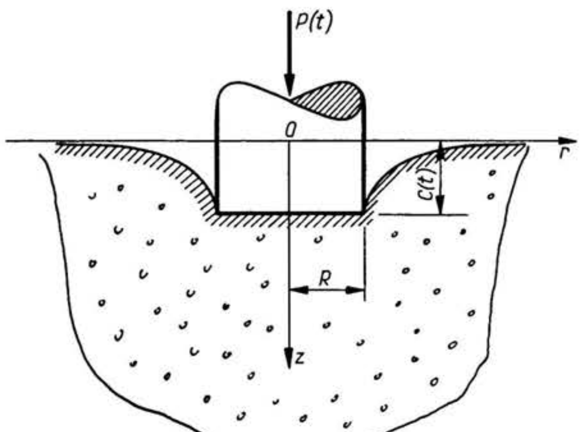


FIG. 1.

Denoting by $u(r, z, t)$, $w(r, z, t)$ the displacements respectively in r- and z-directions, we shall write the Eqs. (2.4), (2.3) in the form

$$
\begin{gathered}
N\left(\Delta u-\frac{u}{r^2}\right)+(N+M)\varepsilon_{,r} = Ap_{,r},\\
N\Delta w+(N+M)\varepsilon_{,z} = Ap_{,z},\\
\frac{k}{\gamma}\Delta p = \frac{3n}{\alpha_w}\dot{p}+\dot{\varepsilon}, \qquad \varepsilon = \frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}.
\end{gathered}
\tag{3.1}
$$

The boundary conditions of the problem are following:

$$
\begin{aligned}
\sigma_{rz}(r, 0, t) &= 0, && r > 0,\\
w(r, 0, t) &= c(t), && 0 < r < R,\\
\sigma_z(r, 0, t) &= 0, && r > R,
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
p(r, 0, t) &= 0 && \text{for permeable edge,}\\
p_{,z}(r, z, t)|_{z=0} &= 0 && \text{for impermeable edge.}
\end{aligned}
$$

Simultaneously, for the uniqueness of solution we demand $u, w, p \to 0$ for $r \to \infty$ and $z \to \infty$.

Assuming that the force $P(t)$ increases slowly from zero to the limit value, we take homogeneous initial conditions

$$
u(r, z, 0) = w(r, z, 0) = p(r, z, 0) = 0
\tag{3.3}
$$

which are consistent with the structure of the Eqs. (3.1) for the given type of load (the so-called "consistence conditions" which must be satisfied in case of non-normal system of equations, fulfilled automatically).

To begin with, we perform on the system (3.1) the Hankel integral transform with respect to the variable r and the Laplace transform with respect to t. Including the form of operators (2.2) and the convolution theorem, it will read

$$
\begin{gathered}
\bar{N}(\bar{\bar{u}}''-\omega^2\bar{\bar{u}}) = \omega[(\bar{N}+\bar{M})\bar{\bar{\varepsilon}}-\bar{A}\bar{\bar{p}}],\\
\bar{N}(\bar{\bar{w}}''-\omega^2\bar{\bar{w}}) = -[(\bar{N}+\bar{M})\bar{\bar{\varepsilon}}'-\bar{A}\bar{\bar{p}}'],\\
s\bar{\bar{\varepsilon}} = \frac{k}{\gamma}(\bar{\bar{p}}''-\omega^2\bar{\bar{p}})-\frac{3n}{\alpha_w}s\bar{\bar{p}},
\end{gathered}
\tag{3.4}
$$

where

$$
\begin{bmatrix} \bar{\bar{u}} \\ \bar{\bar{\varepsilon}} \\ \\ \bar{\bar{w}} \\ \bar{\bar{p}} \end{bmatrix}
\begin{matrix} = \int_0^\infty\int_0^\infty \begin{bmatrix} u(r,z,t) \\ \varepsilon(r,z,t) \end{bmatrix} rJ_1(\omega r)e^{-st}dr\,dt, \\ \\ = \int_0^\infty\int_0^\infty \begin{bmatrix} w(r,z,t) \\ p(r,z,t) \end{bmatrix} rJ_0(\omega r)e^{-st}dr\,dt, \end{matrix}
$$

$\bar{N}, \bar{M}, \bar{A}$ — Laplace transforms of operators N, M, A, $(') = d/dz$.

Solving the system of ordinary equations (3.4) and taking into account the fact that the hydrostatics pressure p must vanish when z tends to infinity, we obtain

$$\bar{\bar{u}} = C_3 e^{-\omega z} + C_1 \frac{\bar{N}+\bar{M}}{2} \bar{N}^{-1} \left[\frac{3n}{\alpha_w} + \bar{A}(\bar{N}+\bar{M})^{-1} \right] z e^{-\omega z} - C_2 \frac{\omega}{s} \bar{A}\bar{B}^{-1}(2\bar{N}+\bar{M})^{-1} e^{-mz},$$

$$\bar{\bar{w}} = C_4 e^{-\omega z} + C_1 \frac{\bar{N}+\bar{M}}{2} \bar{N}^{-1} \left[\frac{3n}{\alpha_w} + \bar{A}(\bar{N}+\bar{M})^{-1} \right] z e^{-\omega z} - C_2 \frac{m}{s} \bar{A}\bar{B}^{-1}(2\bar{N}+\bar{M})^{-1} e^{-mz}, \tag{3.5}$$

$$\bar{\bar{p}} = C_1 e^{-\omega z} + C_2 e^{-mz},$$

where

$$m = \sqrt{\omega^2 + s\frac{\gamma}{k}\left[\frac{3n}{\alpha_w} + \bar{A}(2\bar{N}+\bar{M})^{-1} \right]},$$

$$\bar{B} = \frac{\gamma}{k}\left[\frac{3n}{\alpha_w} + \bar{A}(2\bar{N}+\bar{M})^{-1} \right].$$

