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Dynamic coupled thermoelastic problems in micropolar theory. II

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GENERAL solution of the dynamic micropolar coupled thermoelastic equations has been obtained for arbitrary distribution of body forces, body couples and heat sources in an infinite body by the use of Laplace-Fourier transforms. Short time solutions have been obtained for the cases of suddenly applied body force, body couple and heat source acting at a point. The corresponding classical coupled thermoelastic solutions have been derived by letting the parameter α approach zero. Some numerical results have been illustrated graphically.

Ogólne rozwiązanie równań dynamicznej mikropolarnej sprzężonej termosprężystości uzyskane zostało metodami transformacji Laplace'a-Fouriera dla dowolnego rozkładu sił masowych, obciążeń momentowych i źródeł ciepła w ciele nieograniczonym. Rozwiązania dla krótkiego okresu czasu otrzymane zostały dla przypadków nagłego obciążenia siłami masowymi, siłami momentowymi i źródłem ciepła działającym w punkcie. Odpowiednie klasyczne rozwiązania sprzężonej termosprężystości otrzymano jako graniczny przypadek zdążając z parametrem α do zera. Niektóre wyniki liczbowe przedstawiono na wykresach.

Общее решение уравнений динамической сопряженной микрополярной термоупругости получено методами преобразования Лапласа-Фурье для произвольного распределения массовых сил, моментных нагрузок и теплоисточников в неограниченном теле. Решения для короткого отрезка времени получены для случаев внезапного нагружения массовыми силами, моментными силами и теплоисточником действующим в точке. Соответствующие классические решения сопряженной термоупругости получены как предельный случай при стремлении с параметром α к нулю. Некоторые числовые результаты представлены на графиках.

1. Introduction

THE CLASSICAL theory of elasticity does not explain certain discrepancies that occur in the case of problems involving elastic vibrations of high frequency and short wave length, that is, vibrations due to the generation of ultrasonic waves. The reason lies in the microstructure of the material which exerts a special influence at high frequencies and short wave lengths.

W. VOIGT [1] attempted to eliminate these discrepancies by suggesting that the transmission of interaction between two particles of a body through an elementary area lying within the material was effected not solely by the action of a force vector but also by a moment (couple) vector. This led to the existence of couple stress in elasticity. Later E. and F. COSSERAT [2] gave a unified theory in which every material particle is capable

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of both linear displacement and rotation during the deformation of the material. Thus in this Cosserat theory, the deformation of the body is determined by a displacement vector and independently of this by a rotation vector.

The Cosserat continuum was unnoticed for a long time. This theory was revoked in various special forms called Cosserat pseudo-continuum by GÜNTHER [3], GRIOLI [4], AERO and KUVSHINSKII [5] and SCHAEFER [6]. Other investigators who considered the linear and non-linear theory of elasticity for this Cosserat pseudo-continuum are TRUESDELL and TOUPIN [7], TOUPIN [8] and MINDLIN and TIERSTEN [9].

The general theory of non-linear and linear microelastic continuum was given by ERINGEN and SUHUBI [10, 11]. This theory, in special cases, contains the Cosserat continuum and the indeterminate couple stress theory. A similar theory was also given by PALMOV [12]. ERINGEN and SUHUBI [10, 11] renamed their theory and it is known now as micropolar elasticity or asymmetric elasticity. Thus the micropolar elasticity deals with such materials whose constituents are dumbbell-type molecules, and the elements are allowed to rotate independently without stretch.

Only recently this micropolar elasticity was further extended to include thermal effects by NOWACKI [13]. TAUCHERT, CLAUS, and ARIMAN [14] had also given the basic equations of linear theory of micropolar thermoelasticity. Owing to the newness of the theory of micropolar thermoelasticity, very few problems [15, 16, 17, 18, 19 and 20] have been solved so far.

In this paper we consider the problem of determining the displacements, rotations and temperature in an infinite micropolar thermoelastic medium under the action of time-dependent body forces, body couples and heat sources. In Sec. 2 we have listed the basic equations of coupled micropolar thermoelasticity as derived by NOWACKI [13]. In Sec. 3, we obtain the general solution of these equations by using Fourier-Laplace transforms for any arbitrary distribution of body forces, body couples and heat sources in an infinite medium. In Secs. 4, 5 and 6, respectively, we derive solutions for a sudden body force, body couple and heat source acting at a point in an infinite medium. Exact inversions have been obtained in the space domain but Laplace inversions only for small time approximations which are quite appropriate for the problems under consideration. Some numerical results have been illustrated graphically. Results for an impulsively applied body force, body couple and heat source have also been obtained by the authors recently [21].

2. Basic equations of thermoelasticity

For a homogeneous isotropic centrosymmetric body occupying the region V, we have the following linearized equations of thermoelasticity [1]:

$$(2.1)\quad \begin{aligned} &(\lambda+2\mu)\nabla\nabla\cdot\mathbf{u}-(\mu+\alpha)\nabla\times\nabla\times\mathbf{u}+2\alpha\nabla\times\mathbf{w}+\varrho\mathbf{X} = \varrho\ddot{\mathbf{u}}+\nu\nabla\theta,\\ &(\beta+2\gamma)\nabla\nabla\cdot\mathbf{w}-(\gamma+\varepsilon)\nabla\times\nabla\times\mathbf{w}+2\alpha\nabla\times\mathbf{u}-4\alpha\mathbf{w}+J\mathbf{Y} = J\ddot{\mathbf{w}},\\ &\nabla^2\theta-\frac{1}{k}\theta-\eta_0\nabla\cdot\ddot{\mathbf{u}} = -\frac{Q}{k}, \end{aligned}$$

where the kinematic relations and linear constitutive equations are given by

$$
\begin{aligned}
&\beta_{ij} = w_{j,i}, \gamma_{ij} = u_{j,i}+\varepsilon_{kji}w_k,\\
&\sigma_{ij} = 2\mu\gamma_{(ij)}+2\alpha\gamma_{[ij]}+(\lambda\gamma_{kk}-\nu\theta)\delta_{ij},\\
&\mu_{ij} = 2\gamma\beta_{(ij)}+2\varepsilon\beta_{[ij]}+\beta\beta_{kk}\delta_{ij},\\
&\bar{s} = \nu\gamma_{kk}+\frac{C_E}{T_0}\theta.
\end{aligned}
\tag{2.3}
$$

In the foregoing equations we have used the following notations:

σ_{ij} the stress tensor components,
μ_{ij} the couple-stress tensor components,
u_i the displacement field components,
w_i the rotation field components,
X_i the body force components,
Y_i the body couple components,
γ_{ij} the strain tensor components,
β_{ij} the curvature twist tensor components,
ε_{ijk} the unit anti-symmetric tensor,
$\bar{s}$ entropy per unit volume,
C_E specific heat at constant deformation,
θ deviation from an equilibrium temperature T_0,
k coefficient of thermal diffusivity,
Q heat source of the body,
$\eta_0 \quad = \dfrac{\nu T_0}{C_E k}$.

Here, λ, μ are Lamé's constants and $\alpha, \beta, \gamma, \varepsilon$ are new constants of elasticity referring to the isothermal state. The constant ν depends on the mechanical as well as on the thermal properties of the body. The symbols ( ) and [ ] denote symmetric and skew-symmetric parts of a tensor, respectively; ϱ is the density, J is the rotational inertia and dots denote the time derivatives.

3. The general solution of the basic equations

In this section we shall find the displacement $\mathbf{u}(x_1, x_2, x_3, t)$, rotation $\mathbf{w}(x_1, x_2, x_3, t)$ and temperature field $\theta(x_1, x_2, x_3, t)$ in an infinite micropolar thermoelastic body under the action of time-dependent body forces $\mathbf{X}$, body couples $\mathbf{Y}$ and heat sources Q, i.e. we shall find the solution of equations given by (2.1) for $-\infty < x_1, x_2, x_3 < \infty$, $t \geqslant 0$, under prescribed body forces, body couples and heat sources.

To solve these equations we shall first reduce the equations (2.1) to a simpler form by decomposing the vectors $\mathbf{u}, \mathbf{w}, \mathbf{X}$ and $\mathbf{Y}$ into their potential and solenoidal parts, i.e.:

$$
\begin{aligned}
&\mathbf{u} = \operatorname{grad}\phi+\operatorname{rot}\boldsymbol{\Psi},\ \operatorname{div}\boldsymbol{\Psi} = 0,\\
&\mathbf{w} = \operatorname{grad}\textstyle\sum+\operatorname{rot}\mathbf{H},\ \operatorname{div}\mathbf{H} = 0,
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
&\mathbf{X} = \operatorname{grad}v+\operatorname{rot}\boldsymbol{\chi},\ \operatorname{div}\boldsymbol{\chi} = 0,\\
&\mathbf{Y} = \operatorname{grad}\phi+\operatorname{rot}\boldsymbol{\eta},\ \operatorname{div}\boldsymbol{\eta} = 0.
\end{aligned}
\tag{3.2}
$$

Substitution of (3.1) and (3.2) into the basic equations (2.1) yields

$$
\begin{aligned}
&(\Box_1 D - \omega\partial_t \nabla^2)\phi = -\varrho D v - \frac{\nu}{k} Q,\\
(3.3)\qquad &(\Box_2\Box_4 + 4\alpha^2\nabla^2)\boldsymbol{\Psi} = 2\alpha J \operatorname{rot}\boldsymbol{\eta} - \varrho\Box_4\boldsymbol{\chi},\\
&(\Box_2\Box_4 + 4\alpha^2\nabla^2)\mathbf{H} = 2\alpha\varrho \operatorname{rot}\boldsymbol{\chi} - J\Box_2\boldsymbol{\eta},\\
&\Box_3 \textstyle\sum + J\sigma = 0,
\end{aligned}
$$

$$
(3.4)\qquad (\Box_1 D - \omega\partial_t\nabla^2)\theta = -\varrho\eta_0\partial_t\nabla^2 v - \frac{1}{k}\Box_1 Q,
$$

where

$$
\begin{aligned}
D &= \nabla^2 - \frac{1}{k}\partial_t, & \omega &= 2\eta_0 k/(\lambda+2\mu),\\
\Box_1 &= (\lambda+2\mu)\nabla^2 - \varrho\partial_t^2, & \Box_2 &= (\mu+\alpha)\nabla^2 - \varrho\partial_t^2,\\
\Box_3 &= (\beta+2\gamma)\nabla^2 - 4\alpha - J\partial_t^2, & \Box_4 &= (\gamma+\varepsilon)\nabla^2 - 4\alpha - J\partial_t^2,
\end{aligned}
$$

$$
\nabla^2 = \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_l}, \qquad \partial_t^2 = \frac{\partial^2}{\partial t^2}.
$$

To solve these wave equations, let us introduce the Laplace transform $F^{(1)}(x_1, x_2, x_3, p)$ of the function $F(x_1, x_2, x_3, t)$ by the relation

$$
(3.5a)\qquad F^{(1)}(x_1, x_2, x_3, p) = \int_0^\infty F(x_1, x_2, x_3, t) e^{-pt} dt
$$

and also introduce the Fourier-Laplace transform

$F^{(2)}(\zeta_1, \zeta_2, \zeta_3, p)$ of the function $F(x_1, x_2, x_3, t)$ by

$$
(3.5b)\qquad F^{(2)}(\zeta_1, \zeta_2, \zeta_3, p) = \frac{1}{(2\pi)^{3/2}} \int_{E_3} F^{(1)}(x_1, x_2, x_3, p)\exp(ix_k\zeta_k) dE,
$$

where $dE = dx_1 dx_2 dx_3$ and E_3 is the x_1, x_2, x_3-space.

