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Optimal design of plates loaded by two sets of lateral loads

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OPTIMAL design problem of a circular plate with built-in edge is considered for the case of loading by two independent sets of lateral loadings, one acting downwards, the other upwards. The problem is solved for a rigid-plastic material satisfying the Tresca yield condition.

W pracy rozpatrzono optymalne projektowanie okrągłej płyty utwierdzonej na brzegu i obciążonej przez dwa układy obciążeń poprzecznych, działających w przeciwnych kierunkach. Podano analityczne rozwiązanie problemu dla materiału sztywno-plastycznego, spełniającego warunek plastyczności Treski.

В работе исследуется оптимальное проектирование круглой пластинки, защемленной по контуру и находящейся под действием двух равномерно распределенных систем поперечных нагрузок, действующих вниз и вверх. Задача решается для жестко-пластического материала с условием текучести Треска.

1. Introduction

PROBLEMS of optimal design of plastic plates and shells of minimum weight loaded transversely by loads directed in one direction were investigated in numerous papers (see, for instance, [1, 2]). The optimality condition due to DRUCKER and SHIELD [3, 4] requiring constant rate of dissipation per unit area of the median surface and unit plate thickness was usually utilized in constructing the optimal solutions. However, as it was shown by MRÓZ [5], this condition corresponds to a global minimum of volume only for sandwich plates or shells: for solid plates, the condition of constant specific rate of dissipation may correspond neither to global nor local minimum. For such structures, the problem of optimal design should be reformulated by introducing constraints on maximum and minimum thickness and considering rib-reinforced designs [6]. In [7], the present writer considered the problem of optimal design of a simply-supported plate loaded by downward and upward lateral pressures. In the present work, the case of a circular plate with built-in edge will be treated. The downward loading is assumed to act on the whole plate area, whereas the upward loading acts on the central portion. Similarly as in [7], the sandwich cross-section is considered and h, H denote the thickness of thin sheets and of the central layer. As usually, we assume that $H \gg h$. The material of sheets is homogeneous and satisfies the Tresca yield condition with the associated flow rule. The problem is reduced to determining such variable sheet thickness that the plate volume reaches its minimum within the class of plates of constant limit load.

It is shown that various optimal solutions exist depending on the ratio of intensities of downward and upward loads, and the radius of central loaded portion. These solutions are presented in both the analytical and graphical forms.

2. Optimal solutions

Let the built-in circular plate of radius R be loaded on its lateral surface by a uniform downward loading p_1 and by the uniform upward loading p_2 distributed on the circular area of radius R_0 (Fig. 1). In the cylindrical polar coordinate system r, φ, z, p_1 and p_2 are respectively directed along the *z-axis* in positive and negative directions.

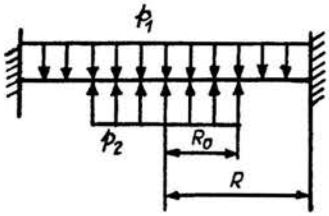

FIG. 1.

The equilibrium equations have the form:

$$r\frac{dM_r}{dr}+M_r-M_\varphi = \begin{cases} \dfrac{p_1-p_2}{2}r^2, & 0 \leqslant r \leqslant R_0, \\ \dfrac{p_2R_0^2-p_1r^2}{2}, & R_0 \leqslant r \leqslant R, \end{cases} \tag{2.1}$$

where M_r and M_φ are the radial and the circumferential bending moments. The curvature rates k_r and k_φ are expressed in term of the rate of deflection w in the following form:

$$k_r = -\frac{d^2W}{dr^2}, \quad k_\varphi = -\frac{1}{r}\frac{dW}{dr}. \tag{2.2}$$

