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On a mathematical theory of elastic-plastic materials

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THE FIRST part of this paper deals with the constitutive equation of the elastic-plastic material, regarded as a particular "rate-type" material. The results of the classical theories of plasticity are re-interpreted in terms of abstract properties of a "plasticity operator" which naturally arises from the analysis of the constitutive equation. The second part of the paper considers the incremental equilibrium problem. Existence and uniqueness of the solution are discussed, as well as the classical direct and complementary variational principles.

Pierwsza część pracy dotyczy równania konstytutywnego dla materiału sprężysto-plastycznego, rozważanego jako szczególny przypadek materiału typu prędkościowego. Rezultaty klasycznych teorii plastyczności są przetłumaczone na język abstrakcyjnych własności "operatora plastyczności", który wynika w sposób naturalny z równania konstytutywnego. W drugiej części pracy rozważa się problem równowagi przyrostowej. Przedyskutowano zarówno istnienie i jednoznaczność rozwiązania, jak również bezpośrednie i uzupełniające zasady wariacyjne.

Первая часть работы касается определяющего уравнения для упруго-пластического материала, рассматриваемого как частный случай материала скоростного типа. Результаты классических теорий пластичности переведены на язык абстрактных свойств "оператора пластичности", который естественным образом следует из определяющего уравнения. Во второй части работы рассмотрена проблема равновесия в приростах. Обсуждены так существование и единственность решения, как и непосредственные и дополнительные вариационные принципы.

1. Introduction

IN RECENT years, considerable effort has been devoted to fitting the plasticity theory into the general framework of Continuum Mechanics. In particular, its relation to TRUESDELL's theory of hypo-elastic materials has been pursued by several Authors, as, for example, GREEN [1], PERZYNA [2], TOKUOKA [3]. Of special interest is a paper by PIPKIN and RIVLIN [4] where a theory of rate-independent materials is developed which includes classical plasticity as a special case. This approach is taken as the starting point of the present paper. The first part analyses some constitutive assumptions and extends PRAGER's kinematical hardening model to three dimensions, under the assumption of infinitesimal deformations. A constitutive equation is obtained in which the response functional depends only on the actual values of the stress, strain and strain-rate tensors, and allows the elastic-plastic material to be regarded as a particular "rate-type" material of the first order, in accordance with the definition given in [2].

The form of the constitutive equation also suggests the introduction of a non-linear operator, denoted as the "plasticity operator". We shall prove that all the classical as-

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sumptions and results can be given a more compact form in terms of this operator and we shall also establish some new results. For instance, the hardening rule, usually assumed completely independent of the constitutive equation, is here shown to be implicitly contained in it. Likewise, a rather natural classification of elastic-plastic materials is possible, in which the work-hardening and the perfectly plastic materials form two very special sub-classes of materials.

The incremental equilibrium problem is next discussed, in strict analogy with the well known analysis due to KOITER [5]. However, the mathematical tools introduced with the definition of the plasticity operator allow us not only to prove the uniqueness theorem and the variational principles as in KOITER's treatment, but also to establish an existence theorem, even if this is restricted to the class of work-hardening materials obeying the classical assumption of normality.

2. Constitutive equations

This Section is devoted to the deduction of the simplest constitutive equation for a plastic material. For this purpose, both geometrical and constitutive linearity is assumed, but, of course, the latter assumption does not hold for the transition from an elastic to a plastic response, which is intrinsically non-linear. We also assume that time effects are unimportant, so that acceleration terms are disregarded in the equations of motion, and time does not appear explicitly in the constitutive equation. The latter hypothesis is not only consistent with the current definitions of plasticity as "a branch of rheology in which..... time effects play a minor role" (DRUCKER [6]), but also conforms with the theory of rate-independent materials developed in [4].

The theory presented here is based on the observation that all elastic-plastic constitutive equations originate from the extrapolation of the familiar one-dimensional diagram given in Fig. 1. It is well known that the simplest representation of this type of material response is the incremental one, which, on denoting by $\dot{\sigma}\,dt$, $\dot{\varepsilon}\,dt$ the stress and strain increments occurring in the time interval $(t, t+dt)$, results in the following equations:

$$\begin{cases} \dot{\sigma} = A\dot{\varepsilon} & \text{for} \quad |\sigma - B\varepsilon| < (A-B)\varepsilon_0, \\ & \text{and for} \quad |\sigma - B\varepsilon| = (A-B)\varepsilon_0, \quad \dot{\varepsilon}(\sigma - B\varepsilon) < 0, \\ \dot{\sigma} = B\dot{\varepsilon} & \text{for} \quad |\sigma - B\varepsilon| = (A-B)\varepsilon_0, \quad \dot{\varepsilon}(\sigma - B\varepsilon) \geqslant 0. \end{cases}$$

The two possible responses are called the elastic and the plastic response. A, B are the elastic and the work-hardening moduli respectively, and ε_0 is the strain corresponding to first yielding.

This formulation can be given a more compact form by introducing the strains ε', ε'' indicated in Fig. 1 and defined by the equations:

$$\varepsilon = \varepsilon' + \varepsilon'', \quad \sigma = A\varepsilon' + B\varepsilon''. \tag{2.1}$$

Then, we may alternatively characterize the elastic response by the conditions:

$$|\varepsilon'| < \varepsilon_0, \quad \text{or} \quad |\varepsilon'| = \varepsilon_0, \quad \dot{\varepsilon}\varepsilon' < 0,$$

and the plastic one by

$$|\varepsilon'| = \varepsilon_0, \quad \dot{\varepsilon}\varepsilon' \geqslant 0.$$

We observe that we have obtained a constitutive equation in which the response functional depends only on the present values of the strain ε' and of the strain rate $\dot{\varepsilon}$:

$$\dot{\sigma} = \dot{\sigma}(\varepsilon', \dot{\varepsilon}). \tag{2.2}$$

Particularly useful is the geometrical interpretation of ε', ε''. To any point P of the (σ, ε) plane, let us associate the segment of the ε-axis

$$\varepsilon'' \pm \varepsilon_0,$$

which will be denoted as the elastic range. The constitutive equation written above ensures that an elastic response occurs as long as the projection of P in the ε-axis falls

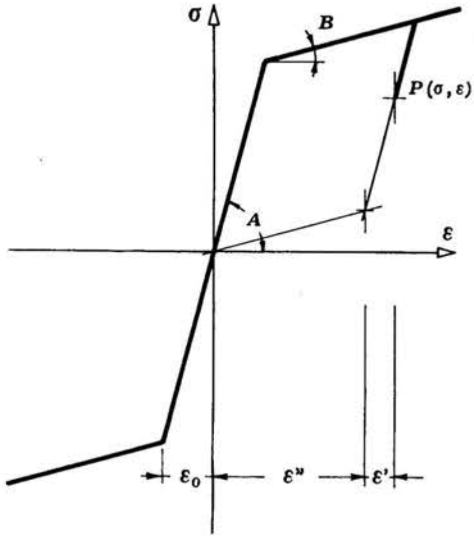

FIG. 1. One-dimensional stress-strain diagram for an elastic-plastic material.

within the elastic range. When it moves outwards, it is "followed" by the elastic range. This is the well known PRAGER kinematical hardening model, in which the strains ε', ε'' represent the position of the projection of P in the elastic range and the translation of the latter with respect to the origin.

These concepts can be extended to the three-dimensional case in a rather spontaneous way. It is worth noting that an incremental form for the constitutive equations is adopted in the flow theories of plasticity, with, however, the additional assumption of dependence of the response functional on the strain history (NAGHDI [7] p. 143, KACHANOV [8] p. 44, 77, 83, BLAND [9], PIPKIN and RIVLIN [4]). Only HILL ([10], p. 24) does not make this assumption.