Application of (3.5) to the Eqs. (3.3) and (3.4) yields a following system of algebraic equations:

$$
\begin{aligned}
&[(\zeta^2+\beta_1^2)(\zeta^2+q)+\omega q\zeta^2]\phi^{(2)} = \frac{1}{c_1^2}(\zeta^2+q)v^{(2)} - \frac{\nu}{\varrho c_1^2 k} Q^{(2)},\\
&[(\zeta^2+\beta_1^2)(\zeta^2+q)+\omega q\zeta^2]\theta^{(2)} = \frac{\eta_0 k}{c_1^2} q\zeta^2 v^{(2)} + \frac{1}{k}(\zeta^2+\beta_1^2)Q^{(2)},\\
&(\zeta^2+\beta_3^2)\textstyle\sum^{(2)} = \frac{1}{c_3^2}\sigma^{(2)},\\
(3.6)\quad &[(\zeta^2+\beta_2^2)(\zeta^2+2+\beta_4^2) - r^* s\zeta^2]\Psi_j^{(2)} = -\frac{r^*}{c_4^2} i\zeta_k\varepsilon_{jkl}\eta_l^{(2)} + \frac{1}{c_2^2}(\zeta^2+2s+\beta_4^2)\chi_j^{(2)},\\
&[(\zeta^2+\beta_2^2)(\zeta^2+2s+\beta_4^2) - r^* s\zeta^2]H_j^{(2)} = -\frac{s}{c_2^2} i\zeta_k\varepsilon_{jkl}\chi_l^{(2)} + \frac{1}{c_4^2}(\zeta^2+\beta_2^2)\eta_j^{(2)},
\end{aligned}
$$

where

$$c_1^2 = \frac{\lambda+2\mu}{\varrho}, \quad c_2^2 = \frac{\mu+\alpha}{\varrho}, \quad c_3^2 = \frac{\beta+2\gamma}{J}, \quad c_4^2 = \frac{\lambda+\varepsilon}{J},$$

$$\beta_1 = \frac{p}{c_1}, \quad \beta_2 = \frac{p}{c_2}, \quad \beta_3 = \left(\frac{p^2+4\alpha/J}{c_3}\right)^{\frac{1}{2}}, \quad \beta_4 = \frac{p}{c_4},$$

$$q = \frac{p}{k}, \quad r^* = \frac{2\alpha}{\varrho c_2^2}, \quad s = \frac{2\alpha}{Jc_4^2}, \quad \zeta^2 = \zeta_1^2+\zeta_2^2+\zeta_3^2,$$

ω is known as thermoelastic coupling constant and ε_{jkl} is the alternating tensor.

Taking the same Fourier-Laplace transform of (3.1) and (3.2) we obtain:

$$\begin{aligned} u_j^{(2)} &= -i\zeta_j\phi^{(2)} - i\zeta_k\varepsilon_{jkl}\Psi_l^{(2)}, \\ w_j^{(2)} &= -i\zeta_j\Sigma^{(2)} - i\zeta_k\varepsilon_{jkl}H_l^{(2)}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} X_j^{(2)} &= -i\zeta_j v^{(2)} - i\zeta_k\varepsilon_{jkl}\chi_l^{(2)}, \\ Y_j^{(2)} &= -i\zeta_j\sigma^{(2)} - i\zeta_k\varepsilon_{jkl}\eta_l^{(2)}. \end{aligned} \tag{3.8}$$

Substituting for $\phi^{(2)}$, $\Psi_l^{(2)}$, $\Sigma^{(2)}$, $H_l^{(2)}$ from (3.6) into (3.7) and taking into account the relation

$$\varepsilon_{jkl}\varepsilon_{lmn} = \delta_{km}\delta_{jn} - \delta_{kn}\delta_{mj},$$

along with the conditions $\operatorname{div}\boldsymbol{\chi} = 0$, $\operatorname{div}\boldsymbol{\eta} = 0$, where δ_{ij} is the Kronecker delta, we obtain

$$\begin{aligned} u_j^{(2)} &= \frac{\nu}{\varrho c_1^2 k}\frac{i\zeta_j}{\Delta_1}Q^{(2)} - \frac{i\zeta_j(\zeta^2+q)}{c_1^2\Delta_1}v^{(2)} + \frac{r^*\zeta^2}{c_4^2\Delta_1}\eta_j^{(2)} - \frac{i\zeta_k\varepsilon_{jkl}}{c_2^2\Delta_2}(\zeta^2+2s+\beta_4^2)\chi_l^{(2)}, \\ w_j^{(2)} &= -\frac{i\zeta_j}{c_3^2(\zeta^2+\beta_3^2)}\sigma^{(2)} + \frac{s\zeta^2}{c_2^2\Delta^2}\chi_j^{(2)} - \frac{(\zeta^2+\beta_2^2)i\zeta_k}{c_4^2\Delta_2}\varepsilon_{jkl}\eta_l^{(2)}, \\ \theta^{(2)} &= \frac{\zeta^2+\beta_1^2}{k\Delta_1}Q^{(2)} + \frac{\eta_0 k}{c_1^2\Delta_1}q\zeta^2 v^{(2)}, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \Delta_1 &= (\zeta^2+\beta_1^2)(\zeta^2+q)+\omega q\zeta^2 = (\zeta^2-\lambda_1^2)(\zeta^2-\lambda_2^2), \\ \Delta_2 &= (\zeta^2+\beta_2^2)(\zeta^2+2s+\beta_4^2)-r^*s\zeta^2 = (\zeta^2-\mu_1^2)(\zeta^2-\mu_2^2) \end{aligned} \tag{3.10}$$

and $\lambda_{1,2}^2$ and $\mu_{1,2}^2$ are the roots of the equations $\Delta_1(\zeta^2) = 0$, $\Delta_2(\zeta^2) = 0$, respectively.

From (3.8) we obtain

$$\begin{aligned} v^{(2)} &= \frac{i\zeta_k}{\zeta^2}X_k^{(2)}, \quad \chi_j^{(2)} = -\frac{i}{\zeta^2}\varepsilon_{jkl}\zeta_k X_l^{(2)}, \\ \sigma^{(2)} &= \frac{i\zeta_k}{\zeta^2}Y_k^{(2)}, \quad \eta_j^{(2)} = -\frac{i}{\zeta^2}\varepsilon_{jkl}\zeta_k Y_l^{(2)}. \end{aligned} \tag{3.11}$$

By substituting the above relations into (3.9) and simplifying we arrive at simple formulae for displacements, rotations and temperature given by

$$
\begin{aligned}
u_j^{(2)} &= \frac{\nu}{\varrho c_1^2 k} i\zeta_j Q^{(2)} + \frac{r^*}{c_4^2 \Delta_2} i\zeta_k \varepsilon_{jkl} Y_l^{(2)} + \frac{\zeta^2+q}{c_1^2 \Delta_1 \zeta^2} \zeta_j \zeta_k X_k^{(2)} \\
&\quad + \frac{\zeta^2+2s+\beta_4^2}{c_2^2 \Delta_2 \zeta^2} (\zeta^2 X_j^{(2)} - \zeta_j \zeta_k X_k^{(2)}), \\
w_j^{(2)} &= -\frac{s}{c_2^2 \Delta_2} i\zeta_k \varepsilon_{jkl} X_l^{(2)} + \frac{\zeta_j \zeta_k Y_k^{(2)}}{c_3^2(\zeta^2+\beta_3^2)\zeta^2} + \frac{\zeta^2+\beta_2^2}{c_4^2 \Delta_2 \zeta^2} (\zeta^2 Y_j^{(2)} - \zeta_j \zeta_k Y_k^{(2)}), \\
\theta^{(2)} &= \frac{\zeta^2+\beta_1^2}{k\Delta_1} Q^{(2)} + \frac{\eta_0 kq}{c_1^2 \Delta_1} i\zeta_k X_k^{(2)}.
\end{aligned}
\tag{3.12}
$$

Thus, the above system of equations gives rise to the general solution for the determination of displacement, rotation and temperature field for any given body forces, body couples and heat sources applied in the infinite medium by first inverting the Fourier transform and then by inverting the Laplace transform.

Let us consider the particular case where $\alpha \to 0$. The Eqs. $(3.12)_1$ and $(3.12)_2$ become independent and give rise to the following classical thermoelastic solution

$$
u_j^{(2)} = \frac{\zeta_j \zeta_k X_k^{(2)}}{c_1^2 \overline{\Delta}_1} + \frac{\zeta^2 X_j^{(2)} - \zeta_j \zeta_k X_k^{(2)}}{c_2^2 \overline{\Delta}_2} + \frac{\nu}{\varrho c_1^2 k} i\zeta_j Q^{(2)},
$$

or

$$
u_j^{(2)} = \frac{\beta^2 \overline{\Delta}_1 \zeta^2 X_j^2 + (\overline{\Delta}_2 - \beta^2 \overline{\Delta}_1 \zeta_j \zeta_k X_k^{(2)})}{c_1^2 \overline{\Delta}_1 \overline{\Delta}_2} + \frac{\nu}{\varrho c_1^2 k} i\zeta_j Q^{(2)},
$$

where

$$
\overline{\Delta}_1 = \zeta^2 + p^2/c, \quad \overline{\Delta}_2 = \zeta^2 + p^2/c_2^2, \quad \beta^2 = \frac{c_1^2}{c_2^2} = \frac{\lambda+2\mu}{\mu}.
$$

It may be noted that $(3.12)_2$ gives $w_j^{(2)} \equiv 0$ and $(3.12)_3$ remains unchanged. In the case of classical elastokinematics, it further reduces to

$$
u_j^{(2)} = \frac{\beta^2(\zeta^2 + p^2/c_1^2) X_j^{(2)} - (\beta^2 - 1)\zeta_j \zeta_k X_k^{(2)}}{c_1^2(\zeta^2 + p^2/c_1^2)(\zeta^2 + p^2/c_2^2)},
$$

which agrees with the previously obtained result [22].