The optimality condition requiring constant specific rate of dissipation within the sheet is expressed as follows:

$$\frac{D}{h} = \frac{M_rk_r+M_\varphi k_\varphi}{h} = \alpha, \tag{2.3}$$

where D denotes the specific dissipation power per unit area of the median surface and α is a positive constant. Introduce the following dimensionless quantities

$$\begin{gathered} \xi = \frac{r}{R}, \quad \varrho_0 = \frac{R_0}{R}, \quad k = \frac{p_2}{p_1}, \\ m_r = \frac{4M_r}{p_1R^2}, \quad m_\varphi = \frac{4M_\varphi}{p_1R^2}, \quad m_0 = \frac{4M_0}{p_1R^2}, \quad w = \frac{4W}{\alpha R^2}, \end{gathered} \tag{2.4}$$

where $M_0 = \sigma_0 Hh$ denotes the limit plastic moment in uniaxial bending.

Let us now consider the particular stress regimes.

i. Let $0 \leqslant k \leqslant 1$. In this case the plastic solution will be the same as for the case of loading by p_1, that is, the solution will correspond to regime A in the central portion

$0 \leqslant \xi \leqslant \varrho_1$ and to regime C in the region $\varrho_1 \leqslant \xi \leqslant 1$. These corner regimes are represented by the lines OA and OC on the stress plane, Fig. 2. Using the boundary conditions $w = 0$ for $\xi = 1$ and $dw/d\xi = 0$ for $\xi = 0$ and $\xi = 1$ together with the continuity conditions of w and $dw/d\xi$ for $\xi = \varrho_1$ from (2.2) and (2.3), we obtain

$$w = \begin{cases} -\xi^2 + 3\varrho_1^2 - 2(1+2\varrho_1), & 0 \leqslant \xi \leqslant \varrho_1, \\ 2(1-\xi)^2, & \varrho_1 \leqslant \xi \leqslant 1, \end{cases} \tag{2.5}$$

where

$$\varrho_1 = \frac{2}{3}. \tag{2.6}$$

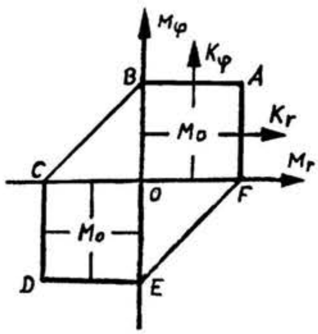

FIG. 2.

The stress field is expressed differently for the two cases $\varrho_0 \leqslant \varrho_1$ and $\varrho_0 \geqslant \varrho_1$. Assume first that $\varrho_0 \leqslant \varrho_1$. Then the equilibrium equation (2.1) furnishes

$$m_0 = \begin{cases} (k-1)\xi^2 + \varrho_1^2 - k\varrho_0^2\left(1 + 2\ln\dfrac{\varrho_1}{\varrho_0}\right), & 0 \leqslant \xi \leqslant \varrho_0, \\ -\xi^2 - 2k\varrho_0^2 \ln\dfrac{\varrho_1}{\xi} + \varrho_1^2, & \varrho_0 \leqslant \xi \leqslant \varrho_1 \\ \dfrac{2(\xi-\varrho_1)(\xi^2+\varrho_1\xi+\varrho_1^2-3k\varrho_0^2)}{3\xi}, & \varrho_1 \leqslant \xi \leqslant 1, \end{cases} \tag{2.7}$$

where the continuity condition of m_r for $\xi = \varrho_0$ and $\xi = \varrho_1$ was used. Since for the regime A we have $m_r = m_0$ and for C there is $m_r = -m_0$, there must be $m_r = 0$ for $\xi = \varrho_1$.