We begin our discussion by introducing some concepts and notation. Henceforth, we let σ, ε denote the CAUCHY stress tensor and the infinitesimal strain tensor, respectively. They are to be regarded as the elements of two six-dimensional vector spaces, the stress space $\mathscr{V}_\sigma$ and the strain space $\mathscr{V}_\varepsilon$. By the usual identification of equidimensional finite vector spaces, we can define the scalar product

$$a \cdot b = \sum_{i,j=1}^{3} a_{ij} b_{ij}, \quad \text{with} \quad a \in \mathscr{V}_\sigma,\ b \in \mathscr{V}_\varepsilon.$$

The sum decompositions (2.1) for σ, ε are still assumed to hold, but now A, B denote fourth-order tensors, called the elastic and the work-hardening tensors respectively, which are regarded as linear mappings of $\mathscr{V}_\sigma$ onto $\mathscr{V}_\varepsilon$. We assume that A, B are symmetric in the sense that

$$Aa \cdot b = Ab \cdot a \quad \text{for any} \quad a, b \in \mathscr{V}_\varepsilon.$$

A restriction is imposed on A, B by the requirement that the Eqs. (2.1) define a one-to-one correspondence between the pairs (σ, ε) and $(\varepsilon', \varepsilon'')$, and implies that the tensor $(A-B)$ be non-singular. This can be seen from the inversion of system (2.1), which gives:

$$\varepsilon' = (A-B)^{-1}(\sigma - B\varepsilon),$$
$$\varepsilon'' = (A-B)^{-1}(A\varepsilon - \sigma).$$

The *elastic range* is defined as the closed domain of $\mathscr{V}_\varepsilon$:

$$f(\varepsilon') \leqslant 0, \tag{2.3}$$

whose boundary is called the *yield surface*. The vector ε'' represents the translation of the elastic range from its initial position corresponding to the natural state. The second-order tensor

$$n(\varepsilon') = \left[\frac{df(\varepsilon')}{d\varepsilon'}\right]_{f(\varepsilon')=0} \tag{2.4}$$

represents the *normal* to the yield surface in the space $\mathscr{V}_\varepsilon$. It is denoted as the *exterior normal* because, due to the Ineq. (2.3), it points outwards from the elastic range. Note also that, in general, it is not a unit vector. We assume that the yield surface is sufficiently smooth to ensure uniqueness of the normal at any one of its points.

We come now to the problem of characterizing the elastic and the plastic response. We suppose that the pair (σ, ε), and hence $(\varepsilon', \varepsilon'')$, is given. Then, we have to express the stress-rate σ as a function of $\dot{\varepsilon}$. But equations:

$$\dot{\varepsilon} = \dot{\varepsilon}' + \dot{\varepsilon}'', \quad \dot{\sigma} = A\dot{\varepsilon}' + B\dot{\varepsilon}'', \tag{2.5}$$

deduced formally by time differentiation of the Eqs. (2.1), show that actually our problem is equivalent to specifying the sum decomposition $(2.5)_1$ of the strain-rate vector. Elastic and plastic responses consist in two different specifications of this decomposition.

We characterize the elastic response by $\dot{\varepsilon}'' = 0$ or, equivalently, by the fact that the elastic range remains fixed in the space $\mathscr{V}_\varepsilon$. This is assumed to occur, firstly, when the point ε' is interior to the elastic range and, secondly, when it is on the boundary, but the vector $\dot{\varepsilon}$ points inwards, i.e. both when $f(\varepsilon') < 0$ and $f(\varepsilon') = 0$, $\dot{\varepsilon} \cdot n < 0$. The remaining possibility is $f(\varepsilon') = 0$, $\dot{\varepsilon} \cdot n \geqslant 0$, and corresponds to a response of the plastic type. We assume that it is characterized by the two conditions

$$\dot{\varepsilon}' n = 0, \quad \dot{\varepsilon}'' = \lambda m. \tag{2.6}$$

The first of these conditions states that the point ε' remains on the yield surface during plastic response, while the second means that the direction of $\dot{\varepsilon}''$ is fixed at any point of the yield surface. Clearly, the vector field $m = m(\varepsilon')$, defined on the yield surface, determines this direction (Fig. 2). The scalar λ can be obtained by substituting the Eq. $(2.6)_2$ into the Eq. $(2.5)_1$ and then multiplying scalarly by n:

$$\lambda = \frac{\dot{\varepsilon}n}{mn}. \tag{2.7}$$

If we direct m outwards from the elastic range, so that

$$mn > 0, \tag{2.8}$$

and recall that during plastic response $\dot{\varepsilon}n \geqslant 0$, we have that λ is always a non-negative number.

The set of constitutive assumptions made above can be collected into a single equation, which will be taken as the constitutive equation of the elastic-plastic material:

$$\dot{\sigma} = A\dot{\varepsilon}' + B\dot{\varepsilon}'', \quad \dot{\varepsilon}'' = \begin{cases} 0 & \text{for } f(\varepsilon') < 0, \\ & \text{and for } f(\varepsilon') = 0,\ \dot{\varepsilon}n < 0, \\ \dfrac{\dot{\varepsilon}n}{mn} m & \text{for } f(\varepsilon') = 0,\ \dot{\varepsilon}n \geqslant 0, \end{cases} \tag{2.9}$$
$$\dot{\varepsilon}' = \dot{\varepsilon} - \dot{\varepsilon}''.$$

It is immediately seen that, since m, n are known functions of ε', this is an equation of the form (2.2) and therefore represents the desired generalization of the one-dimensional case examined before. We observe that $\dot{\varepsilon}n = 0$ implies $\dot{\varepsilon}'' = 0$, and so the corresponding response can be regarded as elastic as well as plastic. This situation is usually designated as *neutral loading*, and represents a continuous transition between the elastic and plastic responses.

It is worth noting that the elastic-plastic material defined by the Eq. (2.9) can be regarded as a particular rate-type material, in accordance with TRUESDELL and NOLL's classification (see [11], Sec. 36).

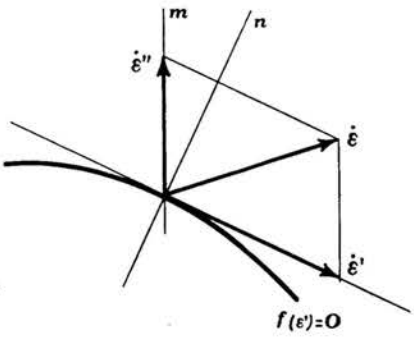

FIG. 2. Decomposition of the strain-rate vector at plastic response.

The same approach may successfully be applied to the cases where constitutive linearization is absent. Figure 3 shows a one-dimensional scheme exhibiting non-linear hardening. Such schemes are often used to account for change in shape of the elastic range during loading (see e.g. BALTOV and SAWCZUK [12]). Although the one-dimensional constitutive equation can obviously be given the form (2.2), we shall not deal here with the problem of its generalization to three dimensions.

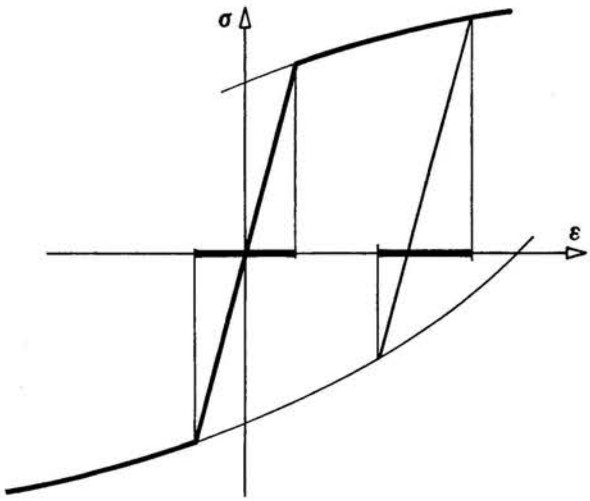

FIG. 3. Non-linear hardening and change in shape of the elastic range.

3. Comparison with the classical theories

The constitutive equations presented above agree with those of PIPKIN and RIVLIN in taking $\dot{\varepsilon}$ as an independent variable and $\dot{\sigma}$ as the dependent one. This contrasts with the choice customarily made in the classical theories. As outlined in the preceding Section, the new choice is motivated by its accordance with the more general theory of rate-type materials. As a consequence, the yield condition has been expressed in terms of strain rather than of stress.

Another difference between the classical theories and this approach is the decomposition $(2.1)_1$ of the strain tensor, which replaces the usual decomposition into an "elastic" and a "plastic" part, given by

$$\varepsilon = \varepsilon^e + \varepsilon^p,$$

where

$$\varepsilon_e \equiv A^{-1}\sigma.$$

Comparison with Eqs. (2.1) shows that the "plastic strain" ε^p is related to the strain ε'' by

$$\varepsilon^p_A = A^{-1}(A-B)\varepsilon''. \tag{3.1}$$

The two decompositions are equivalent, in the sense that they can be derived from each other, provided that A, $(A-B)$ are non-singular.