4. Effect of suddenly applied body force

In this section we seek a solution of the equations of motion when a body force is applied suddenly at the origin. For such a body force we may write

$$
\begin{aligned}
\mathbf{X} &= \frac{1}{\varrho} \delta(x)\delta(y)\delta(z) H(t)\,(0, 0, F), \\
\mathbf{Y} &= 0, \quad Q = 0.
\end{aligned}
\tag{4.1}
$$

The Fourier-Laplace transform of (4.1) gives

$$\mathbf{X}^{(2)}(\zeta_1, \zeta_2, \zeta_3, p) = \frac{F}{\varrho(2\pi)^{3/2}}\left(0, 0, \frac{1}{p}\right), \tag{4.2}$$
$$\mathbf{Y}^{(2)} = 0, \quad Q^{(2)} = 0.$$

Using (4.2) in (3.12) and then integrating with respect to $\zeta_1, \zeta_2, \zeta_3$, we obtain:

$$\{u_1^{(1)}, u_2^{(1)}\} = \frac{iF}{(2\pi)^3\varrho}\frac{1}{p}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}\left(\frac{1}{c_1^2}I_1 - \frac{1}{c^2}I_2\right), \tag{4.3}$$

$$u_3^{(1)}(x, y, z, p) = \frac{F}{(2\pi)^3\varrho}\frac{1}{p}\left[i\frac{\partial}{\partial z}\left(\frac{1}{c_1^2}I_1 - \frac{1}{c_2^2}I_2\right) + \frac{1}{c_2^2}I_3\right], \tag{4.4}$$

$$\left\{w_1^{(1)}, w_2^{(1)}\right\} = \frac{sF}{p(2\pi)^3\varrho c_2^2}\left\{\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right\}I_4, \tag{4.5}$$

$$w_3^{(1)}(x, y, z, p) = 0, \tag{4.6}$$

$$\theta^{(1)}(x, y, z, p) = \frac{\eta_0 kqF}{c_1^2(2\pi)^3\varrho p}\frac{\partial}{\partial z}I_5, \tag{4.7}$$

where I_1 to I_5 are given by

$$I_1 = \int\int\limits_{-\infty}^{\infty}\int \frac{(\zeta^2+q)\zeta_3}{\zeta^2(\zeta^2-\lambda_1^2)(\zeta^2-\lambda_2^2)}e^{-i(\zeta_1 x+\zeta_2 y+\zeta_3 z)}d\zeta_1 d\zeta_2 d\zeta_3, \tag{4.8}$$

$$I_2 = \int\int\limits_{-\infty}^{\infty}\int \frac{(\zeta^2+2s+\beta_4^2)\zeta_3}{\zeta^2(\zeta^2-\mu_1^2)(\zeta^2-\mu_2^2)}e^{-i(\zeta_1 x+\zeta_2 y+\zeta_3 z)}d\zeta_1 d\zeta_2 d\zeta_3, \tag{4.9}$$

$$I_3 = \int\int\limits_{-\infty}^{\infty}\int \frac{\zeta^2+2s+\beta_4^2}{(\zeta^2-\mu_1^2)(\zeta^2-\mu_2^2)}e^{-i(\zeta_1 x+\zeta_2 y+\zeta_3 z)}d\zeta_1 d\zeta_2 d\zeta_3, \tag{4.10}$$

$$I_4 = \int\int\limits_{-\infty}^{\infty}\int \frac{1}{(\zeta^2-\mu_1^2)(\zeta^2-\mu_2^2)}e^{-i(\zeta_1 x+\zeta_2 y+\zeta_3 z)}d\zeta_1 d\zeta_2 d\zeta_3, \tag{4.11}$$

$$I_5 = \int\int\limits_{-\infty}^{\infty}\int \frac{1}{(\zeta^2-\lambda_1^2)(\zeta^2-\lambda_2^2)}e^{-i(\zeta_1 x+\zeta_2 y+\zeta_3 z)}d\zeta_1 d\zeta_2 d\zeta_3. \tag{4.12}$$

To evaluate the integral I_1, we let

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \zeta_1 = \varrho_1\cos\phi, \quad \text{and} \quad \zeta_2 = \varrho_1\sin\phi,$$

so that I_1 takes the following form:

$$I_1 = \int\limits_{-\infty}^{\infty}\zeta_3 e^{-i\zeta_3 z}d\zeta_3\int\limits_0^{\infty}\int\limits_0^{2\pi}\frac{(\varrho_1^2+\zeta_3^2+q)e^{-i\varrho_1 r\cos(\phi-\theta)}}{(\zeta_3^2+\varrho_1^2)(\varrho_1^2+\zeta_3^2-\lambda_1^2)(\varrho_1^2+\zeta_3^2-\lambda_2^2)}\varrho_1 d\varrho_1 d\phi.$$

Performing the ϕ integration we obtain

$$I_1 = 2\pi \int\limits_0^\infty \varrho_1 J_0(\varrho_1 r) d\varrho_1 \int\limits_{-\infty}^{\infty} \frac{\zeta_3(\varrho_1^2+\zeta_3^2+q)e^{-i\zeta_3 z}d\zeta_3}{(\varrho_1^2+\zeta_3^2)(\varrho_1^2+\zeta_3^2-\lambda_1^2)(\varrho_1^2+\zeta_3^2-\lambda_2^2)}.$$

On performing ζ_3 integration we obtain

$$I_1 = -2\pi i \int\limits_0^\infty \left[\frac{\pi q}{\lambda_1^2\lambda_2^2} e^{-|z|\varrho_1} + \frac{(\lambda_1^2+q)\pi}{\lambda_1^2(\lambda_1^2-\lambda_2^2)} e^{-|z|\sqrt{\varrho_1^2-\lambda_1^2}} - \frac{(\lambda_2^2+q)\pi}{\lambda_2^2(\lambda_2^2-\lambda_2^2)} e^{-|z|\sqrt{\varrho_1^2-\lambda_2^2}}\right] \varrho_1 J_0(\varrho_1 r) d\varrho_1,$$

where $J_n(z)$ is the Bessel function of order n and of the first kind. Using the known integral [23], p. 514, we obtain:

$$I_1 = -(2\pi^2 i)\left[\frac{q}{\lambda_1^2\lambda_2^2} \frac{2\Gamma^{(3/2)}}{\sqrt{\pi}} \frac{|z|r}{(z^2+r^2)^{3/2}} + \frac{\lambda_1^2+q}{\lambda_1^2(\lambda_1^2-\lambda_2^2)} \sqrt{\frac{2}{\pi}} \frac{|z|\,(i\lambda_1)^{3/2}}{(z^2+r^2)^{3/4}} \times \right.$$
$$\left. \times K_{3/2}(i\lambda_1\sqrt{z^2+r^2}) - \frac{\lambda_2^2+q}{\lambda_2^2(\lambda_1^2-\lambda_2^2)} \sqrt{\frac{2}{\pi}} \frac{|z|\,(i\lambda_2)^{3/2}}{(z^2+r^2)^{3/4}} K_{3/4}(i\lambda_2\sqrt{z^2+r^2})\right], \tag{4.13}$$

where $K_n(z)$ is the modified Bessel function of the third kind. Also

$$\frac{\partial I_1}{\partial r} = \frac{2\pi^2 i}{\lambda_1^2\lambda_2^2(\lambda_1^2-\lambda_2^2)}\left[q(\lambda_1^2-\lambda_2^2) \frac{4\Gamma^{(5/2)}r|z|}{\sqrt{\pi}(z^2+r^2)^{5/2}} - \lambda_2^2(\lambda_2^2+q)\sqrt{\frac{2}{\pi}}(i\lambda_1)^{5/2} \frac{|z|r}{(z^2+r^2)^{7/4}} \times \right.$$
$$\left. \times K_{5/2}(i\lambda_1\sqrt{z^2+r^2}) + \lambda_1^2(\lambda_2^2+q)\sqrt{\frac{2}{\pi}}(i\lambda_2)^{5/2} \frac{|z|r}{(z^2+r^2)^{7/4}} K_{5/2}(i\lambda_2\sqrt{z^2+r^2})\right].$$

Integral I_2 can be obtained from I_1 directly by replacing $\lambda_{1,2}$ by $\mu_{1,2}$ and q by $2s+\beta_4^2$. Adopting the above procedure we obtain

$$I_3 = (2\pi^2)\sqrt{\frac{2}{\pi}}(i\mu_1)^{1/2}(z^2+r^2)^{-1/4}K_{1/2}(i\mu_1\sqrt{z^2+r^2}), \tag{4.14}$$

$$I_4 = -\frac{(2\pi^2)}{\mu_1^2-\mu_2^2}\sqrt{\frac{2}{\pi}}\left[(i\mu_1)^{1/2}(z^2+r^2)^{-1/4}K_{1/2}(i\mu_1\sqrt{z^2+r^2}) - (i\mu_2)^{1/2}(z^2+r^2)^{-1/4}K_{1/2}(i\mu_2\sqrt{z^2+r^2})\right]. \tag{4.15}$$

Integral I_5 can be obtained from I_4 by replacing $\mu_{1,2}$ by $\lambda_{1,2}$. Now, we have

$$u_1^{(1)}(x,y,z,p) = -\frac{F}{4\pi\varrho p}\frac{x}{r}\left[\frac{1}{c_1^2}\frac{q}{\lambda_1^2-\lambda_2^2}\frac{4\Gamma^{5/2}r|z|}{\sqrt{\pi}(z^2+r^2)^{5/2}} - \frac{\lambda_1^2+q}{\lambda_1^2(\lambda_1^2-\lambda_2^2)} \times \right.$$
$$\times \sqrt{\frac{2}{\pi}}\frac{(i\lambda_1)^{5/2}r|z|}{c_1^2(z^2+r^2)^{7/4}}K_{5/2}(i\lambda_1\sqrt{z^2+r^2}) + \frac{\lambda_2^2+q}{c_1^2\lambda_2^2(\lambda_1^2-\lambda_2^2)}\sqrt{\frac{2}{\pi}}\frac{(i\lambda_1)^{5/2}r|z|}{(z^2+r^2)^{7/4}} \times$$
$$\times K_{5/2}(i\lambda_2\sqrt{z^2+r^2}) + \frac{1}{c_2^2}\frac{2s+\beta_4^2}{\mu_1^2-\mu_2^2}\frac{4\Gamma^{(5/2)}r|z|}{\sqrt{\pi}(z^2+r^2)^{5/2}} - \frac{\mu_1^2+2s+\beta_4^2}{\mu_1^2(\mu_1^2-\mu_2^2)}\sqrt{\frac{2}{\pi}} \times$$
$$\times \frac{(i\mu_1)^{5/2}r|z|}{\pi c_2^2(z^2+r^2)^{7/4}}K_{5/2}(i\mu_1\sqrt{z^2+r^2}) + \frac{\mu_2^2+2s+\beta_4^2}{\mu_2^2(\mu_1^2-\mu_2^2)}\sqrt{\frac{2}{\pi}}\frac{(i\mu_2)^{5/2}r|z|}{c_2^2(z^2+r^2)^{7/4}} \times$$
$$\left. \times K^{5/2}(i\mu_2\sqrt{z^2+r^2})\right], \tag{4.16}$$

$$u_2^{(1)}(x,y,z,p) = \frac{y}{x} u_1^{(1)}(x,y,z,p), \tag{4.17}$$

$$u_3^{(1)}(x,y,z,p) = \frac{r}{x}\frac{\partial}{\partial z} u_1^{(1)}(x,y,z,p) - \frac{1}{pc_2^2}\sqrt{\frac{2}{\pi}} \times \\ \times (i\mu_1)^{1/2}(z^2+r^2)^{-1/4} K_{1/2}\left(i\mu_1\sqrt{z^2+r^2}\right), \tag{4.18}$$

$$w_1^{(1)}(x,y,z,p) = -\frac{Fs}{4\pi\varrho c_2^2(\mu_1^2-\mu_2^2)p}\frac{y}{r}\sqrt{\frac{2}{\pi}}\left[(i\mu_1)^{1/2}(z^2+r^2)^{-1/4} \times \right. \\ \left. \times K_{1/2}\left(i\mu_1\sqrt{z^2+r^2}\right) - (i\mu_2)^{1/2}(z^2+r^2)^{-1/4} K_{1/2}\left(i\mu_2\sqrt{z^2+r^2}\right)\right], \tag{4.19}$$

$$w_2^{(1)}(x,y,z,p) = -\frac{x}{y} w_1^{(1)}(x,y,z,p), \tag{4.20}$$

$$w_3^{(1)}(x,y,z,p) = 0, \tag{4.21}$$

$$\theta^{(1)}(x,y,z,p) = \frac{\eta_0 F\sqrt{2}}{c_1^2 4\pi\varrho\sqrt{\pi}(\lambda_1^2-\lambda_2^2)p}\frac{\partial}{\partial z}\left[(i\lambda_1)^{1/2}(z^2+r^2)^{-1/4}K_{1/2}\left(i\lambda_1\sqrt{z^2+r^2}\right)\right. \\ \left. -(i\lambda_2)^{1/2}(z^2+r^2)^{1/4}K_{1/2}\left(i\lambda_2\sqrt{z^2+r^2}\right)\right]. \tag{4.22}$$