If $\varrho_0 > \varrho_1$, we obtain

$$m_0 = \begin{cases} (k-1)(\xi^2-\varrho_1^2), & 0 \leqslant \xi \leqslant \varrho_1, \\ \dfrac{2(k-1)(\varrho_1^3-\xi^3)}{3\xi}, & \varrho_1 \leqslant \xi \leqslant \varrho_0, \\ \dfrac{2[(\xi-\varrho_0)(\xi^2+\varrho_0\xi+\varrho_0^2-3k\varrho_0^2)+(k-1)(\varrho_1^3-\varrho_0^3]}{3\xi}, & \varrho_0 \leqslant \xi \leqslant 1. \end{cases} \tag{2.8}$$

ii. Assume now that $k \geqslant 1$. Now, there are several regimes within the plate. Assume first that only the regimes $A-C$ occur in the plate. As it follows from (2.7), the condition $m_0 \geqslant 0$ leads to the inequality

$$k\varrho_0^2 \leqslant \frac{\varrho_1^2}{1+2\ln\dfrac{\varrho_1}{\varrho_0}}, \quad 0 \leqslant \varrho_0 \leqslant \frac{2}{3}. \tag{2.9}$$

The formula (2.8) will only be valid when $k\varrho_0^2 \leqslant \varrho_0^2$ $(k \leqslant 1)$. When the inequality (2.9) is violated the negative bending moments occur near the plate centre. It is therefore purposeful to assume that in the central portion both radial and circumferential moments are negative. This assumption implies existence of the regimes *D–F–A–C* or *D–F*. Let us discuss in detail the solutions corresponding to these regimes.

Regime D–F–A–C. Let the regions $0 \leqslant \xi \leqslant \varrho_1$, $\varrho_1 \leqslant \xi \leqslant \varrho_2$, $\varrho_2 \leqslant \xi \leqslant \varrho_3$ and $\varrho_3 \leqslant \xi \leqslant 1$ characterize correspondingly the regimes *D*, *F*, *A* and *C*. Then, from (2.2) and (2.3), we find

$$w = \begin{cases} \xi^2-3(\varrho_1^2+\varrho_2^2-\varrho_3^2)+2\left[2(\varrho_1\varrho_2-\varrho_3)-\varrho_2^2\ln\dfrac{\varrho_3}{\varrho_2}+1\right], & 0 \leqslant \xi \leqslant \varrho_1, \\ 2\xi(2\varrho_2-\xi)+3(\varrho_3^2-\varrho_2^2)-2\left(\varrho_2^2\ln\dfrac{\varrho_3}{\varrho_2}+2\varrho_3-1\right), & \varrho_1 \leqslant \xi \leqslant \varrho_2, \\ -\xi^2+\varrho_3(3\varrho_3-4)+2\left(\varrho_2^2\ln\dfrac{\xi}{\varrho_3}+1\right), & \varrho_2 \leqslant \xi \leqslant \varrho_3, \\ 2(1-\xi)^2, & \varrho_3 \leqslant \xi \leqslant 1, \end{cases} \tag{2.10}$$

where

$$\varrho_1 = \frac{2}{3}\varrho_2, \quad \varrho_3(3\varrho_3-2) = \varrho_2^2. \tag{2.11}$$

Now, let us assume that $\varrho_2 \leqslant \varrho_0 \leqslant \varrho_3$. In this case, from the equilibrium equation (2.1) we obtain, after satisfying the continuity conditions of m_r for $\xi = \varrho_1$ $(m_r = 0)$, $\xi = \varrho_2$, $\xi = \varrho_0$ and $\xi = \varrho_3$ $(m_r = 0)$

$$m_0 = \begin{cases} (k-1)(\varrho_1^2-\xi^2), & 0 \leqslant \xi \leqslant \varrho_1, \\ \dfrac{2(k-1)(\xi^3-\varrho_1^3)}{3\xi}, & \varrho_1 \leqslant \xi \leqslant \varrho_2, \\ (k-1)\xi^2-k\varrho_0^2\left(1+2\ln\dfrac{\varrho_3}{\varrho_0}\right)+\varrho_3^2, & \varrho_2 \leqslant \xi \leqslant \varrho_0, \\ -\xi^3+2k\varrho_0^2\ln\dfrac{\xi}{\varrho_3}+\varrho_3^2, & \varrho_0 \leqslant \xi \leqslant \varrho_3, \\ \dfrac{2(\xi-\varrho_3)(\xi^2+\varrho_3\xi+\varrho_3^2-3k\varrho_0^2)}{3\xi}, & \varrho_3 \leqslant \xi \leqslant 1, \end{cases} \tag{2.12}$$

where

$$3k\varrho_0^2\varrho_2\left(1+2\ln\right)\frac{\varrho_3}{\varrho_0}-\varrho_2[(k-1)\varrho_2^2+3\varrho_3^2]-2(k-1)\varrho_1^3 = 0. \tag{2 13}$$