Making use of the physical relations

$$\begin{aligned} \sigma_r &= 2N\varepsilon_r + M\varepsilon - Ap, \\ \sigma_z &= 2N\varepsilon_z + M\varepsilon - Ap, \\ \sigma_{rz} &= 2N\varepsilon_{rz}, \end{aligned} \tag{3.6}$$

and geometrical relations

$$\begin{aligned} \varepsilon_r &= u_{,r}, \\ \varepsilon_z &= w_{,z}, \end{aligned} \qquad \varepsilon_{rz} = u_{,z} + w_{,r}, \tag{3.7}$$

by (3.5), after performing the integral transforms we obtain the following quantities of stress transforms:

$$\bar{\bar{\sigma}}_z = C_1 \left[-(\bar{N}+\bar{M})\left(-\frac{3n}{\alpha_w} + \frac{\bar{A}}{\bar{N}+\bar{M}} \right) \omega z + \frac{\bar{M}}{2\bar{N}} \bar{A} + \frac{3n}{\alpha_w} \frac{2\bar{N}+\bar{M})\,(\bar{N}+\bar{M})}{2\bar{N}} \right] e^{-wz} + C_2 \frac{2\bar{A}\bar{N}\omega^2}{s\bar{B}(2\bar{N}+\bar{M})} e^{-mz} + C_3 \bar{M}\omega e^{-\omega z} - C_4(2\bar{N}+\bar{M})\omega e^{-\omega z}, \tag{3.8}$$

$$\bar{\bar{\sigma}}_{rz} = C_1 (\bar{N}+\bar{M}) \left(\frac{3n}{\alpha_w} + \frac{\bar{A}}{\bar{N}+\bar{M}} \right) \frac{1-\omega z}{2} e^{-\omega z} + C_2 \frac{2\bar{A}\bar{N}m\omega}{sB(2\bar{N}+\bar{M})} e^{-mz} - C_3 \bar{N}\omega e^{-\omega z} - C_4 \bar{N}\omega e^{-\omega z}.$$

Formulas (3.5), (3.8) contain four unknown parameters C_1, C_2, C_3, C_4. To find these, we need four relations. Three equations yield the boundary conditions (3.2), the fourth follows from the fact that by the solution of the system (3.4), the order of the Eq. $(3.4)_3$ was increased (from two to four). The solutions obtained are therefore integrals of higher order equations. To obtain the solution of system (3.4), the integrals (3.5) must satisfy $(3.4)_3$. Consequently substituting (3.5) into $(3.4)_3$, we come to the fourth condition

$$2\bar{N}\omega(C_3-C_4)+\left[\frac{3n}{\alpha_w}(3\bar{N}+\bar{M})+A\right]C_1 = 0. \tag{3.9}$$

Next, we proceed to include the boundary conditions. Successively we consider first the permeable edge applying conditions $(3.2)_1$–$(3.2)_4$. Performing the Laplace transformation and then substituting (3.5) and (3.8), we obtain

$$\begin{gathered}
\int\limits_0^\infty \left[-\omega(C_3+C_4)+\frac{\bar{N}+\bar{M}}{2\bar{N}}\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{\bar{N}+\bar{M}}\right)C_1+\frac{2\bar{A}m\omega C_2}{s\bar{B}(2\bar{N}+\bar{M})}\right]\omega J_1(\omega r)d\omega = 0,\\
\int\limits_0^\infty \left[C_4-C_2\frac{\bar{A}m}{s\bar{B}(2\bar{N}+\bar{M})}\right]\omega J_0(\omega r)d\omega = \bar{c}(t), \quad r<R,\\
\int\limits_0^\infty \left[\bar{M}\omega C_3-(2\bar{N}+\bar{M})\omega C_4-\bar{A}C_1+2\frac{\omega^2}{s}\frac{\bar{A}\bar{N}}{\bar{B}(2\bar{N}+\bar{M})}C_2\right.\\
\left.+\frac{(\bar{N}+\bar{M})(2\bar{N}+\bar{M})}{2\bar{N}}\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{\bar{N}+\bar{M}}\right)C_1\right]\omega J_0(\omega r)d\omega = 0, \quad r>R,\\
\int\limits_0^\infty (C_1+C_2)\omega J_0(\omega r)d\omega = 0.
\end{gathered} \tag{3.10}$$

On the grounds of (3.9), (3.10) and $(3.10)_4$ we eliminate quantities C_2, C_3, C_4 expressed by means of C_1:

$$\begin{aligned}
C_2 &= -C_1,\\
C_3 &= -\frac{1}{2\omega}\left[\frac{3n}{\alpha_w}+2\frac{m\omega}{s}\frac{\bar{A}}{\bar{B}(2\bar{N}+\bar{M})}\right]C_1,\\
C_4 &= -\frac{1}{2\omega}\left[\frac{3n}{\alpha_w}\frac{2\bar{N}+\bar{M}}{\bar{N}}+\frac{\bar{A}}{\bar{N}}-2\frac{m\omega}{s}\frac{\bar{A}}{\bar{B}(2\bar{N}+\bar{M})}\right]C_1.
\end{aligned} \tag{3.11}$$

