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The stress and displacement functions for the "second" axisymmetric problem of micropolar elastostatics

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IN A FRAME of linear elastostatics of the continuous micropolar medium two methods of the solution of the "second" axisymmetric problem, characterized by the vectors $\mathbf{u} \equiv (0, u_\theta, 0)$, $\boldsymbol{\varphi} \equiv (\varphi_r, 0, \varphi_z)$, are proposed; namely the stress method and the displacement method. On the basis of the nine governing homogeneous differential equations for the stress field in a simply connected region the five solving functions (stress functions) are introduced. It was shown that these stress functions may be reduced to two displacement functions solving the problem. The method of elastic potentials applied by W. NOWACKI in [4] may also be obtained here. The solution of the particular case $\alpha = 0$ is discussed.

W ramach liniowej elastostatyki ciągłego ośrodka mikropolarnego, dla "drugiego" zagadnienia osiowo-symetrycznego, charakteryzowanego przez wektory $\mathbf{u} \equiv (0, u_\theta, 0)$, $\boldsymbol{\varphi} \equiv (\varphi_r, 0, \varphi_z)$, proponuje się dwie drogi rozwiązania: naprężeniową i przemieszczeniową. W oparciu o dziewięć podstawowych, jednorodnych równań różniczkowych pola naprężeń w obszarze jednospójnym, wprowadza się pięć funkcji rozwiązujących (funkcje naprężeń). Pokazano dalej, że pięć funkcji naprężeń można sprowadzić do dwóch funkcji przemieszczeń rozwiązujących zagadnienie i że dochodzi się na tej drodze do metody potencjałów sprężystych W. NOWACKIEGO z pracy [4]. Omówiono rozwiązanie zagadnienia dla przypadku szczególnego $\alpha = 0$.

В рамках линейной эластостатики сплошной микрополярной среды, для „второй" осесимметрической задачи, характеризующейся векторами $\mathbf{u} \equiv (0, u_\theta, 0)$, $\boldsymbol{\varphi} \equiv (\varphi_r, 0, \varphi_z)$, предложены два пути решения: в напряжениях и в перемещениях. Опираясь на девять основных, однородных дифференциальных уравнений поля напряжений в односвязной области, вводится пять решающих функций (функции напряжений). Далее показано, что пять функций напряжений можно свести к двум функциям перемещений, решающих задачу и что приходится по этому пути к методу упругих потенциалов В. Новацкого из работы [4]. Обсуждено решение задачи для частного случая $\alpha = 0$.

1. Introduction

IN LINEAR elastostatics of continuous micropolar medium the basic equations may be reduced to the system of two vector equations from which the displacement vector $\mathbf{u}(\mathbf{x})$ and the rotation vector $\boldsymbol{\varphi}(\mathbf{x})$ are determined [1].

It was shown in [2] that the axisymmetric problem of deformation of a medium may be reduced to two independent problems. In a first problem the displacement field and the field of rotations in cylindrical system of coordinates (r, θ, z) are described by the following vectors $\mathbf{u}$ and $\boldsymbol{\varphi}$, respectively

$$\mathbf{u} \equiv (u_r, 0, u_z), \quad \boldsymbol{\varphi} \equiv (0, \varphi_z, 0). \tag{1.1}$$

In the so called "second" problem the displacement field $\mathbf{u}$ and the field of rotations $\boldsymbol{\varphi}$ are represented by the vectors

$$\mathbf{u} \equiv (0, u_\theta, 0), \quad \boldsymbol{\varphi} \equiv (\varphi_r, 0, \varphi_z). \tag{1.2}$$

In each of the above problems the basic equations for the displacement field and the rotations field are expressed in terms of three coupled differential equations.

Several methods of solution of these problems were discussed in the literature [2–7]. An exhaustive review of them may be found in [8].

In this paper the "second" axisymmetric problem is considered. For its solution both the stress method and the displacement method are proposed.

The starting point for the stress problem considerations is the system of nine basic homogeneous partial differential equations describing the stress field in a simply connected region (see [17]). These equations will be given in Sec. 2. Basing on the stress equations mentioned above we introduce five stress functions solving the problem (Secs. 3, 4). The manner in which these functions were introduced is similar to that used in papers [9 and 10] where the stress functions were applied to the solution of the plane problems of micropolar elasticity.

Next, in Sec. 5, it will be shown that the five stress functions may be reduced to two functions solving the problem (the displacement functions)(¹) (compare [11]). In this way we shall obtain (in Secs. 5, 6) a representation for the displacement u_θ and rotations φ_r, φ_z derived earlier by W. NOWACKI [4] by means of the method of elastic potentials.

In the last section the particular case of the solution of the problem ($\alpha = 0$) is discussed.