If $\varrho_1 \leqslant \varrho_0 \leqslant \varrho_2$, then

$$
(2.14)\qquad m_0 = \begin{cases} (k-1)(\varrho_1^2-\xi^2), & 0 \leqslant \xi \leqslant \varrho_1, \\ \dfrac{2(k-1)(\xi^3-\varrho_1^3)}{3\xi}, & \varrho_1 \leqslant \xi \leqslant \varrho_0, \\ \dfrac{2[(\xi-\varrho_0)(3k\varrho_0^2-\xi^2-\varrho_0\xi-\varrho_0^2)+(k-1)(\varrho_0^3-\varrho_1^3)]}{3\xi}, & \varrho_0 \leqslant \xi \leqslant \varrho_2, \\ -\xi^2+2k\varrho_0^2\ln\dfrac{\xi}{\varrho_3}+\varrho_3^2, & \varrho_2 \leqslant \xi \leqslant \varrho_3, \\ \dfrac{2(\xi-\varrho_3)(\xi^2+\varrho_3\xi+\varrho_3^2-3k\varrho_0^2)}{3\xi}, & \varrho_3 \leqslant \xi \leqslant 1, \end{cases}
$$

where

$$
(2.15)\qquad 6k\varrho_0^2\varrho_2\left(1+\ln\frac{\varrho_3}{\varrho_2}\right)-4k\varrho_0^3+\varrho_2(\varrho_2^2-3\varrho_3^2)-2(k-1)\varrho_1^3 = 0.
$$

However, if we assume that $0 \leqslant \varrho_0 \leqslant \varrho_1$, then

$$
(2.16)\qquad m_0 = \begin{cases} (1-k)\xi^2+k\varrho_0^2\left((1+2\ln\dfrac{\varrho_1}{\varrho_0}\right)-\varrho_1^2, & 0 \leqslant \xi \leqslant \varrho_0, \\ \xi^2+2k\varrho_0^2\ln\dfrac{\varrho_1}{\xi}-\varrho_1^2, & \varrho_0 \leqslant \xi \leqslant \varrho_1, \\ \dfrac{2(\xi-\varrho_1)(3k\varrho_0^2-\xi^2-\varrho_1\xi-\varrho_1^2)}{3\xi}, & \varrho_1 \leqslant \xi \leqslant \varrho_2, \\ -\xi^2+2k\varrho_0^2\ln\dfrac{\xi}{\varrho_3}+\varrho_3^2, & \varrho_2 \leqslant \xi \leqslant \varrho_3, \\ \dfrac{2(\varrho_3-\xi)(3k\varrho_0^2-\xi^2-\varrho_3\xi-\varrho_3^2)}{3\xi}, & \varrho_3 \leqslant \xi \leqslant 1, \end{cases}
$$

where

$$
(2.17)\qquad 6k\varrho_0^2\left[\left(1+\ln\frac{\varrho_3}{\varrho_2}\right)\varrho_2-\varrho_1\right]+\varrho_2(\varrho_2^2-3\varrho_3^2)+2\varrho_1^3 = 0.
$$

It can be checked that the case $\varrho_0 \geqslant \varrho_3$ cannot occur within the plate. Thus, the relations (2.13), (2.15) and (2.17) together with (2.11) enable us to determine the radii ϱ_1, ϱ_2 and ϱ_3. When $\varrho_1 = \varrho_2 = 0$, from (2.11) it follows that $\varrho_3 = 2/3$ and from (2.12) we obtain the expression (2.7), where ϱ_1 should be replaced by ϱ_3, that is, the regimes *D* and *F* vanish and the plastic state *D–F–A–C* is reduced to *A–C* which was already considered.