Including further the mixed boundary condition $(3.10)_{2,3}$ we arrive at the following dual integral equations

$$\begin{gathered}
\int\limits_0^\infty C_1\left[\frac{3n}{\alpha_w}\frac{2\bar{N}+\bar{M}}{\bar{N}}+\frac{\bar{A}}{\bar{N}}\right]J_0(\omega r)d\omega = 2\bar{c}(s), \quad r<R,\\
\int\limits_0^\infty C_1\left[\frac{3n}{\alpha_w}(\bar{N}+\bar{M})+\bar{A}-\frac{2}{s}\frac{\bar{A}\bar{N}\omega(m-\omega)}{\bar{B}(2\bar{N}+\bar{M})}\right]\omega J_0(\omega r)d\omega = 0, \quad r>R.
\end{gathered}$$

Making use of the transformation variables

$$r = Ru, \quad \omega = vR^{-1}$$

and substituting

$$\Phi_1(v, s) = C_1 v\left[\frac{3n}{\alpha_w}(\bar{N}+\bar{M})+\bar{A}-\frac{2}{sR^2}\frac{\bar{A}\bar{N}v(m-v)}{\bar{B}(2\bar{N}+\bar{M})}\right],$$

$$\text{(3.12)} \qquad L_1(v, s) = \frac{1}{v\left[\dfrac{3n}{\alpha_w}(\bar{N}+\bar{M})+\bar{A}-\dfrac{2}{sR^2}\dfrac{\bar{A}\bar{N}v(m-v)}{\bar{B}(2\bar{N}+\bar{M})}\right]},$$

$$f(s) = \frac{2\bar{N}R}{\dfrac{3n}{\alpha_w}(2\bar{N}+\bar{M})+\bar{A}}\bar{c}(s),$$

we finally obtain

$$\text{(3.13)} \qquad \begin{aligned} &\int_0^\infty \Phi_1(v, s)L_1(v, s)J_0(uv)\,dv = f(s), \quad u<1,\\ &\int_0^\infty \Phi_1(v, s)J_0(uv)\,dv = 0, \quad u>1. \end{aligned}$$

Similarly for the impermeable edge, for which in lieu of (3.2) we apply $(3.2)_5$ by means of analogical transformations, we obtain dual equations

$$\text{(3.14)} \qquad \begin{aligned} &\int_0^\infty \Phi_2(v, s)L_2(v, s)J_0(uv)\,dv = f(s), \quad u<1,\\ &\int_0^\infty \Phi_2(v, s)J_0(uv)\,dv = 0, \quad u>1, \end{aligned}$$

in which

$$\text{(3.15)} \qquad \begin{aligned} \Phi_2(v, s) &= C_1 v\left[\frac{3n}{\alpha_w}(\bar{N}+\bar{M})+\bar{A}-\frac{2}{sR^2}\frac{\bar{A}\bar{N}\dfrac{v^2}{m}(m-v)}{\bar{B}(2\bar{N}+\bar{M})}\right],\\ L_2(v, s) &= \frac{1}{v\left[\dfrac{3n}{}(\bar{N}+\bar{M})+\bar{A}-\dfrac{2}{sR^2}\dfrac{\bar{A}\bar{N}\dfrac{v^2}{m}(m-v)}{\bar{B}(2\bar{N}+\bar{M})}\right]}. \end{aligned}$$

Further discussion of the problem is connected with the solution of the equations (3.13) and (3.14).

4. Solution of the dual integral equations of the problem

Both cases of the Eqs. (3.13) and (3.14) may be treated together

$$(4.1)\qquad \begin{aligned} &\int_0^\infty L(v,s)\Phi(v,s)J_0(uv)\,dv = f(s), \qquad u<1,\\ &\int_0^\infty \Phi(v,s)J_0(uv)\,dv = 0, \qquad u>1, \end{aligned}$$

where

$$(4.2)\qquad \begin{aligned} L(v,s) &= \begin{bmatrix} L_1(v,s) & \text{for the permeable edge,}\\ L_2(v,s) & \text{for the impermeable edge,}\end{bmatrix}\\ \Phi(v,s) &= \begin{bmatrix} \Phi_1(v,s) & \text{for the permeable edge,}\\ \Phi_2(v,s) & \text{for the impermeable edge.}\end{bmatrix} \end{aligned}$$

To the solution of the Eqs. (4.1) we apply the LEBEDEV-UFLAND's method [15, 10], the unknown function $\Phi(v,s)$ we find in the form

$$(4.3)\qquad \Phi(v,s) = v\int_0^1 \varphi(\xi,s)\cos v\xi\, d\xi;$$

owing to the equation $(4.1)_2$ they are satisfied identically. This follows immediately from the simple calculation

$$\begin{aligned} \int_0^\infty v\int_0^1 \varphi(\xi,s)\cos v\xi J_0(uv)\,d\xi\, dv &= \int_0^\infty \Big[\varphi(1,s)\sin v - \int_0^1 \varphi'_\xi(\xi,s)\sin v\xi\Big] d\xi J_0(uv)\,dv\\ &= \varphi(1,s)\int_0^\infty \sin v J_0(uv)\,dv - \int_0^1 \varphi'_\xi(\xi,s)\int_0^\infty \sin v\xi J_0(uv)\,dv\,d\xi \end{aligned}$$

and from the property of Weber-Schafheitlin integral [8]

$$(4.4)\qquad \int_0^\infty \sin v\xi J_0(uv)\,dv = \begin{cases} \dfrac{1}{\sqrt{\xi^2-u^2}}, & \xi>u,\\ 0, & \xi<u. \end{cases}$$

Substituting (4.3) into the first of the Eqs. (4.1) we obtain

$$(4.5)\qquad \begin{aligned} &\int_0^\infty v\int_0^1 [\varphi(\xi,s)\cos v\xi L(v,s)J_0(uv)]\,d\xi\, dv\\ &\qquad = \int_0^1 \varphi(\xi,s)\int_0^\infty vL(v,s)\cos v\xi J_0(uv)\,dv\,d\xi = f(s). \end{aligned}$$