The reasons for preferring the new choice will become apparent after a discussion of the two basic assumptions in the classical theories, which are known, respectively, as the *flow* and the *hardening* rule.

The flow rule specifies the direction of the plastic strain-rate $\dot{\varepsilon}^p$ during the plastic response, and is equivalent to our assumption $(2.6)_2$, since $\dot{\varepsilon}^p$ is determined by Eq. (3.1). The most common flow rule is the *associated flow rule*. This is also known as the assumption of *normality*(2). To discuss this rule we must formulate the yield condition in terms of the stress. Hence, let us define in the stress space $\mathscr{V}_\sigma$ the vector

$$\sigma' = A\varepsilon'$$

and the surface

$$g(\sigma') \equiv f(\varepsilon') = 0 \tag{3.2}$$

representing the yield condition in terms of stress. The exterior normal is then

$$n_\sigma = \frac{dg(\sigma')}{d\sigma'} = \frac{df(\varepsilon')}{d(A\varepsilon')} = A^{-1}\frac{df(\varepsilon')}{d\varepsilon'}.$$

By introducing the normal n in $\mathscr{V}_\varepsilon$ defined by the Eq. (2.4) we arrive at the following expression for n_σ:

$$n_\sigma = A^{-1}n. \tag{3.3}$$

Normality means that the vector $\dot{\varepsilon}^p$ is assumed to be parallel to n_σ. With the help of the Eqs. (3.2), (3.3) we may state this assumption in the form

$$(A-B)\dot{\varepsilon}'' = \lambda n \tag{3.4}$$

or, on recalling Eq. $(2.6)_2$, in the form

$$(A-B)m = n. \tag{3.5}$$

In order for the scalar λ appearing in the Eq. (3.4) to be the same as in the Eq. $(2.6)_2$, we must obviously define the modulus of the vector m in an appropriate way.

Besides normality, other flow rules, called non-associated, are sometimes considered. They consist in postulating the existence of a plastic potential, i.e., of a scalar function $h = h(\sigma')$ defined in the elastic range and such that

$$\dot{\varepsilon}^p = \frac{dh(\sigma')}{d\sigma'}. \tag{3.6}$$

From our viewpoint, this assumption turns out to require a certain regularity of the vector field $m(\varepsilon')$, which itself is closely connected to $\dot{\varepsilon}^p$ by the Eqs. $(2.6)_2$, (3.1). Upon regarding $h(\sigma') = 0$ as the equation of a surface in the stress space, the geometrical interpretation of the function $h(\sigma')$ becomes clear. It is also easily verified that for $h \equiv g$ the associated flow rule is obtained.

Let us now examine the hardening rule. This law specifies the translation of the yield surface during the plastic response. In the classical theories, this law has no connection with the constitutive equation, being a completely independent assumption. In this approach, however, the translations of the yield surface in the strain and stress space, being $\dot{\varepsilon}''$, $B\dot{\varepsilon}''$ respectively, are specified by the constitutive equation (2.9) itself, so that no further assumption is needed. Thus, a knowledge of A, B and of the vector field m accounts both for the flow and the hardening rule and reveals the mathematical nature of these assumptions.

(2) The two terms denote assumptions having a different historical origin. Their equivalence was discussed in a paper by BLAND [9].

A priori connections between A, B and m can be established by particular flow and hardening rules. For instance, the assumption of normality leads to the Eq. (3.5). Another interesting relation is given by the PRAGER hardening rule, which states that the yield surface in stress space moves parallel to $n_\sigma \equiv A^{-1}n$:

$$\mu Bm = A^{-1}n, \tag{3.7}$$

μ being a scalar multiplier, called the *work-hardening factor*. When combined with normality, this assumption leads to

$$(A-B)m = \mu ABm.$$

and, as this holds for any m, we deduce

$$A-B = \mu AB,$$

or, pre-multiplying by A^{-1} and post-multiplying by B^{-1}

$$B^{-1}-A^{-1} = \mu I, \tag{3.8}$$

I being the identity operator. Then, the proper vectors of A, B are parallel and their proper values α_i, β_i are connected by the relation

$$\beta_i = \frac{\alpha_i}{1+\mu\alpha_i}.$$

It is interesting to observe that the same result may be obtained by combining ZIEGLER's hardening rule [13]:

$$Bm = \mu A\varepsilon'$$

with the particular non-associated flow rule

$$(A-B)m = \varepsilon'$$

which may be interpreted as the equation of a sphere in stress space, as it results from the Eqs. (3.6), (3.2). The same manipulations as above lead again to the Eq. (3.8).

4. The plasticity operator

In Section 2 the elastic-plastic material has been defined as a material having the constitutive equation (2.9). This equation will be considered here under the equivalent forms

$$\dot{\sigma} = A\dot{\varepsilon}-(A-B)\dot{\varepsilon}'', \quad \dot{\sigma} = B\dot{\varepsilon}+(A-B)\dot{\varepsilon}', \tag{4.1}$$

obtained simply by substitution of Eq. $(2.5)_1$ into Eq. (2.9). It is assumed that the linear operators A, B are symmetric and non-singular, and that $\dot{\varepsilon}''$ is a function of $\dot{\varepsilon}$ and of the point ε' of the elastic range, as specified by Eq. (2.9) itself.

If we suppose that the point ε' is kept fixed, the above equation can be given the form

$$\dot{\sigma} = H\dot{\varepsilon}, \tag{4.2}$$

where $H = H(\varepsilon')$ can be regarded as a non-linear mapping of $\mathscr{V}_\varepsilon$ onto $\mathscr{V}_\sigma$ defined at any point of the yield surface. In the following, H will be denoted as the *plasticity operator*. This Section is devoted to the study of those of its properties, which will be of later use in the discussion of conditions for the existence and the uniqueness of the solution of the "incremental" equilibrium problem.

We begin by recalling some definitions. An operator H is said to be *positive-definite* (*semi-definite*) if the product $H\dot{\varepsilon}\cdot\dot{\varepsilon}$ is positive (non-negative) for any $\dot{\varepsilon} \neq 0$. It is said to be *monotonic* (*weakly monotonic*) if the product $(H\dot{\varepsilon}-H\dot{\varepsilon}^*)\cdot(\dot{\varepsilon}-\dot{\varepsilon}^*)$ is positive (non-negative) for any pair of distinct vectors $\dot{\varepsilon}$, $\dot{\varepsilon}^*$. The notation $H > K$ ($H \geqslant K$) means that the operator $(H-K)$ is positive definite (semi-definite). It is well known that monotonicity implies positiveness; but for the plasticity operator it will be shown that the converse is also true.

THEOREM 1. (*Weak*) *monotonicity of H implies and is implied by the positive* (*semi-*) *definiteness of H.*

Proof. The first assertion follows trivially from the definition of monotonicity, taking for instance $\dot{\varepsilon}^* = 0$. To prove the converse, we assume that H is positive definite and we take two arbitrary vectors $\dot{\varepsilon}$, $\dot{\varepsilon}^*$. No generality is lost in assuming that $\dot{\varepsilon}\cdot n \geqslant \dot{\varepsilon}^* n$, so that

$$\begin{aligned}(\dot{\varepsilon}-\dot{\varepsilon}^*)''n = (\dot{\varepsilon}-\dot{\varepsilon}^*)\cdot n \geqslant (\dot{\varepsilon}''-\dot{\varepsilon}^{*''})n \geqslant 0,\\ (\dot{\varepsilon}^*-\dot{\varepsilon})''n = 0,\end{aligned} \tag{4.3}$$

as can easily be verified with the aid of the constitutive equation (2.9). The positive definiteness of H together with the Eq. $(4.1)_1$ next shows that for the vectors $(\dot{\varepsilon}-\dot{\varepsilon}^*)$, $(\dot{\varepsilon}^*-\dot{\varepsilon})$ we have

$$\begin{aligned}H(\dot{\varepsilon}-\dot{\varepsilon}^*)(\dot{\varepsilon}-\dot{\varepsilon}^*) = A(\dot{\varepsilon}-\dot{\varepsilon}^*)(\dot{\varepsilon}-\dot{\varepsilon}^*) - \frac{(\dot{\varepsilon}-\dot{\varepsilon}^*)n}{mn}(A-B)m(\dot{\varepsilon}-\dot{\varepsilon}^*) > 0,\\ H(\dot{\varepsilon}^*-\dot{\varepsilon})(\dot{\varepsilon}^*-\dot{\varepsilon}) = A(\dot{\varepsilon}-\dot{\varepsilon}^*)(\dot{\varepsilon}-\dot{\varepsilon}^*) > 0.\end{aligned} \tag{4.4}$$

What we have to prove is that

$$(H\dot{\varepsilon}-H\dot{\varepsilon}^*)(\dot{\varepsilon}-\dot{\varepsilon}^*) = A(\dot{\varepsilon}-\dot{\varepsilon}^*)(\dot{\varepsilon}-\dot{\varepsilon}^*) - \frac{(\dot{\varepsilon}''-\dot{\varepsilon}^{*''})n}{mn}(A-B)m(\dot{\varepsilon}-\dot{\varepsilon}^*) > 0.$$

If the product $(A-B)m(\dot{\varepsilon}-\dot{\varepsilon}^*)$ is negative, this follows immediately from comparison with the Ineqs. $(4.3)_3$, $(4.4)_4$; if it is positive, the same result follows from the Ineqs. $(4.3)_2$, $(4.4)_2$, and the theorem is proved. The modifications which are required when H is assumed to be positive semi-definite are obvious.