When $\varrho_2 = \varrho_3 = 1$, we have $\varrho_1 = 2/3$. In this case the regimes *D–F–A–C* are reduced to *D–F*. The corresponding moments fields are obtained from (2.16) and (2.14). These expressions differ from (2.7) and (2.8) only by sign. Similarly the kinematic field will be defined by (2.5) with an opposite sign.

Assuming $\varrho_1, \varrho_2 \geqslant 0$ and $\varrho_2, \varrho_3 \leqslant 1$, we can determine lower and upper limits of variation of $k\varrho_0^2$ in function of ϱ_0 for the regimes *D–F–A–C*. These limits will coincide respectively with the upper limit for A–C and the lower limit for D–F. Thus, the regime A–C occurs when

$$(2.18)\qquad \begin{aligned} &k\varrho_0^2 \leqslant \frac{\varrho_1^2}{1+2\ln\frac{\varrho_1}{\varrho_0}}, \quad 0 \leqslant \varrho_0 \leqslant \frac{2}{3},\\ &k\varrho_0^2 \leqslant \varrho_0^2 \quad \text{or} \quad k \leqslant 1, \quad \frac{2}{3} \leqslant \varrho_0 \leqslant 1. \end{aligned}$$

In view of the formulae (2.17), (2.16) and (2.14) it is seen that the regime D–F occurs when

$$(2.19)\qquad \begin{aligned} &k\varrho_0^2 \geqslant \frac{19}{27}, \quad 0 \leqslant \varrho_0 \leqslant \frac{2}{3},\\ &k\varrho_0^2 \geqslant \frac{19+8k}{27(3-2\varrho_0)}, \quad \frac{2}{3} \leqslant \varrho_0 \leqslant 1. \end{aligned}$$

The expressions (2.18) and (2.19) define respectively lower and upper limits of the regime D–F–A–C.

Let us now determine the intervals of variability of $k\varrho_0^2$ for the regime D–F–A–C, when $\varrho_2 \leqslant \varrho_0 \leqslant \varrho_3$, $\varrho_1 \leqslant \varrho_0 \leqslant \varrho_2$, $0 \leqslant \varrho_0 \leqslant \varrho_1$. For $\varrho_2 \leqslant \varrho_0 \leqslant \varrho_3$, we find the upper limit from (2.11) and (2.13). We have

$$(2.20)\qquad k\varrho_0^2 \leqslant \frac{9\left(1+\sqrt{1+3\varrho_0^2}\right)-8\varrho_0^2}{19-81\ln\dfrac{3\varrho_0}{1+\sqrt{1+3\varrho_0^2}}}, \quad 0 \leqslant \varrho_0 \leqslant 1.$$

For this case, (2.18) defines the lower limit.

For $\varrho_1 \leqslant \varrho_0 \leqslant \varrho_2$, the expression (2.20) provides the lower limit and the upper limit is found from (2.11) and (2.15)

$$(2.21)\qquad \begin{aligned} &k\varrho_0^2 \leqslant \frac{2(1-2\varrho_0^2)+\sqrt{4+27\varrho_0^2}}{6\left(1-3\ln\dfrac{9\varrho_0}{2+\sqrt{4+27\varrho_0^2}}\right)}, \quad 0 \leqslant \varrho_0 \leqslant \frac{2}{3},\\ &k\varrho_0^2 \leqslant \frac{19+8k}{27(3-2\varrho_0)}, \quad \frac{2}{3} \leqslant \varrho_0 \leqslant 1. \end{aligned}$$

Finally, for $0 \leqslant \varrho_0 \leqslant \varrho_1$, the lower limit is given by (2.21), and the upper limit is defined by the first expression (2.19).