On the basis of knowledge of functions $L_1(v,s)$ and $L_2(v,s)$ (formulas $(3.12)_1$, $(3.15)_2$) the expression $vL(v,s)$, easily brings us to the form

$$(4.6)\qquad vL(v,s) = \frac{1}{(\overline{N}+\overline{M})\left(\dfrac{3n}{\alpha_w}+\dfrac{\overline{A}}{2\overline{N}+\overline{M}}\right)}[1-M(v,s)],$$

where

$$M(v,s) = \begin{cases} M_1(v,s) & \text{for permeable edge,} \\ M_2(v,s) & \text{for impermeable edge,} \end{cases}$$

(4.7)
$$M_1(v,s) = sR^2 \frac{\bar{A}\bar{B}\bar{N}}{2\bar{N}+\bar{M}} \left\{ \left[\frac{3n}{\alpha_w}(\bar{N}+\bar{M})+\bar{A} \right](m+v)^2 - \frac{2\bar{A}\bar{N}}{2\bar{N}+\bar{M}} v(m+v) \right\}^{-1},$$

$$M_2(v,s) = sR^2 \frac{\bar{A}\bar{B}\bar{N}}{2\bar{N}+\bar{M}} (m+2v)\times$$
$$\times \left\{ \left[\frac{3n}{\alpha_w}(\bar{N}+\bar{M})+\bar{A} \right] m(m+v)^2 - \frac{2\bar{A}\bar{N}}{2\bar{N}+\bar{M}} v^2(m+v) \right\}^{-1}.$$

Substituting the result (4.6) into (4.5) we obtain

$$\int_0^1 \varphi(\xi,s) \int_0^\infty [1-M(v,s)]\cos v\xi J_0(uv)\,dv\,d\xi = (\bar{N}+\bar{M})\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{2\bar{N}+\bar{M}}\right) f(s).$$

On the basis of series of simple transformations after including (4.4) will be

$$\int_0^u \frac{\varphi(\xi,s)}{\sqrt{u^2-\xi^2}}\,d\xi - \int_0^1 \varphi(\xi,s) \int_0^\infty M(v,s)\cos v\xi J_0(uv)\,dv\,d\xi = g(s). \tag{4.8}$$

Here

$$g(s) = (\bar{N}+\bar{M})\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{2\bar{N}+\bar{M}}\right) f(s). \tag{4.9}$$

Taking the substitution $\xi = u\sin\theta$ and the integral representation for the Bessel function

$$J_0(uv) = \frac{2}{\pi}\int_0^{\pi/2} \cos(uv\sin\theta)\,d\theta,$$

we obtain after transformation

$$\int_0^{\pi/2} \left[\varphi(u\sin\theta,s) - \frac{2}{\pi}\int_0^1 \varphi(\xi,s) \int_0^\infty M(v,s)\cos v\xi \cos(uv\sin\theta)\,dv\,d\xi \right] d\theta = g(s).$$

Using the elementary representation for the product of trigonometric functions and the notation

$$\mathfrak{M}(\xi \pm u\sin\theta, s) = \int_0^\infty M(v,s)\cos v(\xi \pm u\sin\theta)\,dv, \tag{4.10}$$

we come to the expression

$$\int_0^{\pi/2} \left\{ \varphi(u\sin\theta,s) - \frac{1}{\pi}\int_0^1 \varphi(\xi,s)\,[\mathfrak{M}(\xi+u\sin\theta,s)+\mathfrak{M}(\xi-u\sin\theta,s)]\,d\xi \right\} d\theta = g(s).$$

Denoting shortly

$$(4.11)\quad F(u\sin\theta, s) = \varphi(u\sin\theta, s) - \frac{1}{\pi}\int_0^1 \varphi(\xi, s)[\mathfrak{M}(\xi+u\sin\theta, s)+\mathfrak{W}(\xi-u\sin\theta, s)]d\xi,$$

we obtain the simple case of SCHLOMILCH integral equation [15]

$$(4.12)\quad \int_0^{\pi/2} F(u\sin\theta, s)d\theta = g(s);$$

the solution has the known form

$$(4.13)\quad F(u, s) = \frac{2}{\pi}g(s).$$

Returning to (4.11) we obtain for the unknown function $\varphi(u, s)$ the equation

$$\varphi(u, s) - \frac{1}{\pi}\int_0^1 \varphi(\xi, s)K(u, \xi, s)d\xi = F(u, s),$$

in which

$$K(u, \xi, s) = \mathfrak{M}(\xi+u, s)+\mathfrak{W}(\xi-u, s).$$

This is a Fredholm integral equation of the second kind with continuous and bounded kernel, which follows from (4.7) and (4.10). Using the symmetry of kernel $K(u, \xi, s)$ and evenness of function $\varphi(u, s)$ (with respect to u), we finally obtain

$$(4.14)\quad \varphi(u, s) - \frac{1}{\pi}\int_{-1}^{1} \mathfrak{W}(\xi-u, s)\varphi(\xi, s)d\xi = F(u, s)^{u\in[-1,1]} \quad s\in C.$$

Finding from here $\varphi(\xi, s)$, we calculate $\Phi(v, s)$ from (4.3), and in this way the solution of foregoing problem. On the base of the property of kernel $\mathfrak{M}(\xi-u, s)$, the function $\varphi(u, s)$ is continuous and bounded, but its effective calculation encounters difficulties, because the form of the kernel is very complicated. The additional complication is here the fact that the equation must be satisfied for every complex parameter s. This gives an important difference between the punch problem in consolidating condition and the classical contact problems of theory of elasticity.