2. Formulation of the stress problem

In the problem (1.2) the stress field is described by two non-symmetric tensors: the stress tensor and the couple-stress tensor. Their components are, respectively,

$$\boldsymbol{\sigma} \equiv \begin{bmatrix} 0 & \sigma_{r\theta} & 0 \\ \sigma_{\theta r} & 0 & \sigma_{\theta z} \\ 0 & \sigma_{z\theta} & 0 \end{bmatrix}, \quad \boldsymbol{\mu} \equiv \begin{bmatrix} \mu_{rr} & 0 & \mu_{rz} \\ 0 & \mu_{\theta\theta} & 0 \\ \mu_{zr} & 0 & \mu_{zz} \end{bmatrix}. \tag{2.1}$$

The above components should satisfy the following differential equations of equilibrium in which the body forces and moments are neglected:

$$\begin{gathered} \sigma_{\theta z} - \sigma_{z\theta} + \frac{\partial}{\partial r}\mu_{rr} + \frac{\partial}{\partial z}\mu_{zr} + \frac{\mu_{rr} - \mu_{\theta\theta}}{r} = 0, \\ \sigma_{r\theta} - \sigma_{\theta r} + \frac{\partial}{\partial r}\mu_{rz} + \frac{\partial}{\partial z}\mu_{zz} + \frac{\mu_{rz}}{r} = 0, \\ \frac{\partial}{\partial r}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{z\theta} + \frac{\sigma_{r\theta} + \sigma_{\theta r}}{r} = 0. \end{gathered} \tag{2.2}$$

(¹) In the same manner the Love function was introduced by J. H. MITCHELL in classical theory of elasticity (compare A. E. LOVE [13], pp. 274–276).

The strain field is described by two non-symmetric tensors, the strain tensor and torsion-flexural tensor, the components of which are, respectively:

$$\boldsymbol{\gamma} \equiv \begin{bmatrix} 0 & \gamma_{r\theta} & 0 \\ \gamma_{\theta r} & 0 & \gamma_{\theta z} \\ 0 & \gamma_{z\theta} & 0 \end{bmatrix}, \quad \boldsymbol{\varkappa} \equiv \begin{bmatrix} \varkappa_{rr} & 0 & \varkappa_{rz} \\ 0 & \varkappa_{\theta\theta} & 0 \\ \varkappa_{zr} & 0 & \varkappa_{zz} \end{bmatrix}. \tag{2.3}$$

The geometrical equations relating the state of strain and the states of displacements and rotations are

$$\gamma_{r\theta} = \frac{\partial u_\theta}{\partial r} - \varphi_z, \quad \gamma_{\theta r} = -\frac{u_\theta}{r} + \varphi_z, \quad \gamma_{\theta z} = -\varphi_r, \quad \gamma_{z\theta} = \frac{\partial u_\theta}{\partial z} + \varphi_r, \tag{2.4}$$

and

$$\varkappa_{rr} = \frac{\partial \varphi_r}{\partial r}, \quad \varkappa_{\theta\theta} = \frac{\varphi_r}{r}, \quad \varkappa_{zz} = \frac{\partial \varphi_z}{\partial z}, \quad \varkappa_{rz} = \frac{\partial \varphi_z}{\partial r}, \quad \varkappa_{zr} = \frac{\partial \varphi_r}{\partial z}. \tag{2.5}$$

The fields of stress and strain are connected by the following constitutive equations:

$$\sigma_{r\theta} = (\mu+\alpha)\gamma_{r\theta} + (\mu-\alpha)\gamma_{\theta r}, \quad \sigma_{\theta r} = (\mu+\alpha)\gamma_{\theta r} + (\mu-\alpha)\gamma_{r\theta}, \tag{2.6}$$

$$\sigma_{\theta z} = (\mu+\alpha)\gamma_{\theta z} + (\mu-\alpha)\gamma_{z\theta}, \quad \sigma_{z\theta} = (\mu+\alpha)\gamma_{z\theta} + (\mu-\alpha)\gamma_{\theta z} \tag{2.7}$$

and

$$\mu_{rr} = 2\gamma\varkappa_{rr} + \beta\varkappa, \quad \mu_{\theta\theta} = 2\gamma\varkappa_{\theta\theta} + \beta\varkappa, \quad \mu_{zz} = 2\gamma\varkappa_{zz} + \beta\varkappa, \tag{2.8}$$

$$\mu_{rz} = (\gamma+\varepsilon)\varkappa_{rz} + (\gamma-\varepsilon)\varkappa_{zr}, \quad \mu_{zr} = (\gamma+\varepsilon)\varkappa_{zr} + (\gamma-\varepsilon)\varkappa_{rz}, \tag{2.9}$$

where

$$\varkappa = \varkappa_{rr} + \varkappa_{\theta\theta} + \varkappa_{zz}. \tag{2.10}$$

The quantities α, β, γ, ε, μ denote materials constants.