From the Ineq. $(4.4)_4$ also follows the result: H positive definite implies A positive definite. The converse in general is not true.

In the following, positiveness of the plasticity operator will be assumed. The product $H\dot{\varepsilon}\dot{\varepsilon}$ will be called the *incremental energy*, and it will be used to give an energy definition of work-hardening and perfectly plastic materials.

At any point of the yield surface we define a *purely plastic response* as the response characterized by a strain-rate vector parallel to m. For such a response, we have from the Eq. $(2.6)_2$ $\dot{\varepsilon}' = 0$, and therefore the incremental energy becomes

$$H\dot{\varepsilon} \cdot \dot{\varepsilon} = B\dot{\varepsilon}'' \cdot \dot{\varepsilon}''. \tag{4.5}$$

By analogy with the one-dimensional case, we shall use the terms *hardening*, *softening* and *perfect plasticity* to characterize the sign of the incremental energy during a purely plastic response.

DEFINITION 1. *An elastic-plastic material is said to harden, to soften or to behave as perfectly plastic at a given point of the yield surface if the incremental energy evaluated at that point for a purely plastic response is positive, negative or zero.*

DEFINITION 2. *A work-hardening material is an elastic-plastic material which hardens at any point of the yield surface.*

DEFINITION 3. *A perfectly plastic material is an elastic-plastic material which behaves as perfectly plastic at any point of the yield surface.*

Work-hardening and perfectly plastic materials are therefore two very special cases of elastic-plastic materials. They may also be characterized in terms of the operator B. In fact, from the Eq. (4.5) and from the above definitions it follows that a positive definite B defines a work-hardening material, while $B = 0$ defines a perfectly plastic material. Equation (4.5) shows also that if $H > 0$, then the material is work-hardening, but the converse does not generally hold.

We shall now discuss the assumption of normality and its mathematical implications for the plasticity operator. We begin by stating some lemmas.

LEMMA 1. *Normality is equivalent to the two conditions:*

$$(A-B)\dot{\varepsilon}'' \cdot \dot{\varepsilon}' = 0, \quad (A-B)\dot{\varepsilon}'' \cdot \dot{\varepsilon}'' > 0, \quad \textit{for any} \quad \dot{\varepsilon}', \dot{\varepsilon}'' \neq 0. \tag{4.6}$$

Proof. For elastic responses we have $\dot{\varepsilon}'' = 0$, so that normality and the above conditions are compatible with each other. For plastic responses, the Eq. $(2.6)_1$ states that the vector $\dot{\varepsilon}'$ is tangent to the yield surface. Therefore, if the Eq. $(4.6)_1$ holds for any $\dot{\varepsilon}'$, the nvector $(A-B)\dot{\varepsilon}''$ must be normal to the yield surface, and, conversely, if it is normal to the yield surface, the Eq. $(4.6)_1$ holds.

Moreover, the Ineq. $(4.6)_2$ can now be given the form

$$\lambda\dot{\varepsilon}'' \cdot n > 0 \quad \text{for any} \quad \dot{\varepsilon}'' \neq 0,$$

and, as $\dot{\varepsilon}'' \cdot n > 0$ for constitutive reasons, it can be reduced to: $\lambda > 0$. This condition specifies that vector $(A-B)\dot{\varepsilon}''$ points outwards from the elastic range, and, as this is explicitly required in our formulation (3.4), (3.5) of normality, this completes the proof of the equivalence between normality and the statement (4.6).

Note that if the yield surface is sufficiently smooth, vectors $\dot{\varepsilon}''$ span the space $\mathscr{V}_s$. In this case, from the Ineq. $(4.6)_2$ the positive definiteness of $(A-B)$ can be deduced. This will always be assumed in the following.

LEMMA 2. *If normality holds, then*

$$(A-B)\dot{\varepsilon}'' \cdot \dot{\varepsilon}^{*\prime} \leqslant 0 \quad \textit{for any} \quad \dot{\varepsilon}, \dot{\varepsilon}^*.$$

Proof. Substitution of the Eq. (3.4) into the above inequality yields:

$$\lambda\dot{\varepsilon}^{*\prime}n \leqslant 0.$$

But $\lambda \geqslant 0$, $\dot{\varepsilon}^{*\prime}n \leqslant 0$, for constitutive reasons, and this proves the lemma.

LEMMA 3. *Normality implies that*

$$(A-B)\dot{\varepsilon}\dot{\varepsilon}^* \leqslant (A-B)\dot{\varepsilon}'\dot{\varepsilon}^{*\prime}+(A-B)\dot{\varepsilon}''\dot{\varepsilon}^{*\prime\prime},$$

for any $\dot{\varepsilon}, \dot{\varepsilon}^*$. *When* $\dot{\varepsilon} = \dot{\varepsilon}^*$, *the equality sign holds.*

Proof. It is sufficient to replace $\dot{\varepsilon}$, $\dot{\varepsilon}^*$ by $\dot{\varepsilon}'+\dot{\varepsilon}''$, $\dot{\varepsilon}^{*\prime}+\dot{\varepsilon}^{*\prime\prime}$ and to apply Lemmas 1, 2.

LEMMA 4. *If normality holds, then*

$$(A-B)(\dot{\varepsilon}+\dot{\varepsilon}^*)'(\dot{\varepsilon}+\dot{\varepsilon}^*)' \leqslant (A-B)(\dot{\varepsilon}'+\dot{\varepsilon}^{*\prime})(\dot{\varepsilon}'+\dot{\varepsilon}^{*\prime}) \qquad \textit{for any} \quad \dot{\varepsilon}, \dot{\varepsilon}^*.$$

Proof. By using the Lemma 3 the above inequality can be given the form

$$(A-B)(\dot{\varepsilon}+\dot{\varepsilon}^*)''(\dot{\varepsilon}+\dot{\varepsilon}^*)'' \geqslant (A-B)(\dot{\varepsilon}''-\dot{\varepsilon}^{*\prime\prime})(\dot{\varepsilon}''-\dot{\varepsilon}^{*\prime\prime})+2(A-B)\dot{\varepsilon}''\dot{\varepsilon}^*+2(A-B)\dot{\varepsilon}^{*\prime\prime}\dot{\varepsilon}. \tag{4.7}$$

Recalling that, by normality,

$$(A-B)\dot{\varepsilon}'' = \begin{cases} \dfrac{\dot{\varepsilon}n}{mn} & \text{if} \quad \dot{\varepsilon}n > 0, \\ 0 & \text{if} \quad \dot{\varepsilon}n \leqslant 0, \end{cases} \tag{4.8}$$

it is easy to recognize that the Ineq. (4.7) is verified with the equality sign when both responses are elastic and when they are both plastic. When one response, say $\dot{\varepsilon}$, is elastic and the other one is plastic, we have $\dot{\varepsilon}n > 0 \geqslant \dot{\varepsilon}^*n$. Then, if $(\dot{\varepsilon}+\dot{\varepsilon}^*)n > 0$, substitution of the Eq. (4.8) into the Ineq. (4.7) and multiplication by the positive scalar mn gives the inequality:

$$[(\dot{\varepsilon}+\dot{\varepsilon}^*)n]^2 \geqslant (\dot{\varepsilon}n)^2+2(\dot{\varepsilon}n)(\dot{\varepsilon}^*n);$$

if $(\dot{\varepsilon}+\dot{\varepsilon}^*)n \leqslant 0$, the Ineq. (4.7) becomes

$$0 \geqslant (\dot{\varepsilon}n)[\dot{\varepsilon}^*n+(\dot{\varepsilon}+\dot{\varepsilon}^*)n];$$

both inequalities are satisfied for any $\dot{\varepsilon}$, $\dot{\varepsilon}^*$ with $\dot{\varepsilon}n > 0 \geqslant \dot{\varepsilon}^*n$.