In this situation we will find the solution of (4.14) in a series form

$$(4.15)\quad \varphi(u, s) = \sum_{n=0}^{\infty} a_n(s)L_n(u),$$

where $\{L_n(u)\}$ is a complete, orthogonal system of functions defined on the interval $[-1, 1]$. As a basis of so defined series we take the system of Legendre's polynomials. Substituting (4.15) into (4.14) and performing the orthogonalization procedure we obtain for the coefficients $a_n(s)$ the following system of linear algebraic equations

$$(4.16)\quad \sum_{j=0}^{\infty} A_{ij}(s)a_j(s) = B_i(s), \quad i = 0, 1, \ldots,$$

where

$$A_{ij}(s) = -\int_{-1}^{1}\int_{-1}^{1} \mathfrak{M}(\xi-u, s)L_i(\xi)L_j(u)d\xi du, \quad i \neq j,$$

$$A_{kk}(s) = \int_{-1}^{1} L_k^2(u)du - \int_{-1}^{1}\int_{-1}^{1} \mathfrak{M}(\xi-u, s)L_k(\xi)L_k(u)d\xi du,$$

$$B_i(s) = \int_{-1}^{1} F(u, s)L_i(u)du.$$

In our flat punch cases, the function $F(u, s)$ is constant with respect to argument u which follows from the relation (4.13)

$$F(u, s) = F(s), \quad B_i(s) = F(s)\int_{-1}^{1} L_i(u)du.$$

From the orthogonality of Legendre's polynomials and by the fact that $L_0(u) = \text{const}$ it follows

$$\int_{-1}^{1} L_i(u)du = 0, \quad i = 1, 2, \ldots.$$

Hence

$$\begin{aligned} B_i(s) &= 0, \quad i = 1, 2, \ldots. \\ B_0(s) &= 2F(s). \end{aligned} \tag{4.17}$$

On the basis of (4.16) we obtain

$$a_n(s) = \frac{D_{0n}(s)}{W(s)} 2F(s), \tag{4.18}$$

where

$$W(s) = \det[A_{ij}(s)], \quad D_{0n}(s) - \text{cofactor } A_{0n}(s).$$

The desired solution of the Eq. (4.14) has the form

$$\varphi(u, s) = 2F(s)\sum_{n=0}^{\infty} \frac{D_{0n}(s)}{W(s)} L_n(u). \tag{4.19}$$

5. Determination of the contact stresses and displacement of the punch

The Laplace transform of the contact stress we obtain immediately from the form $(4.1)_2$ for $u < 1$

$$\bar{p}(u, s) = \underset{u<1}{\bar{\sigma}_z}(u, s) = \frac{1}{R^2}\int_0^{\infty} \Phi(v, s)J_0(uv)dv$$

$$= \frac{1}{R^2}\int_0^{\infty} v\left[\int_0^1 \varphi(\xi, s)\cos v\xi d\xi\right]J_0(uv)dv$$

$$= \frac{1}{R^2}\varphi(1, s)\int_0^{\infty} \sin vJ_0(uv)dv - \frac{1}{R^2}\int_0^1 \varphi'_\xi(\xi, s)\int_0^{\infty} \sin v\xi J_0(uv)dv\, d\xi.$$

Including furthermore (4.4), it will be

$$\bar{p}(u,s) = \frac{\varphi(1,s)}{R^2\sqrt{1-u^2}} - \frac{1}{R^2}\int_u^1 \frac{\varphi'_\xi(\xi,s)}{\sqrt{\xi^2-u^2}}d\xi. \tag{5.1}$$

Taking into account the known form (4.19) we obtain

$$\bar{p}(u,s) = \frac{2F(s)}{R^2}\sum_{n=0}^{\infty}\frac{D_{0n}(s)}{W(s)}\left[\frac{L_n(1)}{\sqrt{1-u^2}} - \int_u^1 \frac{L'_n(\xi)}{\sqrt{\xi^2-u^2}}d\xi\right]. \tag{5.2}$$

From the relations (4.13), (4.9) and $(3.11)_3$ we have

$$F(s) = \frac{4R}{\pi}\frac{\bar{N}+\bar{M}}{2\bar{N}+\bar{M}}\bar{N}\bar{c}(s).$$

Substituting this value into the formula (5.2) we finally obtain

$$\bar{p}(u,s) = \frac{8}{\pi R}\frac{\bar{N}+\bar{M}}{2\bar{N}+\bar{M}}\bar{N}\bar{c}(s)\sum_{n=0}^{\infty}\frac{D_{0n}(s)}{W(s)}\left[\frac{L_n(1)}{\sqrt{1-u^2}} - \int_u^1 \frac{L'_n(\xi)}{\sqrt{\xi^2-u^2}}d\xi\right]. \tag{5.3}$$

When the punch is pressed into the medium with the known force $P(t)$ (and it is so in our case), then the self-evident relation holds

$$P(t) = 2\pi\int_0^R p(u,t)\,du,$$

which after Laplace transformation and transition to dimensionless variables assumes the form

$$\bar{P}(s) = 2\pi R\int_0^1 \bar{p}(u,s)\,du. \tag{5.4}$$

This relation after including (5.3) gives the formula for the transform of displacement

$$\bar{c}(s) = \frac{\pi}{8R\bar{N}}\frac{2\bar{N}+\bar{M}}{\bar{N}+\bar{M}}\frac{\bar{P}(s)}{\sum\limits_{n=0}^{\infty}\frac{D_{0n}(s)}{W(s)}H_n}. \tag{5.5}$$