The components of the strain tensors (2.3) are not arbitrary but should satisfy the geometrical compatibility equations. On the basis of [12] the following strain differential compatibility equations for the problem considered are obtained:

$$\varkappa_{zr} - r\frac{\partial \varkappa_{\theta\theta}}{\partial z} = 0, \quad \frac{\partial}{\partial z}\varkappa_{rr} - \frac{\partial}{\partial r}\varkappa_{zr} = 0, \tag{2.11}$$

$$\frac{\partial}{\partial z}\gamma_{\theta r} + \frac{1}{r}\gamma_{z\theta} - \varkappa_{zz} - \varkappa_{\theta\theta} = 0, \quad \varkappa_{zr} + \frac{\partial}{\partial z}\gamma_{\theta z} = 0, \tag{2.12}$$

$$\frac{\partial}{\partial r}\varkappa_{zz} - \frac{\partial}{\partial z}\varkappa_{rz} = 0, \tag{2.13}$$

$$\frac{\partial}{\partial z}\gamma_{r\theta} - \frac{\partial}{\partial r}\gamma_{z\theta} + \varkappa_{zz} + \varkappa_{rr} = 0 \tag{2.14}$$

and

$$\begin{gathered} \frac{\partial}{\partial r}(r\varkappa_{\theta\theta}) - \varkappa_{rr} = 0, \\ \frac{\partial}{\partial r}(r\gamma_{\theta z}) + r(\varkappa_{\theta\theta} + \varkappa_{rr}) = 0, \\ \frac{\partial}{\partial r}(r\gamma_{\theta r}) + \gamma_{r\theta} - r\varkappa_{rz} = 0. \end{gathered} \tag{2.15}$$

Equations (2.11)–(2.15) may also be obtained from the geometrical relations (2.4) and (2.5) by elimination of the displacement u_θ and the rotations φ_r, φ_z. Besides, in this way one can get the following three differential or algebraic relations

$$\frac{\partial}{\partial z}(\gamma_{r\theta}+\gamma_{\theta r})+\left(\frac{1}{r}-\frac{\partial}{\partial r}\right)(\gamma_{z\theta}+\gamma_{\theta z})=0, \tag{2.16}$$

$$\varkappa_{rr}+\frac{\partial}{\partial r}\gamma_{\theta z}=0, \qquad \varkappa_{\theta\theta}+\frac{1}{r}\gamma_{\theta z}=0, \tag{2.17}$$

which may be obtained also from the Eqs. (2.11)–(2.15) by integration.

Equations (2.16), (2.17) together with relations

$$\frac{\partial}{\partial z}\gamma_{\theta r}+\frac{1}{r}\gamma_{z\theta}-\varkappa_{zz}-\varkappa_{\theta\theta}=0, \qquad \varkappa_{zr}+\frac{\partial}{\partial z}\gamma_{\theta z}=0, \tag{2.18}$$

$$\frac{\partial}{\partial r}(r\gamma_{\theta r})+\gamma_{r\theta}-r\varkappa_{rz}=0 \tag{2.19}$$

give six relations from which the Eqs. (2.11)–(2.15) may easily be derived. The equations (2.12) and $(2.15)_3$ are identical with the Eqs. (2.18) and (2.19), respectively, while the Eqs. (2.11), (2.13), (2.14) and $(2.15)_{1,2}$ are obtained by algebraic or differential transformations of the Eqs. (2.16)–(2.19). The six relations (2.16), (2.19) constitute the geometrical compatibility equations for the problem considered.

In the following, the strain compatibility equations will be expressed in terms of stresses. For this purpose, the constitutive equations (2.6)–(2.10) solved with respect to strains are used

$$\begin{aligned}
\gamma_{r\theta}&=\frac{1}{4\mu}(\sigma_{r\theta}+\sigma_{\theta r})+\frac{1}{4\alpha}(\sigma_{r\theta}-\sigma_{\theta r}),\\
\gamma_{\theta r}&=\frac{1}{4\mu}(\sigma_{r\theta}+\sigma_{\theta r})-\frac{1}{4\alpha}(\sigma_{r\theta}-\sigma_{\theta r}),\\
\gamma_{z\theta}&=\frac{1}{4\mu}(\sigma_{z\theta}+\sigma_{\theta z})+\frac{1}{4\alpha}(\sigma_{z\theta}-\sigma_{\theta z}),\\
\gamma_{\theta z}&=\frac{1}{4\mu}(\sigma_{z\theta}+\sigma_{\theta z})-\frac{1}{4\alpha}(\sigma_{z\theta}-\sigma_{\theta z})
\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}
&\varkappa_{rr}=\frac{1}{2\gamma}(\mu_{rr}-\beta\varkappa), \qquad \varkappa_{\theta\theta}=\frac{1}{2\gamma}(\mu_{\theta\theta}-\beta\varkappa), \qquad \varkappa_{zz}=\frac{1}{2\gamma}(\mu_{zz}-\beta\varkappa),\\
&\varkappa_{zr}=\frac{1}{4\gamma}(\mu_{zr}+\mu_{rz})+\frac{1}{4\varepsilon}(\mu_{zr}-\mu_{rz}),\\
&\varkappa_{rz}=\frac{1}{4\gamma}(\mu_{zr}+\mu_{rz})-\frac{1}{4\varepsilon}(\mu_{zr}-\mu_{rz}),
\end{aligned} \tag{2.21}$$