It is a formidable task to obtain the inverse Laplace transforms to the above set of quantities. For this reason we have resorted to the case of small time approximations; first, we note that

$$2\lambda_{1,2}^2 = -\left(\frac{p^2}{c_1^2} + \frac{p}{k}(1+\omega)\right) \pm \sqrt{\left[\frac{p^2}{c_1^2}+\frac{p}{k}(1+\omega)\right]^2 - \frac{4p^3}{c_1^2 k}}, \\ 2\mu_{1,2}^2 = -(2s+\beta_4^2+\beta_2^2+rs^*) \pm \sqrt{(2s+\beta_4^2+\beta_2^2-rs^*)-4\beta_2^2(2s+\beta_4^2)}, \tag{4.23}$$

then, expanding these binomially and retaining only the necessary terms, we obtain

$$\lambda_1 \approx \frac{i}{c_1}\sqrt{\frac{ap}{2}}, \qquad \lambda_2 \approx p + \frac{id}{4c_1}, \tag{4.24}$$

$$\mu_1 \approx \frac{ip}{c_4}, \qquad \mu_2 \approx \frac{ip}{c_2}, \tag{4.25}$$

in which only positive roots $\lambda_{1,2}$ and $\mu_{1,2}$ are considered owing to the regularity conditions at infinity and

$$a = \frac{c_1^2}{k}(1+\omega) - \frac{\omega-1}{k},$$

$$d = \frac{c_1^2}{k}(1+\omega) + \frac{\omega-1}{k}.$$

The modified Bessel functions that appear in the above expressions can be expressed in terms of a series by the relation [22]:

$$K_{n+\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}\frac{1}{(2z)}k. \tag{4.26}$$

To obtain the Laplace inverse of the expressions (4.16) to (4.22), we expand the expressions in the inverse powers of p and retain the terms up to the order p^{-3}. Thus on inversion we obtain the following for the displacement, rotation and temperature field for the case of suddenly applied force [24]:

$$
\begin{aligned}
(4.27)\qquad u_1(x,y,z,t)\frac{4\pi\varrho}{F} = & -\frac{x|z|}{h^{5/2}}\frac{4\Gamma(5/2)}{\sqrt{\pi}}\left[\left(\frac{c_1^2}{ak}+\frac{1}{c_2^2}\frac{t^2}{2}\right)\right] \\
& -l\frac{x|z|}{c_1^2h^2}\left[\left\{\left(t+\frac{1}{4}\frac{ah}{c_1^2}\right)\operatorname{Erfc}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2}}t^{-1/2}\right)-\frac{1}{\sqrt{\pi}}\frac{1}{c_1}\sqrt{\frac{ah}{2}}t^{1/2}e^{-\frac{ah}{8c_1^2t}}\right\}\right. \\
& +\frac{4c_1t^{3/2}}{\sqrt{\frac{a}{2}}\sqrt{h}}e^{-\frac{ah}{16c_1^2t}}D_{-4}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2}}t^{-1/2}\right)+\left(\frac{16c_1^2}{ah}-\frac{d}{2}+\frac{a}{2}\right)\times \\
& \left.\times\frac{2^{5/2}t^2}{\sqrt{\pi}}e^{-\frac{ah}{16c_1^2t}}D_{-5}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2}}t^{-1/2}\right)\right] \\
& -\frac{|z|x}{c_1^2h^2}e^{-\frac{d\sqrt{h}}{4c_1}}\left[\left(1+\frac{3c_1}{\sqrt{h}}\right)H\left(t-\frac{\sqrt{h}}{c_1}\right)-l\left(1+\frac{3c_1}{\sqrt{h}}\right)A(t)\right. \\
& \left.+\left(\frac{3c_1}{h}-\frac{3dc_1}{a\sqrt{h}}+\frac{3mc_1}{\sqrt{h}}+m\right)B(t)\right]-\frac{|z|x}{c_2^2h^2}\left[H\left(t-\frac{\sqrt{h}}{c_2}\right)\right. \\
& \left.-\frac{3c_2}{h}A^*(t)+\left(\frac{3c_2^2}{h}+\frac{2sc_2^2c_4^2}{c_2^2-c_4^2}\right)B^*(t)\right]+\frac{2sc_4^2}{c_2^2-c_4^2}\frac{|z|x}{h^2}B^{**}(t),
\end{aligned}
$$

$$
(4.28)\qquad u_2(x,y,z,t)=\frac{y}{x}u_1(y,x,z,t),
$$

$$
\begin{aligned}
(4.29)\qquad u_3(x,y,z,t)\frac{4\pi\varrho}{F} = & \frac{2r\Gamma(3/2)}{\sqrt{\pi}}\left(\frac{z}{|z|h^{3/2}}-\frac{3z|z|}{h^{5/2}}\right)\left(1+\frac{2c_1^2}{ak}\right)\frac{t^2}{2} \\
& +l\left[\frac{z|z|}{c_1^2h^{3/2}}\left\{\left(t+\frac{ah}{4c_1^2}\right)\operatorname{Erfc}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2}}t^{-1/2}\right)-\frac{1}{\sqrt{\pi}}\frac{1}{c_1}\sqrt{\frac{ah}{2}}t^{1/2}\times\right.\right. \\
& \left.\times e^{-\frac{ah}{8c_1^2t}}\right\}-\frac{1}{c_1}\left(\frac{z}{|z|h}-\frac{3z|z|}{h^2}\right)\frac{4t^{3/2}}{\sqrt{\pi}}e^{-\frac{ah}{16c_1^2t}}D_{-4}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2}}t^{-1/2}\right) \\
& \left.-\left(\frac{d-a}{2c_1^2}\frac{z|z|}{h^{3/2}}+\frac{2z}{a|z|h^{3/2}}-\frac{6z|z|}{ah^{5/2}}\right)\frac{2^{5/2}t^2}{\sqrt{\pi}}e^{-\frac{ah}{16c_1^2t}}D_{-5}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2}}t^{-1/2}\right)\right] \\
& +e^{-\frac{d\sqrt{h}}{4c_1}}\left[\left(\frac{1}{c_1^2}\frac{z|z|}{h^{3/2}}-\frac{z}{c_1|z|h}+\frac{3z|z|}{c_1h^2}\right)H\left(t-\frac{\sqrt{h}}{c_1}\right)\right. \\
& +\left(\frac{lz}{c_1|z|h}-\frac{3lz|z|}{c_1h^2}-\frac{lz|z|}{c_1^2h^{3/2}}-\frac{z}{|z|h^{3/2}}+\frac{3z|z|}{h^{3/2}}\right)A(t)
\end{aligned}
$$

$$+\left\{\frac{d-4m}{4c_1}\left(\frac{z}{|z|h}-\frac{3z|z|}{h^2}\right)+\frac{m}{c_1^2}\frac{z|z|}{h^{3/2}}\right\}B(t)\Bigg]$$

$$+\left[\left(\frac{1}{c_2^2}\frac{z|z|}{h^{3/2}}+\frac{1}{c_2^2\sqrt{h}}\right)H\left(t-\frac{\sqrt{h}}{c_2}\right)-\frac{1}{c_2}\left(\frac{z}{|z|h}-\frac{3z|z|}{h^2}\right)A^*(t)\right.$$

$$\left.+\left(\frac{2sc_4^2}{c_2^2-c_4^2}\frac{z|z|}{h^{3/2}}+\frac{2sc_4^2}{c_2^2-c_4^2}\frac{1}{\sqrt{h}}-\frac{z}{|z|h^{3/2}}-\frac{3z|z|}{h^{5/2}}\right)B^*(t)\right]-\frac{2sc_4^2}{c_2^2-c_4^2}\left(\frac{1}{\sqrt{h}}+\frac{z|z|}{h^{3/2}}\right)B^{**}(t),$$

$$w_1(x,y,z,t)\frac{4\pi\varrho c_2(c_2^2-c_4^2)}{c_4Fs}=\frac{-yc_4}{h^{7/4}}\left[A^*(t)+\frac{c_2}{\sqrt{h}}B^*(t)\right]+\frac{yc_2}{h^{7/4}}\left[A^{**}(t)+\frac{c_4}{\sqrt{h}}B^{**}(t)\right], \tag{4.30}$$

$$w_2(x,y,z,t)=-\frac{x}{y}w_1(x,y,z,t), \tag{4.31}$$

$$w_3(x,y,z,t)\equiv 0, \tag{4.32}$$

$$\theta(x,y,z,t)\frac{4\pi}{\eta_0F}=-\left[\frac{\sqrt{a}}{\sqrt{2}c_1}\frac{z}{h}\left(\frac{2t^{1/2}}{\sqrt{\pi}}e^{-\frac{ah}{8c_1^2t}}-\sqrt{\frac{ah}{2}}\operatorname{Erfc}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2}}t^{-1/2}\right)\right)\right. \tag{4.33}$$

$$+\frac{2(a-d)}{\sqrt{\pi}}t^{3/2}e^{-\frac{gh}{16c_1^2t}}D_{-4}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2}}t^{-1/2}\right)$$

$$+\frac{z}{h^{3/2}}\left\{\left(t+\frac{ah}{4c_1^2}\right)\operatorname{Erfc}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2}}t^{-1/2}\right)-\frac{1}{\sqrt{\pi}}\frac{1}{c_1}\sqrt{\frac{ah}{2}}t^{1/2}e^{-\frac{ah}{8c_1^2t}}\right\}$$

$$\left.+\frac{z}{h^{3/2}}\frac{a-d}{2}\frac{2^{5/2}t^2}{\sqrt{\pi}}e^{-\frac{ah}{16c_1^2t}}D_{-5}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2}}t^{-1/2}\right)\right]$$

$$-e^{-\frac{d\sqrt{h}}{4c_1}}\left[\frac{1}{c_1}\frac{z}{h}H\left(t-\frac{\sqrt{h}}{c_1}\right)+\left(\frac{z}{c_1h}\left(\frac{a}{2}-\frac{d}{4}\right)+\frac{z}{h^{3/2}}\right)A(t)\right.$$

$$\left.+\left\{\frac{z}{c_1h}\left(\frac{(a-d)^2}{4}-\frac{3d^2}{16}+\frac{ad}{8}\right)+\frac{z(a-d)}{2h^{3/2}}\right\}B(t)\right].$$