Here

$$H_n = \int_0^1\left[\frac{L_n(1)}{\sqrt{1-u^2}} - \int_u^1 \frac{L'_n(\xi)}{\sqrt{\xi^2-u^2}}d\xi\right]du. \tag{5.6}$$

Substituting (5.5) into (5.3), it will be

$$\bar{p}(u,s) = \frac{\bar{P}(s)}{R\sum\limits_{n=0}^{\infty}(-1)^n H_n D_{0n}(s)}\sum_{n=0}^{\infty}D_{0n}(s)\left[\frac{L_n(1)}{\sqrt{1-u^2}} - \int_u^1 \frac{L'_n(\xi)}{\sqrt{\xi^2-u^2}}d\xi\right]. \tag{5.7}$$

Finally, after performing the Laplace inverse transformation, we obtain the final closed form for contact stresses and punch displacement

$$(5.8)\quad p(u,t) = \frac{1}{2\pi i}\int\limits_L \frac{\overline{P}(s)}{R\sum\limits_{n=0}^{\infty}(-1)^n H_n D_{0n}(s)} \sum_{n=0}^{\infty} D_{0n}(s)\times$$

$$\times\left[\frac{L_n(1)}{\sqrt{1-u^2}} - \int\limits_u^1 \frac{L_n'(\xi)}{\sqrt{1-u^2}}d\xi\right]e^{st}ds,$$

$$c(t) = \frac{1}{2\pi i}\int\limits_L \frac{\pi}{8R\overline{N}}\,\frac{2\overline{N}+\overline{M}}{\overline{N}+\overline{M}}\,\frac{\overline{P}(s)}{\sum\limits_{n=0}^{\infty}\frac{D_{0n}(s)}{W(s)}H_n}e^{st}ds.$$

These formulas constitute the complete exact solution of the posed problem.

6. Numerical example

The obtained formulas give possibility to do the qualitative analysis of the problem. As follows from $(5.8)_1$, the character of singularity of contact stresses under the punch in consolidating medium is the same as in one-phase elastic solid. The quantitative information as well as the additional consolidation process need an exact analysis of functions (5.8) for given parameters of the medium and given functions $P(t)$. The formulas (5.8) have a complicated structure and demand series of labour-consuming operations [calculation of the integrals (5.6) and (4.10), computation of coefficients of the system (4.16), calculation of Laplace inverse transforms], but all these calculations can be done effectively. For example, the results of calculation for a medium with elastic skeleton and loading

$$(6.1)\qquad P(t) = \begin{cases} P_0\dfrac{t}{t_0}, & 0 \leqslant t < t_0,\\ P_0, & t \geqslant t_0,\end{cases}$$

are given in the sequel. In this solution, the system of the first four Legendre's polynomials

$$L_0(u) = 1,\quad L_1(u) = u,\quad L_2(u) = \frac{1}{2}(3u^2-1),\quad L_3(u) = \frac{1}{2}(5u^3-3u)$$

is used.

The coefficients of the system (4.16) with the integrals (4.10) are computed in analytical form

$$A_{ij}(s) = -\int\limits_0^1\int\limits_0^1 \mathfrak{M}(\xi-u,s)L_i(\xi)L_j(u)d\xi\,du = -\int\limits_0^1\int\limits_0^1\int\limits_0^{\infty} M(v,s)\cos v(\xi-u)\times$$

$$\times L_i(\xi)L_j(u)dv\,d\xi\,du = -\int\limits_0^{\infty} M(v,s)\Big(\int\limits_0^1\int\limits_0^1 \cos v(\xi-u)L_i(\xi)L_j(u)d\xi\,du\Big)\,dv$$

$$= -\int\limits_0^{\infty} M(v,s)\,\alpha_{ij}(v)dv,$$

where

$$\alpha_{ij} = \int_0^1 \int_0^1 \cos v(\xi - u) L_i(\xi) L_j(u) d\xi du.$$

By means of a structure

$$\int_0^\infty M(v, s)\alpha_{ij}(v)dv = \lim_{\eta\to 0} \int_0^\infty M(v, s)\alpha_{ij}(v)\cos\eta v dv$$

the interpolation approximation method of KRYLOV [5] to calculate the Fourier integrals may be applied.

Analogically to the Laplace inverse-transforms calculation, the effective method of interpolation [6] was used. To this aim, the formulas (5.8) are written in the form

$$\begin{aligned} p(u, t) &= \frac{1}{2\pi i} \int_L T(u, s)\overline{P}(s)e^{st}ds, \\ c(t) &= \frac{1}{2\pi i} \int_L S(s)\overline{P}(s)e^{st}ds, \end{aligned} \tag{6.2}$$

where

$$T(u, s) = \frac{1}{\sum_{n=0}^{N} (-1)^n D_{0n}(s) H_n R} \sum_{n=0}^{N} D_{0n}(s) \left[\frac{L_n(1)}{\sqrt{1-u^2}} - \int_u^1 \frac{L_n'(\xi)}{\sqrt{\xi^2 - u^2}} d\xi\right],$$

$$S(s) = \frac{\pi}{8R\mu} \frac{2\mu+\lambda}{\mu+\lambda} \frac{W(s)}{\sum_{n=0}^{N} H_n D_{0n}(s)},$$

$$\overline{P}(s) = \int_0^\infty P(t)e^{-st}dt = \frac{1-e^{-t_0 s}}{t_0 s^2}.$$

From the structure of expressions $T(u, s)$ and $S(s)$ it follows that they are bounded for $|s| \to \infty$. This is the necessary condition to apply the Krylov's method.