A fundamental property of operator H connected with the assumption of normality is given by the following

THEOREM 2. *Normality is equivalent to the limitations* $A \geqslant H \geqslant B$.

Proof. By multiplying the Eqs. (4.1) by $\dot{\varepsilon}$ and by putting $\dot{\sigma} = H\dot{\varepsilon}$, we have

$$\begin{aligned} (A-H)\dot{\varepsilon}\dot{\varepsilon} &= (A-B)\dot{\varepsilon}''\dot{\varepsilon} = (A-B)\dot{\varepsilon}''\dot{\varepsilon}'+(A-B)\dot{\varepsilon}''\dot{\varepsilon}'', \\ (H-B)\dot{\varepsilon}\dot{\varepsilon} &= (A-B)\dot{\varepsilon}'\dot{\varepsilon} = (A-B)\dot{\varepsilon}'\dot{\varepsilon}'+(A-B)\dot{\varepsilon}'\dot{\varepsilon}''. \end{aligned} \tag{4.9}$$

Let us assume normality, or equivalently, that the Eqs. (4.6) hold. Then, the right-hand sides of the Eqs. (4.9) are non-negative, and $A \geqslant H \geqslant B$. Conversely, if $A \geqslant H \geqslant B$, right-hand sides are non-negative for every $\dot{\varepsilon}'\dot{\varepsilon}''$ and the Eqs. (4.6) hold.

A first consequence of Theorem 2 is that, in the case of normality, positive definiteness of B implies positive definiteness of H. In accordance with the definition of work-hardening given above, this means that, in the case of normality, work-hardening materials are characterized by $H > 0$ and *vice versa*.

We shall now investigate the existence of the inverse operator to H, i.e., the existence of the operator H^{-1} defined by

$$\dot{\varepsilon} = H^{-1}\dot{\sigma}. \tag{4.9}$$

For the elastic response we have simply $\dot{\varepsilon} = A^{-1}\dot{\sigma}$. For the plastic one, the constitutive equation

$$\dot{\sigma} = A\dot{\varepsilon} - \frac{\dot{\varepsilon}n}{mn}(A-B)m \tag{4.10}$$

multiplied by $n_\sigma \equiv A^{-1}n$ gives:

$$\dot{\sigma}n_\sigma = \frac{Bmn_\sigma}{mn}\dot{\varepsilon}n. \tag{4.11}$$

If $Bmn_\sigma \neq 0$, substitution into the Eq. (4.10) and multiplication by A^{-1} give

$$\dot{\varepsilon} = A^{-1}\dot{\sigma} + \frac{\dot{\sigma}n_\sigma}{Bmn_\sigma}A^{-1}(A-B)m, \tag{4.12}$$

and this defines H^{-1} in the case of plastic response. From this analysis it follows that:

THEOREM 3. *A sufficient condition for the existence of H^{-1} is that the product Bmn_σ is different from zero at every point of the yield surface.*

It is interesting to observe that this condition is satisfied for a material obeying PRAGER's hardening rule (3.7), which gives:

$$Bmn_\sigma = n_\sigma n_\sigma > 0.$$

On the other hand, for ZIEGLER's rule we have

$$Bmn_\sigma = \varepsilon' n.$$

It can be shown that positiveness of $\varepsilon' n$ leads to PRAGER's "consistency" condition on the shape of the yield surface (see e.g. [7], p. 141]. This condition requires that the points $\alpha\varepsilon'$ with $0 \leqslant \alpha < 1$ must be interior to the elastic range whenever the point ε' belongs to the elastic range.

In the case of normality, we have from the Eq. (3.5)

$$Bmn_\sigma = BmA^{-1}(A-B)m = (B^{-1}-A^{-1})BmBm.$$

As a well known result of matrix theory (see e.g. [14], p. 59) ensures that positive definiteness of B, $(A-B)$ implies positive definiteness of $(B^{-1}-A^{-1})$, we deduce that existence of H^{-1} is ensured for all work-hardening materials obeying normality. This result can be obtained more directly by referring to Theorem 1, which concerns the monotonicity of H, and recalling that a monotonic operator is invertible.

5. The incremental problem

Let us consider a body occupying the region $\mathscr{B}$ at time t, in equilibrium under given body forces and under surface tractions specified at some portion $\partial_1\mathscr{B}$ of the boundary. The complementary part $\partial_2\mathscr{B}$ of the boundary is subject to geometrical constraints.

Let now external forces and geometrical constraints be varied during the time interval $(t, t+dt)$. We denote as the *incremental equilibrium problem* the problem of determining the stress, strain and displacement fields $\dot{\sigma}dt$, $\dot{\varepsilon}dt$, $\dot{u}dt$ for the equilibrium problem under the perturbed data. In other words, the stress field is required to satisfy the equilibrium conditions

$$\operatorname{div}\dot{\sigma}+\dot{f} = 0 \quad \text{in } \mathscr{B}, \qquad \dot{\varepsilon}\nu = \dot{F} \quad \text{on } \partial_1\mathscr{B}, \tag{5.1}$$

where $\dot{f}dt$, $\dot{F}dt$ are the perturbed body forces and surface tractions, and ν is the exterior normal to $\partial_1\mathscr{B}$. The strain and displacement fields are required to satisfy the compatibility equations

$$2\dot{\varepsilon} = \operatorname{grad}\dot{u}+(\operatorname{grad}\dot{u})^T \quad \text{in } \mathscr{B}, \qquad \dot{u} = \dot{v} \quad \text{on } \partial_2\mathscr{B}, \tag{5.2}$$

where $\dot{v}dt$ are the perturbed geometrical constraints on $\partial_2\mathscr{B}$. Furthermore, the constitutive equation $\dot{\sigma} = H\dot{\varepsilon}$ must be satisfied.

In this Section we consider the conditions under which the uniqueness of the solution of the incremental problem is ensured, and also under which variational principles can be stated. In the next Section we shall deal with the existence problem. We begin with a very standard uniqueness theorem.

THEOREM 4. *If H is monotonic, the incremental problem admits no more than one solution.*

Proof. Suppose that there are two solutions, $\dot{\sigma}, \dot{\varepsilon}, \dot{u}$ and $\dot{\sigma}^*, \dot{\varepsilon}^*, \dot{u}^*$. The virtual work equation for the fields $\dot{\sigma}-\dot{\sigma}^*, \dot{\varepsilon}-\dot{\varepsilon}^*, \dot{u}-\dot{u}^*$ is

$$\int_{\mathscr{B}} (\dot{\sigma}-\dot{\sigma}^*)\cdot(\dot{\varepsilon}-\dot{\varepsilon}^*)dx = 0. \tag{5.3}$$

But, as $\dot{\sigma}, \dot{\sigma}^*$ are solutions, they obey the constitutive equation. Therefore, $\dot{\sigma}-\dot{\sigma}^* = H\dot{\varepsilon}-H\dot{\varepsilon}^*$ and, by the monotonicity of H, the integrand is positive unless $\dot{\varepsilon} = \dot{\varepsilon}^*$ through $\mathscr{B}$.

In particular, Theorems 1, 2 discussed in the preceding Section ensure uniqueness for all work-hardening materials obeying normality.

We shall now discuss the minimum principles. For this purpose, we introduce the notation

$$\langle a, b\rangle = \int_{\mathscr{B}} a(x)b(x)dx, \qquad \langle a, b\rangle_{\partial_1} = \int_{\partial_1\mathscr{B}} a(x)b(x)dx.$$

Then we define the functional

$$W(\dot{u}^*) = \langle H\dot{\varepsilon}^*, \dot{\varepsilon}^*\rangle-2\langle\dot{f}, \dot{u}^*\rangle-2\langle\dot{F}, \dot{u}^*\rangle_{\partial_1} \tag{5.4}$$

over the linear manifold of the functions $\dot{u}^*(x)$, $\dot{\varepsilon}^*(x)$ satisfying the compatibility equations (5.2). By introducing the strain field $\dot{\varepsilon}$ corresponding to the solution of the incremental problem and by using the virtual work equation, we may express the functional W in the form

$$W(\dot{u}^*) = \langle H\dot{\varepsilon}^*, \dot{\varepsilon}^*\rangle-2\langle H\dot{\varepsilon}, \dot{\varepsilon}^*\rangle+2\langle\dot{T}, \dot{v}\rangle_{\partial_1},$$

where $\dot{T}$ are the surface tractions on $\partial_2\mathscr{B}$ corresponding to the solution, and $\dot{v}$ are the data on $\partial_2\mathscr{B}$.