The corresponding classical solutions can be derived for this case by putting $\alpha=\beta=$ $=\gamma=\varepsilon=0$ and ignoring w_1, w_2 and w_3. Thus

$$u_1^c(x,y,z,t)\frac{4\pi\varrho}{F}=-\frac{x|z|}{h^{5/2}}\frac{4\Gamma^{(5/2)}}{\sqrt{\pi}}\left[\left(\frac{c_1^2}{ak}+\frac{1}{c_2^{*2}}\right)\frac{t^2}{2}\right] \tag{4.34}$$

$$-l\frac{x|z|}{c_1^2h^2}\left[\left\{\left(t+\frac{ah}{4c_1^2}\right)\operatorname{Erfc}\left(\frac{1}{2c_1}\sqrt{\frac{a}{2}}\sqrt{h}t^{-1/2}\right)-\frac{1}{c_1}\sqrt{\frac{ah}{2\pi}}t^{1/2}e^{-\frac{ah}{8c_1^2t}}\right\}\right.$$

$$+\frac{4c_1 t^{3/2}}{\sqrt{\frac{a}{2}}\sqrt{h}} e^{-\frac{ah}{16c_1^2 t}} D_{-4}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2}}\, t^{-1/2}\right)+\left(\frac{16c_1^2}{ah}-\frac{d}{2}+\frac{a}{2}\right)\times$$

$$\times\frac{2^{5/2}t^2}{\sqrt{\pi}} e^{-\frac{ah}{16c_1^2 t}} D_{-5}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2}}\, t^{-1/2}\right)\Bigg]-\frac{x|z|}{c_1^2 h^2} e^{-\frac{d\sqrt{h}}{4c_1}}\Bigg[\left(1+\frac{3c_1}{\sqrt{h}}\right) H\left(t-\frac{\sqrt{h}}{c_1}\right)$$

$$-l\left(1+\frac{3c_1}{\sqrt{h}}\right) A(t)+\left(\frac{3c_1}{h}-\frac{3dc_1}{a\sqrt{h}}+\frac{3mc_1}{\sqrt{h}}+m\right) B(t)\Bigg]-\frac{x|z|}{c_2^{*2} h^2}\Bigg[H\left(t-\frac{\sqrt{h}}{c_2^*}\right)$$

$$-\frac{3c_2^*}{\sqrt{h}} A_c^*(t)+\frac{3c_2^{*2}}{h} B_c^*(t)\Bigg],$$

(4.35) $$u_2^c(x, y, z, t) = u_1^c(y, x, z, t),$$

(4.36) $$u_3^c(x, y, z, t)\frac{4\pi\varrho}{F} = \frac{2r\Gamma^{(3/2)}}{\sqrt{\pi}}\left(\frac{z}{|z|h^{3/2}}-\frac{3z|z|}{h^{5/2}}\right)\left(1+\frac{2c_1^2}{ak}\right)\frac{t^2}{2}$$

$$+l\Bigg[\frac{z|z|}{c_1^2 h^{3/2}}\Bigg\{\left(t+\frac{ah}{4c_1^2}\right)\operatorname{Erfc}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2}}\, t^{-1/2}\right)-\frac{1}{\sqrt{\pi}}\frac{1}{c_1}\sqrt{\frac{ah}{2}}\, t^{1/2} e^{-\frac{ah}{8c_1^2 t}}\Bigg\}$$

$$-\frac{1}{c_1}\left(\frac{z}{|z|h}-\frac{3z|z|}{h^2}\right)\frac{4t^{3/2}}{\sqrt{\pi}} e^{-\frac{ah}{16c_1^2 t}} D_{-4}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2}}\, t^{-1/2}\right)$$

$$-\left(\frac{d-a}{2c_1^2}\frac{z|z|}{h^{3/2}}+\frac{2z}{a|z|h^{3/2}}-\frac{6z|z|}{ah^{5/2}}\right)\frac{2^{5/2}t^2}{\sqrt{\pi}} e^{-\frac{ah}{16c_1^2 t}} D_{-5}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2}}\, t^{-1/2}\right)\Bigg]$$

$$+e^{-\frac{d\sqrt{h}}{4c_1}}\Bigg[\left(\frac{1}{c_1^2}\frac{z|z|}{h^{3/2}}-\frac{z}{c_1|z|h}+\frac{3z|z|}{c_1 h^2}\right) H\left(t-\frac{\sqrt{h}}{c_1}\right)+\left(\frac{lz}{c_1|z|h}-\frac{3lz|z|}{c_1 h^2}-\frac{lz|z|}{c_1^2 h^{3/2}}\right.$$

$$\left.-\frac{z}{|z|h^{3/2}}+\frac{3z|z|}{h^{5/2}}\right) A(t)+\left\{\frac{d-4m}{4c_1}\left(\frac{z}{|z|h}-\frac{3z|z|}{h^2}\right)+\frac{m}{c_1^2}\frac{z|z|}{h^{3/2}}\right\} B(t)\Bigg]+\Bigg[\left(\frac{1}{c_2^*}\frac{z|z|}{h^{3/2}}\right.$$

$$\left.+\frac{1}{c_2^{*2}\sqrt{h}}\right) H\left(t-\frac{h}{c_2^*}\right)-\frac{1}{c_2^*}\left(\frac{z}{|z|h}-\frac{3z|z|}{h^2}\right) A_c^*(t)-\left(\frac{z}{|z|h^{3/2}}+\frac{3z|z|}{h^{5/2}}\right) B_{c.}^*(t)\Bigg]$$

and the temperature θ for this case is the same as that given by (4.33), where

(4.37) $$A(t) = \begin{cases} 0, & \text{for } 0 < t < \dfrac{\sqrt{h}}{c_1}, \\ t-\dfrac{\sqrt{h}}{c_1}, & \text{for } t > \dfrac{\sqrt{h}}{c_1}, \end{cases}$$

(4.38) $$B(t) = \begin{cases} 0, & \text{for } 0 < t < \dfrac{\sqrt{h}}{c_1}, \\ \dfrac{1}{\Gamma^{(3)}}\left(t-\dfrac{\sqrt{h}}{c_1}\right)^2, & \text{for } t > \dfrac{\sqrt{h}}{c_1}, \end{cases}$$

$A^*(t)$, $B^*(t)$ are obtained from (4.50) and (4.51), respectively, by replacing c_1 by c_2. Similarly $A^{**}(t)$, $B^{**}(t)$ are obtained from these equations by replacing c_1 by c_4; H is the Heaviside step function and Erfc is the complementary error function, and

$$A_c^*(t) = \begin{cases} 0, & \text{for} \quad 0 < t < \dfrac{\sqrt{h}}{c_2^*}, \\[2ex] t - \dfrac{\sqrt{h}}{c_2^*}, & \text{for} \quad t > \dfrac{\sqrt{h}}{c_2^*}, \end{cases}$$

$$B_c^*(t) = \begin{cases} 0, & \text{for} \quad 0 < t < \dfrac{\sqrt{h}}{c_2^*}, \\[2ex] \dfrac{1}{\Gamma^{(3)}} \left(t - \dfrac{h}{c_2^*}\right)^2, & \text{for} \quad t > \dfrac{\sqrt{h}}{c_2^*}, \end{cases}$$

(4.41)
$$l = \frac{c_1^2}{k} - \frac{a}{2}, \quad h = x^2+y^2+z^2,$$
$$m = \frac{a^2}{2} - \frac{ad}{4} - \frac{3d^2}{8} + \frac{c_1^2}{2k}(d-a), \quad c_2^{*2} = \frac{\mu}{\varrho}$$

and

(4.42)
$$D_\nu(\tau) = 2^{\frac{1}{2}\nu+\frac{1}{4}} \tau^{-\frac{1}{2}} W_{\frac{1}{2}\nu+\frac{1}{4}, \frac{1}{4}} \left(\frac{1}{2}\tau^2\right),$$

(4.43)
$$W_{k,\mu}(\tau) = \frac{\Gamma(-2\mu) M_{k,\mu}(\tau)}{\Gamma\left(\frac{1}{2} - \mu - k\right)} + \frac{\Gamma(2\mu) M_{k,-\mu}(\tau)}{\Gamma\left(\frac{1}{2} + \mu - k\right)},$$

(4.44)
$$M_{k,\mu}(\tau) = \tau^{\frac{1}{2}+\mu} e^{-\frac{1}{2}\tau} {}_1F_1\left(\frac{1}{2} + \mu - k; 2\mu+1; \tau\right),$$

(4.45)
$${}_1F_1(a; b; \tau) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(b+n)} \frac{\tau^n}{n!},$$

is the confluence hypergeometric function.

5. Effect of suddenly applied body couple

In this section we consider the case where body couple is applied suddenly at the origin

(5.1)
$$\mathbf{Y}(x, y, z, t) = \frac{M}{J} \delta(x)\delta(y)\delta(z) H(t) (0, 0, 1),$$
$$\mathbf{X} = 0, \quad Q = 0,$$

and thus

(5.2)
$$\mathbf{Y}^{(2)}(\zeta_1, \zeta_2, \zeta_3, p) = \frac{M}{(2\pi)^{3/2} J} \frac{1}{p} (0, 0, 1),$$
$$\mathbf{X}^{(2)} = 0, \quad Q^{(2)} = 0.$$