The complete procedure of calculation of nonelementary integrals (Fourier and Laplace inverse transforms) consists in the computation of values of functions in interpolation nodes — because the residual operations are reduced to simple and closed formulas given in [5, 6].

Calculations were performed for the following physical constants of the medium

$$\mu = 125\,\text{kG/cm}^2, \quad \alpha_v = 500\,\text{kG/cm}^2, \quad \alpha_p = 25\,\text{kG/cm}^2,$$
$$k = 2\cdot 10^{-1}\,\text{cm/day}, \quad n = 0.4, \quad \alpha_w = 6\cdot 10^4\,\text{kG/cm}^2.$$

These values were found in the paper [13], and they are approximate quantities only, obtained in the way of selected laboratory tests for clay.

Moreover, it was assumed that $R = 500\,\text{cm}$, $P_0 = 667\,\text{T}$.

The calculation was performed by means of computer making one common programme for permeable and impermeable edge. The results of these calculations are presented in Figs. 2, 3, 4, 5, 6, 7, 8.

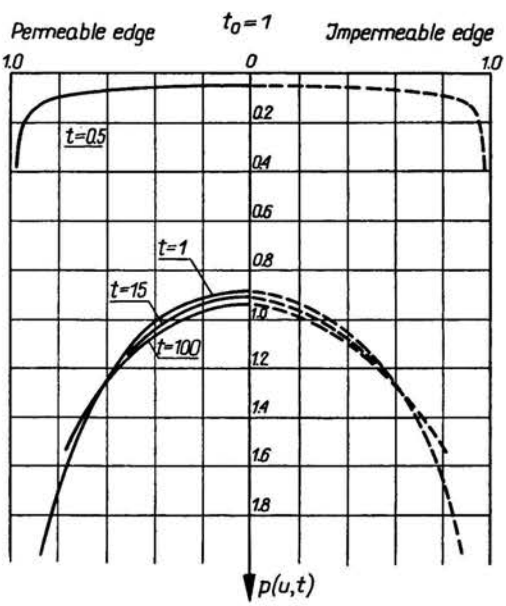


FIG. 2.

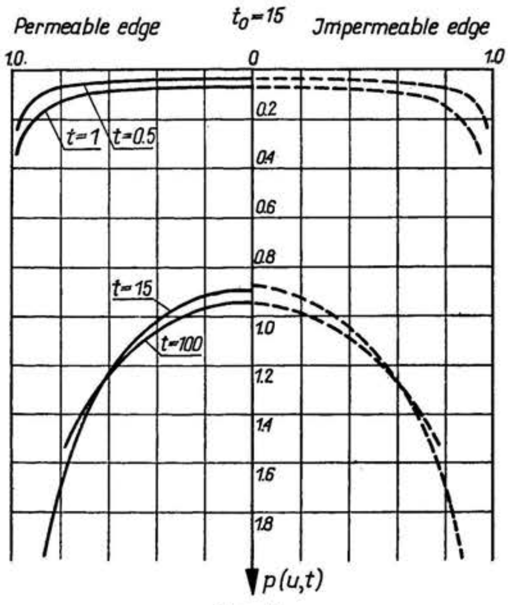


FIG. 3.

In the Figs. 2, 3, 4, 5 the contact stress distribution for a given load program is given. Figures give the analogical distribution but for a different time point t_0. Figure 7 demonstrates the displacement of the punch in time, Fig. 8 shows the same phenomenon but for different values of t_0 and in logarythmical scale.

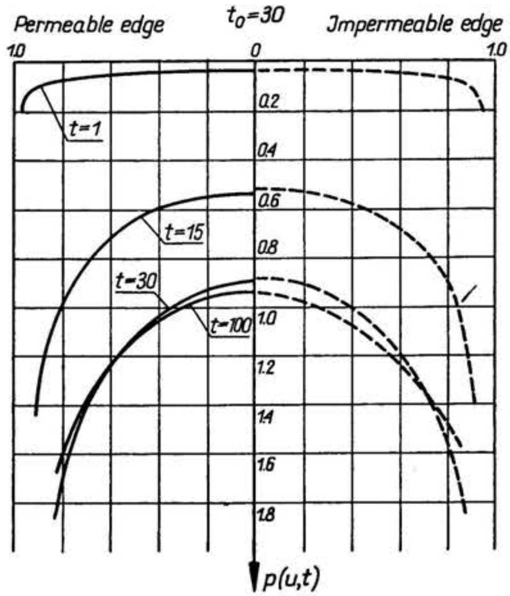


FIG. 4.

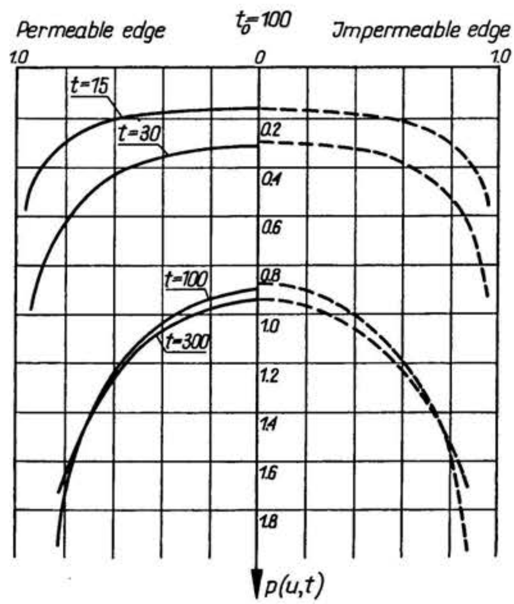
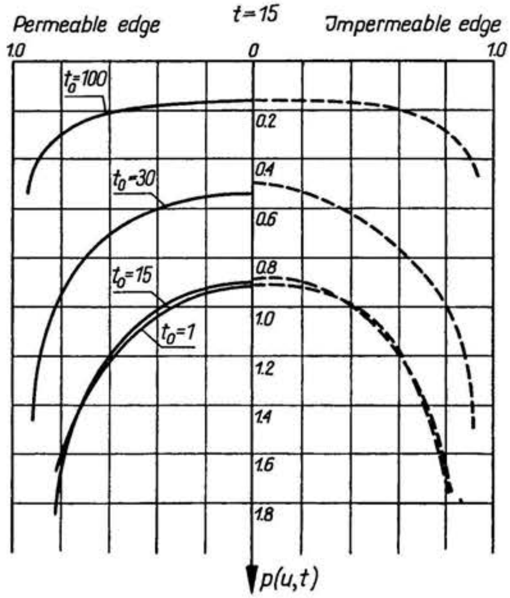