The first of the minimum principles states that

THEOREM 5. *If A, B are positive definite and if normality holds, then the functional $W(\dot{u}^*)$ achieves an absolute minimum at the solution $\dot{u}$ of the incremental problem.*

Proof. Let us consider the difference

$$W(\dot{u}^*)-W(\dot{u}) = \langle H\dot{\varepsilon}^*, \dot{\varepsilon}^*\rangle - 2\langle H\dot{\varepsilon}, \dot{\varepsilon}^*\rangle + \langle H\dot{\varepsilon}, \dot{\varepsilon}\rangle. \tag{5.5}$$

The theorem asserts that this difference is positive, unless $\dot{u}^* = \dot{u}$. By substitution of the constitutive equation $(4.1)_2$ and by normality we have:

$$\begin{aligned}(5.6)\quad & W(\dot{u}^*)-W(\dot{u}) \\ &= \langle B\dot{\varepsilon}^*, \dot{\varepsilon}^*\rangle - 2\langle B\dot{\varepsilon}, \dot{\varepsilon}^*\rangle + \langle B\dot{\varepsilon}, \dot{\varepsilon}\rangle + \langle (A-B)\dot{\varepsilon}', \dot{\varepsilon}\rangle - 2\langle (A-B)\dot{\varepsilon}', \dot{\varepsilon}^*\rangle + \langle (A-B)\dot{\varepsilon}^{*\prime}, \dot{\varepsilon}^*\rangle \\ &= \langle B(\dot{\varepsilon}^* - \dot{\varepsilon}), \dot{\varepsilon}^* - \dot{\varepsilon}\rangle + \langle (A-B)(\dot{\varepsilon}^{*\prime} - \dot{\varepsilon}'), \dot{\varepsilon}^{*\prime} - \dot{\varepsilon}'\rangle - 2\langle (A-B)\dot{\varepsilon}', \dot{\varepsilon}^{*\prime\prime}\rangle.\end{aligned}$$

If $\dot{\varepsilon}^* \neq \dot{\varepsilon}$, the first term is positive by positive definiteness of B. The remaining two are non-negative by Lemmas 1, 2 of Sec. 4.

To state the complementary minimum principle we introduce the functional

$$V(\dot{\sigma}^*) = \langle \dot{\sigma}^*, H^{-1}\dot{\sigma}^*\rangle - 2\langle \dot{\sigma}^* \cdot \nu, \dot{v}\rangle_{\partial_2} \tag{5.7}$$

defined over the set of the stress fields $\dot{\sigma}^*(x)$ satisfying the equilibrium equations (5.1). By introducing the fields $\dot{\sigma}(x)$, $\dot{u}(x)$ corresponding to the solution of the incremental problem and by using the virtual work equation, we have that

$$V(\dot{\sigma}^*) = \langle \dot{\sigma}^*, H^{-1}\dot{\sigma}^*\rangle - 2\langle \dot{\sigma}^*, H^{-1}\dot{\sigma}\rangle + 2\langle \dot{f}, \dot{u}\rangle + 2\langle \dot{F}, \dot{u}\rangle_{\partial_1}.$$

The complementary minimum principle can now be stated as follows:

THEOREM 6. *If A, B are positive definite and if normality holds, the functional $V(\dot{\sigma}^*)$ has an absolute minimum at the solution of the incremental problem.*

Proof. We observe that the assumptions made above are sufficient to ensure the existence of H^{-1}. Next, we again consider the difference

$$\begin{aligned}V(\dot{\sigma}^*)-V(\dot{\sigma}) &= \langle \dot{\sigma}^*, H^{-1}\dot{\sigma}^*\rangle - 2\langle \dot{\sigma}^*, H^{-1}\dot{\sigma}\rangle + \langle \dot{\sigma}, H^{-1}\dot{\sigma}\rangle = \\ &= \langle H\dot{\varepsilon}^*, \dot{\varepsilon}^*\rangle - 2\langle H\dot{\varepsilon}^*, \dot{\varepsilon}\rangle + \langle H\dot{\varepsilon}, \dot{\varepsilon}\rangle,\end{aligned}$$

where we define $\dot{\varepsilon}^* = H^{-1}\dot{\sigma}^*$. This difference is identical with the right-hand side of the Eq. (5.5), except for $\dot{\varepsilon}$, $\dot{\varepsilon}^*$ which are interchanged. But in proving its positiveness the fact that $\dot{\varepsilon}$ is the solution was not used. Therefore, by repeating the proof of the preceding theorem we can state that $V(\dot{\sigma}^*)-V(\dot{\sigma})$ is positive for any equilibrated stress field different from $\dot{\sigma}(x)$.

6. The existence theorem

We have proved in the preceding Section that the solution of the incremental equilibrium problem minimizes the functional

$$W(\dot{u}) = \langle H\dot{\varepsilon}, \dot{\varepsilon}\rangle - 2\langle \dot{f}, \dot{u}\rangle - 2\langle \dot{F}, \dot{u}\rangle_{\partial_1}$$

over the linear manifold of the functions obeying the compatibility conditions (5.2). Here, we shall prove that such a solution exists. It is well known that to give a satisfactory response to the existence problem it is necessary that the domain of definition of W be suitably enlarged, to become a complete function space. Hence, we introduce the HILBERT space $\mathscr{H}$ obtained by the completion of the set of the compatible functions with respect to the scalar product

$$(a, b) = \int\limits_{\mathscr{B}} (ab + \operatorname{grad} a \operatorname{grad} b) dx,$$

and we take as the norm of $\mathscr{H}$ the usual norm

$$||a|| = \sqrt{(a, a)}.$$

We shall prove that there exists a function minimizing W throughout $\mathscr{H}$. It may happen that this function does not belong to the original domain of W; in this case, it will be referred to as a "generalized solution".

We assume that

$$\alpha^2 I \geqslant A \geqslant H \geqslant B \geqslant \beta^2 I > 0, \tag{6.1}$$

I being the identity operator. By Theorem 2, the second and third inequalities correspond to normality; the first one postulates the boundedness of the elastic operator A and the last one requires the coerciveness of B and implies that the material is work-hardening. Furthermore, we assume that $\partial\mathscr{B}$ is sufficiently regular to admit the *a priori* limitations(3):

$$\langle \dot{\varepsilon}, \dot{\varepsilon} \rangle \geqslant \gamma^2 ||\dot{u}||^2, \quad \langle \dot{u}, \dot{u} \rangle_{\partial_1} \leqslant \delta^2 ||\dot{u}||^2. \tag{6.2}$$

The following inequality is also easily verified(4)

$$\langle \dot{u}, \dot{u} \rangle + \langle \dot{\varepsilon}, \dot{\varepsilon} \rangle \leqslant ||\dot{u}||^2. \tag{6.3}$$

Under the assumptions made above, the functional W results to be continuous, convex and bounded from below. Boundedness is a consequence of the coerciveness of B and of KORN's inequality $(6.2)_1$, and may easily be deduced with the aid of the Ineq. (6.3) and of the CAUCHY-SCHWARZ inequality, as follows:

$$\begin{aligned} W(\dot{u}) &= \langle H\dot{\varepsilon}, \dot{\varepsilon} \rangle - 2\langle \dot{f}, \dot{u} \rangle - 2\langle \dot{F}, \dot{u} \rangle \\ &\geqslant \beta^2 \langle \dot{\varepsilon}, \dot{\varepsilon} \rangle - 2\sqrt{\langle \dot{f}, \dot{f} \rangle \langle \dot{u}, \dot{u} \rangle} - 2\sqrt{\langle \dot{F}, \dot{F} \rangle_{\partial_1} \langle \dot{u}, \dot{u} \rangle_{\partial_1}} \\ &\geqslant \beta^2 \gamma^2 ||\dot{u}||^2 - 2\left[\sqrt{\langle \dot{f}, \dot{f} \rangle} + \delta \sqrt{\langle \dot{F}, \dot{F} \rangle_{\partial_1}}\right] \cdot ||\dot{u}|| \\ &\geqslant \frac{-1}{\beta^2 \gamma^2} \left[\sqrt{\langle \dot{f}, \dot{f} \rangle} + \delta \sqrt{\langle \dot{F}, \dot{F} \rangle_{\partial_1}}\right]^2 \equiv \frac{-1}{\beta^2 \gamma^2} \Phi^2. \end{aligned} \tag{6.4}$$