Substituting (5.2) into (3.12) and integrating with respect to $\zeta_1, \zeta_2, \zeta_3$ we obtain

$$(5.3)\qquad [u_1^{(1)}(x, y, z, p), u_2^{(1)}(x, y, z, p)] \frac{c_4^2(2\pi)^3 J}{Mr^*} = \frac{1}{p}\left[\frac{\partial}{\partial y} I_4 - \frac{\partial}{\partial x} I_4\right],$$

$$(5.4)\qquad [w_1^{(1)}(x, y, z, p), w_2^{(1)}(x, y, z, p)] \frac{(2\pi)^3 J}{M} = \frac{1}{p}\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\left(\frac{1}{c_3^2} I_6 - \frac{1}{c_4^2} I_7\right)\right],$$

$$(5.5)\qquad w_3^{(1)}(x, y, z, p) \frac{(2\pi)^3 J}{M} = \frac{1}{p}\frac{\partial}{\partial z}\left[\frac{1}{c_3^2} I_6 - \frac{1}{c_4^2} I_7\right] + \frac{1}{p}\frac{1}{c_4^2} I_8,$$

$$(5.6)\qquad u_3^{(1)} = 0, \quad \theta^{(1)} = 0,$$

where I_4 is given by (4.11) and

$$(5.7)\qquad I_6 = \int\int_{-\infty}^{\infty}\int \frac{i\zeta_3}{\zeta^2(\zeta^2+\beta_3^2)} e^{-i(\zeta_1 x+\zeta_2 y+\zeta_3 z)} d\zeta_1 d\zeta_2 d\zeta_3,$$

$$(5.8)\qquad I_7 = \int\int_{-\infty}^{\infty}\int \frac{i\zeta_3(\zeta^2+\beta_2^2)}{\zeta^2(\zeta^2-\mu_1^2)(\zeta^2-\mu_2^2)} e^{-i(\zeta_1 x+\zeta_2 y+\zeta_3 z)} d\zeta_1 d\zeta_2 d\zeta_3,$$

$$(5.9)\qquad I_8 = \int\int_{-\infty}^{\infty}\int \frac{\zeta^2+\beta_2^2}{(\zeta^2-\mu_1^2)(\zeta^2-\mu_2^2)} e^{-i(\zeta_1 x+\zeta_2 y+\zeta_3 z)} d\zeta_1 d\zeta_2 d\zeta_3,$$

in which $\mu_{1,2}^2$ are the roots of the equation $\Delta_2(\zeta^2) = 0$. The value of the integral I_4 is already known from the previous section and, after evaluating the remaining integrals, we find

$$(5.10)\qquad I_6 = \frac{2\pi^2}{\beta_3^2}\left[\frac{2}{\sqrt{\pi}}\Gamma^{(3/2)}\frac{|z|}{h^{3/2}} - \sqrt{\frac{2}{\pi}}\frac{\beta_3^{3/2}}{h^{3/4}} K_{3/2}(\beta_3\sqrt{h})\right],$$

$$(5.11)\qquad \frac{\partial I_6}{\partial r} = -\frac{2\pi^2}{\beta_3^2}\left[\frac{4\Gamma^{(5/3)} r|z|}{\sqrt{\pi} h^{5/2}} - \sqrt{\frac{2}{\pi}}\frac{|z| r\beta_3^{5/2}}{h^{7/4}} K_{5/2}(\beta_3\sqrt{h})\right],$$

$$(5.12)\qquad I_7 = 2\pi^2\left[\frac{\beta_3^2}{\mu_1^2\mu_2^2}\frac{2\Gamma^{3/2} r|z|}{\sqrt{\pi} h^{3/2}} \frac{\mu_1^2+\beta_3^2}{\mu_1^2(\mu_1^2-\mu_2^2)}\left(\frac{\pi}{2}\right)^{-1/2}\frac{|z|(i\mu_1)^{3/2}}{h^{3/4}} K_{3/2}(i\mu_1\sqrt{h})\right.$$
$$\left. - \frac{\mu_2^2+\beta_3^2}{\mu_2^2(\mu_1^2-\mu_2^2)}\left(\frac{\pi}{2}\right)^{-1/2}\frac{|z|(i\mu_2)^{3/2}}{h^{3/4}} K^{3/2}(i\mu_2\sqrt{h})\right],$$

$$(5.13)\qquad \frac{\partial I_7}{\partial r} = -\frac{2\pi^2}{\mu_1^2\mu_2^2(\mu_1^2-\mu_2^2)}\left[\beta_3^2(\mu_1^2-\mu_2^2)\frac{4\Gamma^{(5/2)} r|z|}{\sqrt{\pi} h^{5/2}} - \mu_2^2(\mu_1^2+\beta_3^2)\left(\frac{\pi}{2}\right)^{-1/2} \times\right.$$
$$\left. \times (i\mu_1)^{5/2}\frac{r|z|}{h^{7/4}} K_{5/2}(i\mu_1\sqrt{h}) + \mu_1^2(\mu_2^2+\beta_3^2)\left(\frac{\pi}{2}\right)^{-1/2}(i\mu_2)^{5/2}\frac{r|z|}{h^{7/4}} K_{5/2}(i\mu_2\sqrt{h})\right],$$

$$I_8 = 2\pi^2\left[\frac{\mu_1^2+\beta_2^2}{\mu_1^2-\mu_2^2}\sqrt{\frac{2}{\pi}}\,\frac{(i\mu_1)^{1/2}}{h^{1/4}}K_{1/2}(i\mu_1\sqrt{h}) - \frac{\mu_2^2+\beta_2^2}{\mu_1^2-\mu_2^2}\sqrt{\frac{2}{\pi}}\,\frac{(i\mu_2)^{1/2}}{h^{1/4}}K_{1/2}(i\mu_2\sqrt{h})\right], \tag{5.14}$$

where $x = r\cos\theta$, $y = r\sin\theta$, $h = r^2+z^2$.

The method used in evaluating the above integrals is similar to what has been adopted in the previous section.

Converting the modified Bessel functions occuring in the above expressions into exponential form by the use of the formula (4.26) and making use of the relations (4.24), (4.25), the above expressions reduce to

$$u_1^{(1)}(x,y,z,p)\frac{4\pi J c_4(c_2^2-c_4^2)}{r^* M c_2} = -\frac{yc_4}{h^{7/4}}\left[1+\frac{c_2}{\sqrt{h}p}\right]\frac{e^{-\frac{p\sqrt{h}}{c_2}}}{p^2} + \frac{yc_2}{h^{7/4}}\left[1+\frac{c_4}{\sqrt{h}p}\right]\frac{e^{-\frac{p\sqrt{h}}{c_4}}}{p^2}, \tag{5.15}$$

$$u_2^{(1)}(x,y,z,p) = -\frac{x}{y}u_1^{(1)}(x,y,z,p), \tag{5.16}$$

$$\begin{aligned} w_1^{(1)}(x,y,z,p)\frac{4\pi J}{M} &= \frac{x}{r}\left(\frac{2\Gamma^{(3/2)}}{c_3\sqrt{\pi}}\frac{|z|}{h^{3/2}} - \frac{4\Gamma^{(5/2)}c_2^2}{c_3\sqrt{\pi}}\frac{r|z|}{h^{5/2}}\right)\frac{1}{p^3} + \frac{c_3^2-c_2^2}{c_3c_4^2(c_2^2-c_4^2)}\times \\ &\times\frac{x|z|}{h^2}\left(1+\frac{3c_2}{\sqrt{h}}\frac{1}{p}+\frac{3c_2^2}{h}\frac{1}{p^2}\right)\frac{e^{-\frac{p\sqrt{h}}{c_2}}}{p} - \frac{x|z|}{rc_3h}\left(\frac{1}{c_3^{1/2}}\frac{1}{p}+\frac{1}{\sqrt{h}}\frac{1}{p^2}\right)\frac{e^{-\frac{p\sqrt{h}}{\sqrt{c_3}}}}{p} \\ &- \frac{c_3^2-c_4^2}{c_2^2c_4^2c_3(c_2^2-c_4^2)}\frac{x|z|}{h^2}\left(1+\frac{3c_2}{\sqrt{h}}\frac{1}{p}+\frac{3c_4^2}{h}\frac{1}{p^2}\right)\frac{e^{-\frac{p\sqrt{h}}{c_3}}}{p}, \end{aligned} \tag{5.17}$$

$$w_2^{(1)}(x,y,z,p) = \frac{y}{x}w_1^{(1)}(x,y,z,p), \tag{5.18}$$

$$\begin{aligned} w_3^{(1)}(x,y,z,p)\frac{4\pi J}{M} &= \frac{2\Gamma^{(3/2)}}{c_3\sqrt{\pi}}\left[\frac{z}{|z|h^{3/2}} - \frac{3z|z|}{h^{5/2}} + \frac{c_2^2rz}{|z|h^{3/2}} - \frac{3c_2^2rz|z|}{h^{5/2}}\right]\frac{1}{p^3} \\ &+ \frac{c_2(c_3-c_2^2)}{c_3c_4^2(c_2^2-c_4^2)}\left[\left(\frac{z}{|z|h}-\frac{2z|z|}{h^2}\right)\frac{1}{p} + c_2\left(\frac{z}{|z|h^{3/2}}-\frac{3z|z|}{h^{5/2}}\right)\frac{1}{p^2}\right]\frac{e^{-\frac{p\sqrt{h}}{c_2}}}{p} \\ &- \left[\frac{1}{c_3^{3/2}}\left(\frac{z}{|z|h}-\frac{|z|}{h^2}\right)\frac{1}{p} + \frac{1}{c_3}\left(\frac{z}{|z|h^{3/2}}-\frac{3z|z|}{h^{5/2}}\right)\frac{1}{p^2}\right]\frac{e^{-\frac{p\sqrt{h}}{\sqrt{c_3}}}}{p} \\ &- \left[\frac{1}{c_4^2\sqrt{h}} - \frac{c_3-c_4^2}{c_2^2c_3c_4(c_2^2-c_4^2)}\left\{\left(\frac{z}{|z|h}-\frac{3z|z|}{h^2}\right)\frac{1}{p} + c_4\left(\frac{z}{|z|h^{3/2}}-\frac{3z|z|}{h^{5/2}}\right)\frac{1}{p^2}\right\}\right]\frac{e^{-\frac{p\sqrt{h}}{c_4}}}{p}, \end{aligned} \tag{5.19}$$

$$u_3^{(1)} \equiv 0, \quad \theta^{(1)} \equiv 0. \tag{5.20}$$

The above expressions are quite suitable for the Laplace inversion. Thus on inverting we obtain

$$(5.21)\quad u_1(x,y,z,t)\frac{4\pi J c_4(c_2^2-c_4^2)}{r^* M c_2} = -\frac{y c_4}{h^{7/4}}\left[A^*(t)+\frac{c_2}{\sqrt{h}}B^*(t)\right] + \frac{y c_2}{h^{7/4}}\left[A^{**}(t)+\frac{c_4}{\sqrt{h}}B^{**}(t)\right],$$

$$(5.22)\quad u_2(x,y,z,t) = -\frac{x}{y}u_1(x,y,z,t),$$

$$(5.23)\quad w_1(x,y,z,t)\frac{4\pi J}{M} = \frac{x}{r}\left(\frac{\Gamma^{(3/2)}}{c_3\sqrt{\pi}}\frac{|z|}{h^{3/2}} - \frac{2\Gamma^{(5/2)}c_2^2}{c_3\sqrt{\pi}}\frac{r|z|}{h^{5/2}}\right)t^2 + \frac{c_3^2-c_2^2}{c_3c_4^2(c_3^2-c_4^2)}\times$$
$$\times\frac{x|z|}{h^2}\left(H\left(t-\frac{\sqrt{h}}{c_2}\right)+\frac{3c_2}{\sqrt{h}}A^*(t)+\frac{3c_2}{h}B^*(t)\right) - \frac{x|z|}{rc_3h}\left(\frac{1}{c_3^{1/2}}F(t)+\frac{1}{\sqrt{h}}G(t)\right)$$
$$-\frac{c_3^2-c_4^2}{c_2^2c_4^2(c_2^2-c_4^2)}\frac{x|z|}{h^2}\left(H\left(t-\frac{\sqrt{h}}{c_4}\right)+\frac{3c_2}{\sqrt{h}}A^{**}(t)+\frac{3c_4^2}{h}B^{**}(t)\right),$$