The dependence of $k\varrho_0^2$ on ϱ_0 for all regimes is given in Fig. 3. The lower boundaries of the domains I, II and III are expressed by the formulae (2.18), (2.20) and the first expression of (2.21). The variable plate thickness corresponding to these regimes for $\varrho_0 = 0.5$ is shown in Fig. 4a (regimes $A-C$ and D–F) and 4b (regime D–F–A–C).

When $k = 0$, all cases are reduced to that considered in [8], for which only p_1 acts downward on the plate. The varying plate thickness for this case is shown in Fig. 4a by the dashed line.

In view of our analysis, it can be concluded that for the two-parameter loading the optimal design problem becomes quite involved as compared to single loading case treated in [8]. Practically, the present case occurs for plates resting on a foundation and loaded

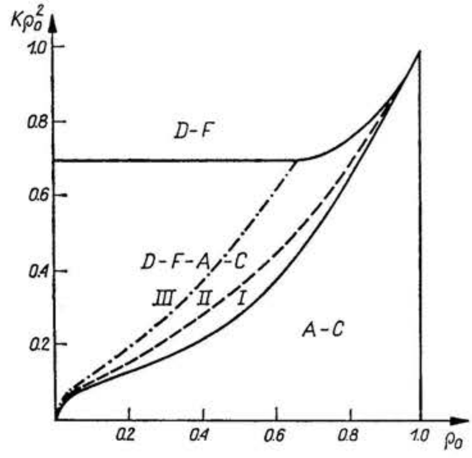
$K\rho_0^2$
1.0
0.8
0.6
0.4
0.2
D-F
D-F-A-C
III
II
I
A-C
0.2
0.4
0.6
0.8
1.0
ρ_0

FIG. 3.

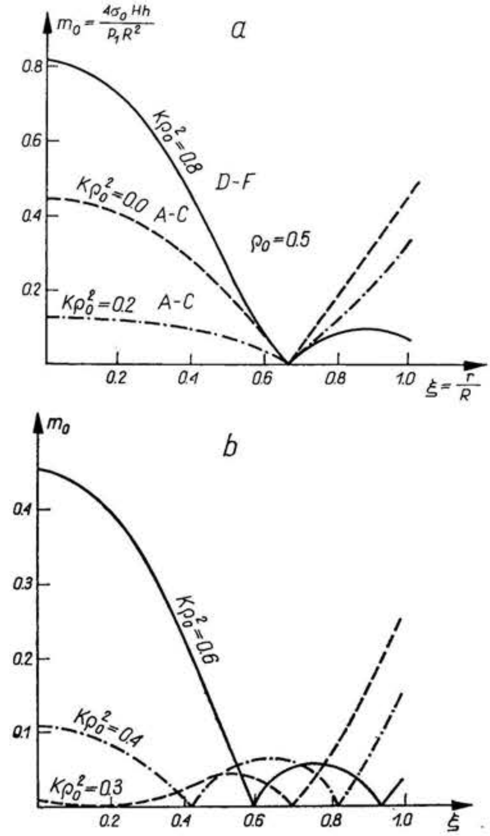
$m_0=\frac{4\sigma_0 Hh}{p_1 R^2}$
a
0.8
0.6
0.4
0.2
$K\rho_0^2=0.8$
D-F
$K\rho_0^2=0.0$ A-C
$\rho_0=0.5$
$K\rho_0^2=0.2$ A-C
0.2
0.4
0.6
0.8
1.0
$\xi=\frac{r}{R}$
m_0
b
0.4
0.3
0.2
0.1
$K\rho_0^2=0.6$
$K\rho_0^2=0.4$
$K\rho_0^2=0.3$
0.2
0.4
0.6
0.8
1.0
ξ

FIG. 4.

laterally or for closures of pressure vessels with additional loading. The analytical solution discussed in this paper may be useful in designing plates of piece-wise constant thickness and in testing numerical algorithms for optimal design in more complicated cases.

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