Boundedness is then ensured whenever the number Φ defined above and depending only on the data, is finite. To prove continuity, we take two elements $\dot{u}, \dot{u}^*$ of $\mathscr{H}$ and consider the difference

$$W(\dot{u}^*) - W(\dot{u}) = \langle H\dot{\varepsilon}^*, \dot{\varepsilon}^* \rangle - \langle H\dot{\varepsilon}, \dot{\varepsilon} \rangle - 2\langle \dot{f}, \dot{u} - \dot{u}^* \rangle - 2\langle \dot{F}, \dot{u} - \dot{u}^* \rangle_{\partial_1}. \tag{6.5}$$

(3) The first one is KORN's inequality (see e.g. FICHERA [15], Sec. 12). A proof of it for the various types of boundary conditions can be found in MIKHLIN [16], Secs. 40 to 42. For the second inequality, see e.g. [15], Sec. 2.

(4) By decomposing the displacement gradient into the sum of its symmetric and skew-symmetric parts $\dot{\varepsilon}, \dot{\omega}$ we have:

$$||\dot{u}||^2 - \langle \dot{u}, \dot{u} \rangle = \langle \operatorname{grad} \dot{u}, \operatorname{grad} \dot{u} \rangle = \langle \dot{\varepsilon} + \dot{\omega}, \dot{\varepsilon} + \dot{\omega} \rangle = \langle \dot{\varepsilon}, \dot{\varepsilon} \rangle + \langle \dot{\omega}, \dot{\omega} \rangle \geqslant \langle \dot{\varepsilon}, \dot{\varepsilon} \rangle.$$

By putting $\dot{\varepsilon}^* = \dot{\varepsilon}+\dot{\eta}$ and by using the constitutive equation $(4.1)_2$ and Lemmas 1, 4, we have

$$\langle H\dot{\varepsilon}^*, \dot{\varepsilon}^*\rangle-\langle H\dot{\varepsilon}, \dot{\varepsilon}\rangle = \langle B(\dot{\varepsilon}+\dot{\eta}), \dot{\varepsilon}+\dot{\eta}\rangle-\langle B\dot{\varepsilon}, \dot{\varepsilon}\rangle+\langle (A-B)(\dot{\varepsilon}+\dot{\eta})', (\dot{\varepsilon}+\dot{\eta})'\rangle$$
$$-\langle (A-B)\dot{\varepsilon}', \dot{\varepsilon}'\rangle \leqslant 2\langle B\dot{\varepsilon}, \dot{\eta}\rangle+\langle B\dot{\eta}, \dot{\eta}\rangle+2\langle (A-B)\dot{\varepsilon}', \dot{\eta}'\rangle+\langle (A-B)\dot{\eta}', \dot{\eta}'\rangle,$$

and, by the CAUCHY-SCHWARZ inequality combined with Lemma 1:

$$\langle H\dot{\varepsilon}^*, \dot{\varepsilon}^*\rangle-\langle H\dot{\varepsilon}, \dot{\varepsilon}\rangle$$
$$\leqslant 2\sqrt{\langle B\dot{\varepsilon}, \dot{\varepsilon}\rangle\langle B\dot{\eta}, \dot{\eta}\rangle}+\langle B\dot{\eta}, \dot{\eta}\rangle+2\sqrt{\langle (A-B)\dot{\varepsilon}, \dot{\varepsilon}\rangle\langle (A-B)\dot{\eta}, \dot{\eta}\rangle}+\langle (A-B)\dot{\eta}, \dot{\eta}\rangle$$
$$\leqslant \alpha^2\langle \dot{\eta}, \dot{\eta}\rangle+4\alpha^2\sqrt{\langle \dot{\varepsilon}, \dot{\varepsilon}\rangle\langle \dot{\eta}, \dot{\eta}\rangle} \leqslant \alpha^2\|\dot{u}-\dot{u}^*\|^2+4\alpha^2\|\dot{u}\|\cdot\|\dot{u}-\dot{u}^*\|.$$

By majorizing the remaining part of the right-hand side of the Eq. (6.5) just as in the Ineq. (6.4) we may conclude that

$$W(\dot{u}^*)-W(\dot{u}) \leqslant [\alpha^2\|\dot{u}-\dot{u}^*\|+4\alpha^2\|\dot{u}\|+\Phi]\cdot\|\dot{u}-\dot{u}^*\|.$$

This implies semicontinuity of W at $\dot{u}$, provided that $\|\dot{u}\|$ and Φ are finite. As a similar inequality holds when $\dot{u}$, $\dot{u}^*$ are interchanged, the continuity of W is ensured and we have that

$$\lim_{\|\dot{u}-\dot{u}^*\|\to 0} |W(\dot{u}^*)-W(\dot{u})| = 0. \tag{6.6}$$

Convexity means that the number

$$J(\dot{u}, \dot{u}^*) = W(\dot{u})-2W\left(\frac{\dot{u}+\dot{u}^*}{2}\right)+W(\dot{u}^*)$$

is positive whenever $u \neq \dot{u}^*$. In our case, by using the Eq. $(4.1)_2$ and Lemmas 1, 4, we have:

$$J(\dot{u}, \dot{u}^*) = \langle B\dot{\varepsilon}, \dot{\varepsilon}\rangle-\frac{1}{2}\langle B(\dot{\varepsilon}+\dot{\varepsilon}^*), \dot{\varepsilon}+\dot{\varepsilon}^*\rangle+\langle B\dot{\varepsilon}^*, \dot{\varepsilon}^*\rangle+\langle (A-B)\dot{\varepsilon}', \dot{\varepsilon}'\rangle \tag{6.7}$$
$$-\frac{1}{2}\langle (A-B)(\dot{\varepsilon}+\dot{\varepsilon}^*)', (\dot{\varepsilon}+\dot{\varepsilon}^*)'\rangle+\langle (A-B)\dot{\varepsilon}^{*\prime}, \dot{\varepsilon}^{*\prime}\rangle$$
$$\geqslant \frac{1}{2}\langle B(\dot{\varepsilon}-\dot{\varepsilon}^*), \dot{\varepsilon}-\dot{\varepsilon}^*\rangle+\frac{1}{2}\langle (A-B)(\dot{\varepsilon}'-\dot{\varepsilon}^{*\prime}), \dot{\varepsilon}'-\dot{\varepsilon}^{*\prime}\rangle$$
$$\geqslant \frac{1}{2}\beta^2\gamma^2\|\dot{u}-\dot{u}^*\|^2,$$

and this is a sufficient condition for convexity.

Once these properties of W have been stated, the proof of the existence theorem is very close to the one concerning the case of linearelasticity. The boundedness of W implies the existence of an exact lower bound, which will be denoted by d, and implies the possibility of constructing a minimizing sequence for W, i.e., a sequence $\{\dot{u}_n\}$ of elements of $\mathscr{H}$ such that

$$d \leqslant W(\dot{u}_n) < d+1/n. \tag{6.8}$$

Thus, taken a positive integer N, for any integers $m, n > N$ we have that

$$J(\dot{u}_m, \dot{u}_n) = W(\dot{u}_n)-2W\left(\frac{\dot{u}_n+\dot{u}_m}{2}\right)+W(\dot{u}_m) \leqslant d+\frac{1}{n}-2d+d+\frac{1}{m} < \frac{2}{N},$$

and, by Ineq. (6.7),

$$\frac{1}{2}\beta^2\gamma^2||\dot{u}_m-\dot{u}_n||^2 < \frac{2}{N} \quad \text{for any} \quad m, n > N.$$

Then, $\{\dot{u}_n\}$ is a Cauchy sequence and, as $\mathscr{H}$ is complete, $\{\dot{u}_n\}$ converges in the norm $||\cdot||$ to an element $\dot{u}_0$ of $\mathscr{H}$. Furthermore, by substituting $\dot{u}_n, \dot{u}_0$ into the Eq. (6.6) we have that the numerical sequence $\{W(\dot{u}_n)\}$ converges to $W(\dot{u}_0)$, so that, by the Ineqs. (6.8), we have $W(\dot{u}_0) = d$.