$$(5.24)\quad w_2(x,y,z,t) = \frac{y}{x}w_1(y,x,z,t),$$

$$(5.25)\quad w_3(x,y,z,t)\frac{4\pi J}{M} = \frac{2\Gamma^{(3/2)}}{c_3\sqrt{\pi}}\left[\frac{z}{|z|h^{3/2}} - \frac{3z|z|}{h^{5/2}} + \frac{c_2^2rz}{|z|h^{3/2}} - \frac{3c_2^2rz|z|}{h^{5/2}}\right]\frac{t^2}{2}$$
$$+\frac{c_2(c_3-c_2^2)}{c_3c_4^2(c_2^2-c_4^2)}\left[\left(\frac{z}{|z|h}-\frac{2z|z|}{h^2}\right)A^*(t)+c_2\left(\frac{z}{|z|h^{3/2}}-\frac{3z|z|}{h^{5/2}}\right)B^*(t)\right]$$
$$-\left[\frac{1}{c_3^{3/2}}\left(\frac{z}{|z|h}-\frac{|z|}{h^2}\right)F(t)+\frac{1}{c_3}\left(\frac{z}{|z|h^{3/2}}-\frac{3z|z|}{h^{5/2}}\right)G(t)\right]-\left[\frac{1}{c_4^2\sqrt{h}}H\left(t-\frac{\sqrt{h}}{c_4}\right)\right.$$
$$\left.-\frac{c_3-c_4^2}{c_2^2c_3c_4(c_2^2-c_4^2)}\left\{\left(\frac{z}{|z|h}-\frac{3z|z|}{h^2}\right)A^{**}(t)+c_4\left(\frac{z}{|z|h^{3/2}}-\frac{3z|z|}{h^{5/2}}\right)B^{**}(t)\right]\right],$$

$$(5.26)\quad u_3(x,y,z,t) = 0,\quad \theta(x,y,z,t) = 0,$$

where

$$(5.27)\quad G(t) = \begin{cases} 0, & \text{for}\quad 0<t<\dfrac{\sqrt{h}}{\sqrt{c_3}}, \\ \dfrac{1}{\Gamma(3)}\left(t-\dfrac{\sqrt{h}}{\sqrt{c_3}}\right)^2, & \text{for}\quad t>\sqrt{\dfrac{h}{c_3}}, \end{cases}$$

$$(5.28)\quad F(t) = \begin{cases} 0, & \text{for}\quad 0<t<\sqrt{\dfrac{h}{c_3}}, \\ t-\sqrt{\dfrac{h}{c_3}}, & \text{for}\quad t>\sqrt{\dfrac{h}{c_3}}. \end{cases}$$

6. Effect of suddenly applied heat source

Consider a heat source of constant strength S applied suddenly at the origin. Then

$$Q(x, y, z, t) = S\delta(x)\delta(y)\delta(z)H(t), \tag{6.1}$$
$$\mathbf{X} = \mathbf{Y} = 0.$$

In our transform notation this can be written as

$$Q^{(2)}(\zeta_1, \zeta_2, \zeta_3, p) = \frac{S}{(2\pi)^{3/2}}\frac{1}{p}, \tag{6.2}$$
$$\mathbf{X}^{(2)} = \mathbf{Y}^{(2)} = 0.$$

Substituting from (6.2) in (3.12) we obtain:

$$(u_1^{(2)}, u_2^{(2)}, u_3^{(2)}) = \frac{\nu S}{p\varrho c_1^2(2\pi)^{3/2}}\frac{i}{\Delta_1}(\zeta_1, \zeta_2, \zeta_3), \tag{6.3}$$

$$\theta^{(2)}(\zeta_1, \zeta_2, \zeta_3, p) = \frac{S}{pk(2\pi)^{3/2}}\frac{\zeta^2+\beta_1^2}{\Delta_1}. \tag{6.4}$$

Performing the Fourier-inversion with respect to $\zeta_1, \zeta_2, \zeta_3$ to the above set of equations, we obtain:

$$(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}) = \frac{-\nu S}{p\varrho c_1^2 k(2\pi)^{3/2}}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)I_5, \tag{6.5}$$

$$\theta^{(1)}(x, y, z, p)\frac{k(2\pi)^3}{\nu S} = \frac{1}{p}I_9, \tag{6.6}$$

where I_5 is defined in (4.12) and

$$I_9 = \iiint\limits_{-\infty}^{\infty} \frac{\zeta^2+\beta_1^2}{(\zeta^2-\lambda_1^{(2)})(\zeta^2-\lambda_1^{(2)})} e^{-(\zeta_1 x+\zeta_2 y+\zeta_3 z)}d\zeta_1 d\zeta_2 d\zeta_3. \tag{6.7}$$

On performing the integration, we find that

$$I_9 = 2\pi^2\left[\frac{\lambda_1^2+\beta_1^2}{\lambda_1^2-\lambda_2^2}\sqrt{\frac{2}{\pi}}\frac{(i\lambda_1)^{1/2}}{(z^2+r^2)^{1/4}}K_{1/2}\left(i\lambda_1\sqrt{z^2+r^2}\right) - \frac{\lambda_2^2+\beta_1^2}{\lambda_2^1-\beta_2^2}\sqrt{\frac{2}{\pi}}\frac{(i\lambda_2)^{1/2}}{(z^2+r^2)^{1/4}}K_{1/2}\left(i\lambda_2\sqrt{z^2+r^2}\right)\right], \tag{6.8}$$

where $x = r\cos\theta$, $y = r\sin\theta$.

It may be noticed that it is a spherically symmetric case and we will find the radial displacement u_R instead of u_1, u_2, u_3. To obtain the Laplace inversion, we as usual expand the right-hand terms of the equations (6.5) and (6.6) in the ascending powers of p^{-1}. Thus we obtain

$$U_R^{(1)}(h, p) = -\frac{\nu S}{4\pi\varrho k}\frac{1}{\sqrt{h}}\left[\left(\frac{1}{c_1}\sqrt{\frac{a}{2}}\frac{1}{p^2\sqrt{p}} + \frac{1}{\sqrt{h}p^3}\right)e^{-\frac{1}{c_1}\sqrt{\frac{aph}{2}}} - \left\{\frac{1}{c_1}\frac{1}{p^2} + \left(\frac{2+a-d}{2c_1} + \frac{1}{\sqrt{h}}\right)\frac{1}{p^3}\right\}e^{\frac{d\sqrt{h}}{4c_1}-\frac{p\sqrt{h}}{c_1}}\right] + 0\left(\frac{1}{p^4}\right), \tag{6.9}$$

$$
(6.10)\quad \theta^{(1)}(h,p) = \frac{S}{4\pi k}\frac{1}{\sqrt{h}}\Bigg[\left\{\frac{1}{p} - \frac{d}{2}\frac{1}{p^2} + \left(\frac{3d^2}{16} - \frac{ad}{4}\right)\frac{1}{p^3}\right\} e^{-\frac{1}{c_1}\sqrt{\frac{aph}{2}}}
+ \frac{d}{2}\left\{\frac{1}{p^2} - \left(\frac{5d}{8} - \frac{a}{2}\right)\frac{1}{p^3}\right\} e^{-\frac{d\sqrt{h}}{4c_1} - \frac{p\sqrt{h}}{c_1}}\Bigg] + 0\left(\frac{1}{p^4}\right).
$$

On inverting we obtain displacement and temperature field for the case of suddenly applied heat source. Thus

$$
(6.11)\quad U_R(h,t) = -\frac{\nu S}{4\pi\varrho k}\frac{1}{\sqrt{h}}\Bigg\{\Bigg[\frac{4}{c_1}\sqrt{\frac{at}{2\pi}}\, t D_{-4}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2t}}\right)
+ \sqrt{\frac{2}{\pi h}}\, 4t^2 D_{-5}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2t}}\right)\Bigg] e^{-\frac{ah}{16c_1^2 t}} + \left[\frac{1}{c_1}A(t) + \left(\frac{2+a-d}{2c_1} + \frac{1}{\sqrt{h}}\right)B(t)\right] e^{-\frac{d\sqrt{h}}{4c_1}}\Bigg\},
$$

$$
(6.12)\quad \theta(h,t) = \frac{S}{4\pi k}\frac{1}{\sqrt{h}}\Bigg\{\mathrm{Erf}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2t}}\right) - \frac{d}{2}\Bigg[\left(t + \frac{ah}{4c_1}\right)\mathrm{Erfc}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{2t}}\right)
- \frac{1}{c_1}\sqrt{\frac{aht}{2\pi}}\, e^{-\frac{ah}{8c_1^2 t}}\Bigg] + d\left(\frac{3}{4}d - a\right)^4 \sqrt{\frac{2}{\pi}}\, t^2 e^{-\frac{ah}{16c_1^2 t}} \times
\times D_{-5}\left(\frac{1}{2c_1}\sqrt{\frac{ah}{t}}\right) + \frac{d}{2}\left[A(t) - \left(\frac{5d}{8} - \frac{a}{2}\right)B(t)\right] e^{-\frac{d\sqrt{h}}{4c_1}}\Bigg\}.
$$

7. Numerical results and conclusions

We have calculated the displacements, rotations and temperature for two typical values of time, namely $t = 1.0$, 1.5 and at a plane $z = 1$, for the results of sections 4, 5 and section 6 are spherically symmetric. Similar calculations are also carried out for the case of classical coupled thermoelasticity and compared with the micropolar theory. In all these calculations we have assumed $c_1 = 5.2$, $c_2 = 3.8$, $c_3 = 3.45$, $c_4 = 1.5$ and $c_2^* = 3.8$ with thermoelastic coupling constant $\omega = 0.0729$ for the sake of convenience. We may say that since the temperature field is independent of the micropolar effect, the solutions thus obtained for temperature in our analysis are in fact the solutions for the classical coupled thermoelasticity.

In the case of suddenly applied body force we observe that the radial displacement U_r undergoes a jump at $t = \sqrt{h}/c_1$ and at $t = \sqrt{h}/c_2$. The magnitude of these jumps at these points, respectively, are

$$
\frac{|z|}{c_1^2 h^2}\left(1 + \frac{3c_1}{\sqrt{h}}\right) e^{-\frac{d\sqrt{h}}{4c_1}}
$$

and $-|z|/c_2^2 h^2$.