FIG. 5.

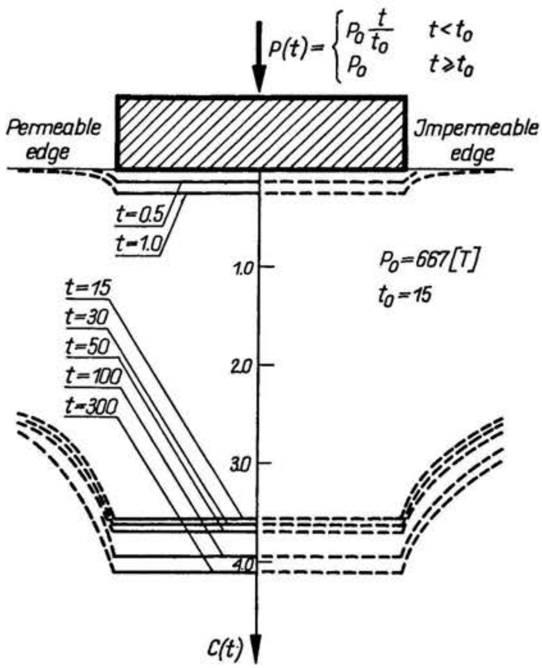
7. Conclusions

The presented considerations lead to formulation of the following remarks. The solution of the problem of a punch, which is pressed into consolidating medium, was presented with success in a closed form in a class of nonelementary integrals. As it was



Permeable edge
t=15
Jmpermeable edge
1.0
0
1.0
t_0=100
t_0=30
t_0=15
t_0=1
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
p(u,t)

FIG. 6.



$P(t)=\begin{cases} P_0\frac{t}{t_0} & t<t_0 \\ P_0 & t\geqslant t_0 \end{cases}$
Permeable edge
Jmpermeable edge
t=0.5
t=1.0
1.0
P_0=667[T]
t_0=15
t=15
t=30
t=50
t=100
t=300
2.0
3.0
4.0
C(t)

FIG. 7.

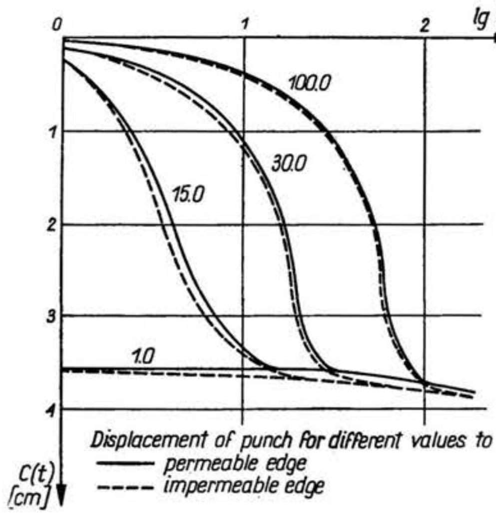


FIG. 8.

proved, the solution is a function with the same class of regularity as the elastic solution (for one-phase medium). In the final state of consolidating process, that is after the outflow of liquid from pores (i.e. for $t \to \infty$), the solution has the same properties as for the one-phase case. In course of consolidation we observe the levelling of stresses in the middle zone under the punch. The solution for permeable edge is not far from that for the impermeable one. Though the numerical calculation for medium with elastic skeleton was performed, the method can effectively be realized also for viscoelastic skeleton. It is worth to make a note that formulas (5.11), which solved the problem, hold for the most general, linear viscoelastic material without aging. These include the rheological properties of the skeleton, different in the laws of shear, dilatation and due to the pressure of fluid strains. The simplest rheological models, in particular all differential models (Kelvin-Voigt's, Maxwell's, Zerner's) belong here automatically, owing to the required form of kernels $K(t-\tau)$, $K_v(t-\tau)$ and $K_p(t-\tau)$. The problem was solved by the programme permitting force increase from zero to the limit value. An example for only the linear increase was considered. The homogeneous initial conditions were discussed consistently. In this way the formulated problem holds for the equations of Biot's consolidation theory — the well-posed initial-boundary problem. The question of immediate loading excluded the homogeneous initial conditions. Then from the non-normality of the system of equations it follows that the initial conditions cannot be arbitrary but must satisfy the so-called "consistence conditions". This question is not discussed in the present paper, taking the opinion that the load increasing from zero better corresponds to the real situation.

Different values of parameter t_0 considered in the numerical example have the aim to find the influence of velocity of the load. As it is shown, this influence is essential.

From the practical point of view it is important to know not only the contact stress distribution but also, in a great part, the settlement of the punch in time. This problem, in our opinion, has also found a sufficient attention.

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