Therefore, the minimum of $W(\dot{u})$ in $\mathscr{H}$ exists and is attained at $\dot{u}_0\dot{u}_0$. Then, $\dot{u}_0$ is the generalized solution of the incremental equilibrium problem.

We observe that the Eq. (6.6) holds only if $||\dot{u}_0||$ is finite. The problem of establishing some *a priori* restrictions on the data which ensure the finiteness of $||\dot{u}_0||$ will not be dealt with here.

7. Perfect plasticity

The perfectly plastic material has been defined in Sec. 4 as an elastic-plastic material characterized by $B = 0$. This property can also be regarded, from the viewpoint of the classical theories, as a particular "hardening rule" imposing that the elastic range remains fixed in the stress space. It is also worth noting that, as shown by the Eq. (3.1), for $B = 0$ the strain ε'' is coincident with the "plastic strain" ε^P, so that in this particular case the sum decomposition (2.1) of ε is identical with the classical one.

In perfect plasticity, according to Definition 3, the incremental energy corresponding to a purely plastic response is zero. Hence, the operator H cannot be positive definite, and the existence and uniqueness theorems proved in the preceding Sections are not available. However, it will be shown that the uniqueness theorem and the minimum principles given above can be retained in a weakened form with slight modifications, whereas the same will not to possible for the existence theorem. Another important property of the perfectly plastic materials is that, for them, normality is a consequence of the positive semi-definiteness of H. To see this, we first write the constitutive equation of the perfectly plastic material, obtained by putting $B = 0$ in the Eq. (2.9):

$$\dot{\sigma} \equiv H\dot{\varepsilon} \equiv A\dot{\varepsilon}', \tag{7.1}$$

and we recall that, by Lemma 1, normality is now equivalent to:

$$A > 0; \quad A\dot{\varepsilon}'\dot{\varepsilon}'' = 0 \quad \text{for any } \dot{\varepsilon}', \dot{\varepsilon}''. \tag{7.2}$$

Then, we state the following

Theorem 7. *In perfect plasticity, normality implies and is implied by positive semi-definiteness of H.*

Proof. We have by Theorem 2 that normality is equivalent to $A \geqslant H \geqslant B$ and, as $B = 0$, this proves the first assertion. Conversely, by assuming $H \geqslant 0$ we have from the Eq. (7.1) that

$$0 \leqslant H\dot{\varepsilon}\dot{\varepsilon} = A\dot{\varepsilon}'\dot{\varepsilon}' + A\dot{\varepsilon}'\dot{\varepsilon}'' \quad \text{for any } \varepsilon', \varepsilon''.$$

But this implies the Eqs. (7.2) and, consequently, normality.

This result allows us to state the uniqueness theorem for the perfectly plastic material:

THEOREM 8. *For a perfectly plastic material the stress-rate field resolving the incremental equilibrium problem is unique if H is weakly monotonic.*

P r o o f. Let $\dot{\varepsilon}$, $\dot{\varepsilon}^*$ be two solutions of the incremental problem. By the virtual work equation we have

$$\langle H\dot{\varepsilon} - H\dot{\varepsilon}^*, \dot{\varepsilon} - \dot{\varepsilon}^* \rangle = 0,$$

and, by the weak monotonicity of H,

$$(H\dot{\varepsilon} - H\dot{\varepsilon}^*)(\dot{\varepsilon} - \dot{\varepsilon}^*) = 0$$

throughout $\mathscr{B}$. Then, the Eq. (7.1) yields:

$$A(\dot{\varepsilon}' - \dot{\varepsilon}^{*\prime})(\dot{\varepsilon}' - \dot{\varepsilon}^{*\prime}) + A(\dot{\varepsilon}' - \dot{\varepsilon}^{*\prime})(\dot{\varepsilon}'' - \dot{\varepsilon}^{*\prime\prime}) = 0.$$

By Theorem 1, weak monotonicity is equivalent to positive semidefiniteness and, by Theorem 7, the latter is equivalent to normality. Hence, the Eqs. (7.2) and Lemma 2 can be used to show that both terms of the above equation are positive unless $\dot{\varepsilon}' = \dot{\varepsilon}^{*\prime}$, and we have proved the uniqueness of the field $\dot{\varepsilon}'$ or, equivalently, of the field $\dot{\sigma}$ resolving the incremental equilibrium problem.

For the first minimum principle proved in Sec. 5 we have a similar modification: the functional $W(\dot{u}^*)$ has again an absolute minimum at the solution $\dot{u}$, but the value of $W(\dot{u})$ is unaltered by adding to $\dot{\varepsilon}$ any distribution of plastic strains. In fact, by putting $B = 0$ in the Eq. (5.6) we have

$$W(\dot{u}^*) - W(\dot{u}) = \langle A(\dot{\varepsilon}^{*\prime} - \dot{\varepsilon}'), \dot{\varepsilon}^{*\prime} - \dot{\varepsilon}' \rangle - 2\langle A\dot{\varepsilon}', \dot{\varepsilon}^{*\prime\prime} \rangle,$$

and this is a positive number unless $\dot{\varepsilon}' = \dot{\varepsilon}^{*\prime}$, as it results from Lemmas 1, 2.

As to the reciprocal variational principle, we have that the functional $V(\dot{\sigma}^*)$ cannot be defined as in the Eq. (5.7) because the condition required by Theorem 3 for inverting the operator H is not satisfied for $B = 0$. This difficulty can be removed by defining $V(\dot{\sigma}^*)$ as follows:

$$V(\dot{\sigma}^*) = \langle \dot{\sigma}^*, A^{-1}\dot{\sigma}^* \rangle - 2\langle \dot{\sigma}^* \nu, \dot{v} \rangle_{\partial_2}.$$

Besides the equilibrium equations (5.1), $\dot{\sigma}^*(x)$ must now obey the restrictions following the fact that the yield surface is fixed in the stress space. Hence, whenever the point σ^* belongs to the yield surface, the vector $\dot{\sigma}^*$ must be such that

$$\dot{\sigma}^* n_{\sigma^*} \leqslant 0. \tag{7.3}$$

Therefore, under given increments of the data, it may happen that the set of the admissible stress rate fields $\dot{\sigma}^*$ be empty. We assume here that it is non-empty, and that a solution $(\dot{u}, \dot{\varepsilon}, \dot{\sigma})$ of the incremental problem exists. The virtual work equation gives then

$$V(\dot{\sigma}^*) = \langle \dot{\sigma}^*, A^{-1}\dot{\sigma}^* \rangle - 2\langle \dot{\sigma}^*, \dot{\varepsilon} \rangle + 2\langle \dot{f}, \dot{u} \rangle + 2\langle \dot{F}, \dot{u} \rangle_{\partial_1}$$

and, consequently,

$$V(\dot{\sigma}^*) - V(\dot{\sigma}) = \langle \dot{\sigma}^*, A^{-1}\dot{\sigma}^* \rangle - 2\langle \dot{\sigma}^*, \dot{\varepsilon} \rangle - \langle \dot{\sigma}, A^{-1}\dot{\sigma} \rangle + 2\langle \dot{\sigma}, \dot{\varepsilon} \rangle$$
$$= \langle \dot{\sigma} - \dot{\sigma}^*, A^{-1}(\dot{\sigma} - \dot{\sigma}^*) \rangle - 2\langle \dot{\sigma}^*, \dot{\varepsilon}'' \rangle + 2\langle \dot{\sigma}, \dot{\varepsilon}'' \rangle,$$

having replaced $\dot{\varepsilon}$ by $A^{-1}\dot{\sigma}+\dot{\varepsilon}''$, by virtue of the constitutive equation (7.1). Recalling that, by normality $\dot{\varepsilon}''$ is parallel to n_σ, we have from the Eqs. (7.2), and (7.3) that $V(\dot{\sigma}^*)$ is greater than $V(\dot{\sigma})$ whenever $\dot{\sigma}^* \neq \dot{\sigma}$. This is the complementary variational principle. As to the existence theorem proved in Sec. 6, we observe that the coerciveness of H played an essential role in proving boundedness and convexity of the functional $W(\dot{u})$. As, obviously, a semi-definite operator cannot be coercive, we conclude that the technique used in Sec. 6 to prove existence cannot be applied to the case of perfect plasticity.

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