We note that these discontinuities diminish with distances from the origin of disturbance. At $t = \sqrt{h}/c_4$ we observe that U_r does not suffer any jump. This is due to the fact that the functions of the type A^{**}, B^{**} vanish identically when $t = \sqrt{h}/c_4$. The same phenomenon is observed for the displacement component U_z. The magnitude of the jumps at $t = \sqrt{h}/c_1$ and $t = \sqrt{h}/c_2$ for U_z are, respectively,

$$\left(\frac{z|z|}{c_1^2 h^{3/2}} - \frac{z}{c_1^2|z|h} + \frac{3z|z|}{c_1 h^2}\right) e^{-\frac{d\sqrt{h}}{4c_1}}$$

and

$$\left(\frac{z|z|}{c_2^2 h^{3/2}} + \frac{1}{c_2^2 h^{1/2}}\right),$$

where rotation components w_1 and w_2 are continuous. The temperature θ undergoes a jump at $t = \sqrt{h}/c_1$ of magnitude $-\frac{z}{c_1 h} e^{-\frac{d\sqrt{h}}{4c}}$ and decreases with distance from the origin of disturbances. It is worth mentioning here that the corresponding classical thermoelastic problem was considered by E. SOOS [25] (the body force is applied suddenly) by a different method for small times. Our results are in agreement with E. SOOS [25]. We shall also make a comment that the same classical results were derived for temperature field by ACHENBACH [26] who solved the above problem for small times in spherical coordinates.

In the case of suddenly applied body force, the displacement component U_r along r-axis and U_z along z-axis are plotted in Figs. 1, 2, for two values of time along the radial

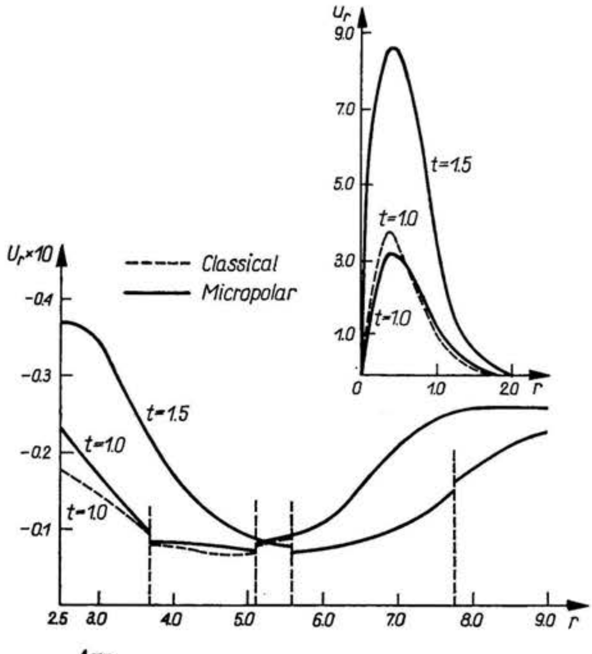

FIG. 1. $u_r = \frac{4\pi p}{F} u_r$ *vs.* r for the case of suddenly; applied body force.

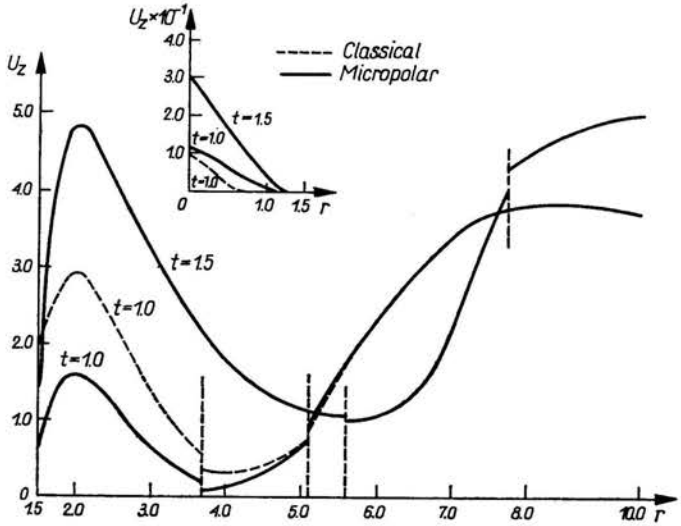

FIG. 2. $U_z = \dfrac{4\pi p}{F} u_z$ *vs.* r for the case of suddenly applied; body force.

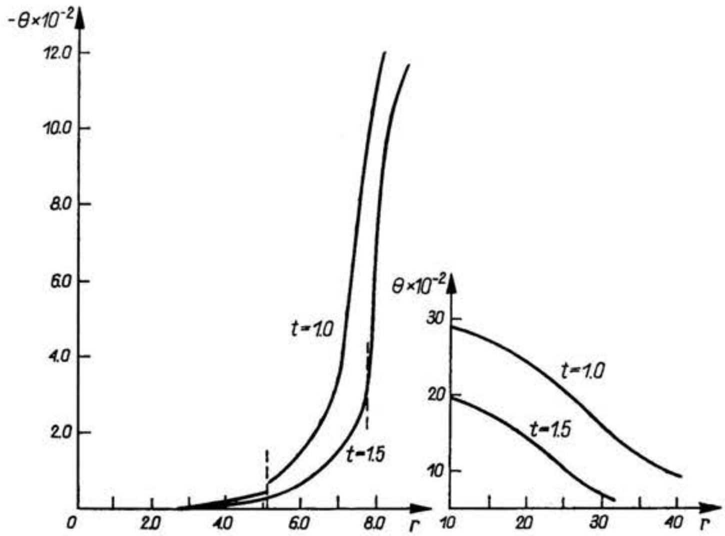

FIG. 3. Temperature $\theta = \dfrac{4\pi}{n_0 F} \theta$ *vs.* r for the case of suddenly applied body force.

distance r. Figure 3 shows the graph of the temperature distribution. The corresponding classical thermoelastic solutions are shown in these graphs by dotted curves and the points of discontinuity are shown by vertical dotted lines. We have compared the classical and micropolar solutions for only one value of $t = 1.0$ for the sake of clarity of the graphs.

From Fig. 1 we see that the disturbances are quite high behind the first wave front and are small behind the second wave front, whereas we observe from Fig. 2 that the disturbances are quite small behind the first wave front and are high behind the second

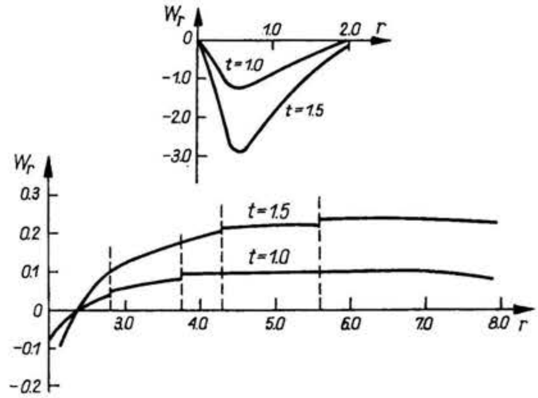

FIG. 4. $W_r = \dfrac{4\pi J}{M} w_r$ *vs.* r for suddenly applied body couple.

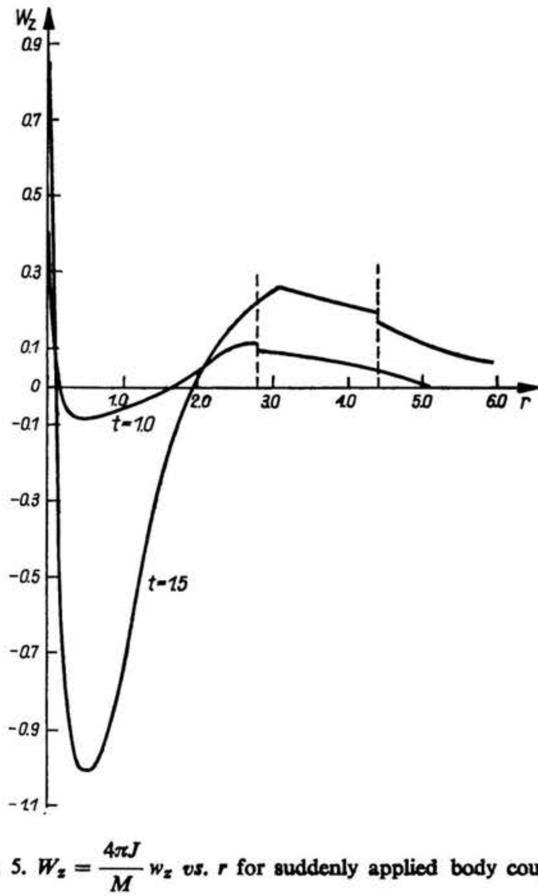

FIG. 5. $W_z = \dfrac{4\pi J}{M} w_z$ *vs.* r for suddenly applied body couple.

wave front. These disturbances slowly damp out as r increases. From these figures we also note that the micropolar effect is more pronounced behind the first and second wave front. This micropolar effect is negligibly small after the second wave front.

When body couple is applied suddenly we see that w_r has discontinuities at $t = \sqrt{h}/c_2$ and $t = \sqrt{h}/c_4$, whereas w_z has only one discontinuity at $t = \sqrt{h}/c_4$ of magnitude $1/c_4\sqrt{h}$ and this discontinuity diminishes with distance. The displacement components u_r and u_z are continuous function, since at $t = \sqrt{h}/c_2$ and $t = \sqrt{h}/c_4$, the functions of type A^*, B^* and A^{**}, B^{**} vanish identically at these points.

Thus we see that the displacement components undergo a jump if the body couple is impulsive and are continuous when a continuous body couple is applied.

Figures 4 and 5 show the plot of the rotation components w_r and w_z for two values of t along the radial distance for suddenly applied body couple.

In the case of suddenly applied body couple we see from Fig. 4 that w_r is negative behind the first wave front and positive ahead of this wave front. From Fig. 5 we observe that w_z has a single discontinuity at the wave front.

In the case of suddenly applied heat source we observe that U_R and θ are both continuous. Figures 6 and 7 are the plots of U_R and θ, respectively.

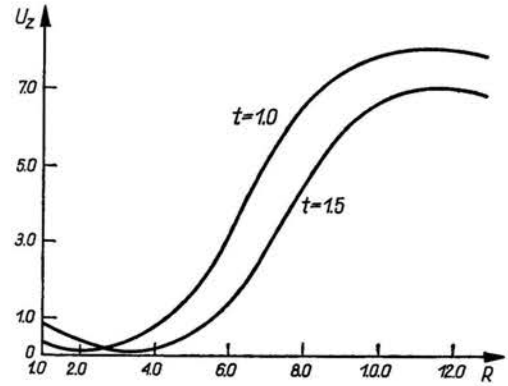

FIG. 6. $U_z = \dfrac{4\pi kp}{vs.} u_s$ *vs.* R for the case of sudden heat source.

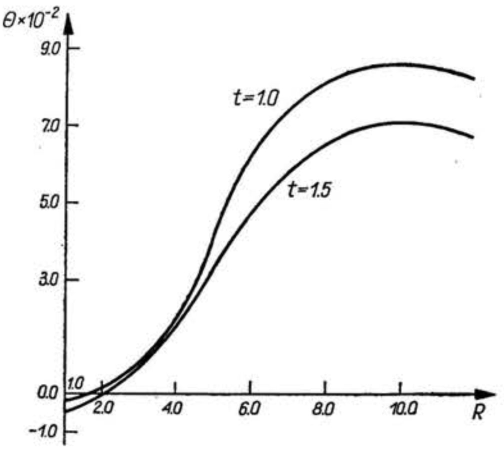

FIG. 7. Temperature $\theta = \dfrac{4\pi k}{S} \theta$ *vs.* R for the case of sudden heat source.

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