$\varkappa$ is expressed here in terms of stresses. From the Eqs. (2.8) and (2.10) we obtain

$$\varkappa=\frac{1}{2\gamma+3\beta}(\mu_{rr}+\mu_{\theta\theta}+\mu_{zz}). \tag{2.22}$$

Substituting the Eqs. (2.20)–(2.22) into the Eqs. (2.16)–(2.19), we have:

$$\frac{\partial}{\partial z}(\sigma_{r\theta}+\sigma_{\theta r})+\left(\frac{1}{r}-\frac{\partial}{\partial r}\right)(\sigma_{z\theta}+\sigma_{\theta z})=0 \tag{2.23}$$

and

$$\begin{gathered}
\frac{1}{2\gamma}(\mu_{rr}-\beta\varkappa)+\frac{\partial}{\partial r}\left[\frac{1}{4\mu}(\sigma_{z\theta}+\sigma_{\theta z})-\frac{1}{4\alpha}(\sigma_{z\theta}-\sigma_{\theta z})\right]=0,\\
\frac{1}{2\gamma}(\mu_{\theta\theta}-\beta\varkappa)+\frac{1}{r}\left[\frac{1}{4\mu}(\sigma_{z\theta}+\sigma_{\theta z})-\frac{1}{4\alpha}(\sigma_{z\theta}-\sigma_{\theta z})\right]=0,\\
\frac{\partial}{\partial z}\left[\frac{1}{4\mu}(\sigma_{r\theta}+\sigma_{\theta r})-\frac{1}{4\alpha}(\sigma_{r\theta}-\sigma_{\theta r})\right]+\frac{1}{r}\left[\frac{1}{4\mu}(\sigma_{z\theta}+\sigma_{\theta z})\right.\\
\left.+\frac{1}{4\alpha}(\sigma_{z\theta}-\sigma_{\theta z})\right]-\frac{1}{2\gamma}(\mu_{\theta\theta}+\mu_{zz}-2\beta\varkappa)=0,\\
\frac{1}{4\gamma}(\mu_{zr}+\mu_{rz})+\frac{1}{4\varepsilon}(\mu_{zr}-\mu_{rz})+\frac{\partial}{\partial z}\left[\frac{1}{4\mu}(\sigma_{z\theta}+\sigma_{\theta z})-\frac{1}{4\alpha}(\sigma_{z\theta}-\sigma_{\theta z})\right]=0,\\
\frac{1}{2\mu}(\sigma_{r\theta}+\sigma_{\theta r})+r\frac{\partial}{\partial r}\left[\frac{1}{4\mu}(\sigma_{r\theta}+\sigma_{\theta r})-\frac{1}{4\alpha}(\sigma_{r\theta}-\sigma_{\theta r})\right]\\
-r\left[\frac{1}{4\gamma}(\mu_{zr}+\mu_{rz})-\frac{1}{4\varepsilon}(\mu_{zr}-\mu_{rz})\right]=0.
\end{gathered} \tag{2.24}$$

Formulating the problem in terms of stresses (for the simply connected region), we require that the components of the state of stress [Eqs. (2.1)] should satisfy three equations of equilibrium (2.2), six equations of geometrical compatibility expressed in terms of stresses (Eqs. (2.23–2.24)), and given boundary conditions expressed also in terms of stresses. The state of deformation in a body is determined by means of the constitutive equations (2.20)–(2.22), while the displacement u_θ and the rotations φ_r, φ_z may be obtained by integration of the geometrical relations (2.4) and (2.5).

3. Introducing of the stress functions

Basing on the stress equations derived in a previous section we shall introduce five stress functions. As a starting point we have the equilibrium equations (2.2) and geometrical compatibility equations (2.23). The stress functions will be introduced in such a way that after relating them with the stresses, the four above-mentioned equations (2.2), (2.23) will be satisfied identically.

Let us introduce first the function $\Phi(r, z)$ assuming

$$\sigma_{r\theta}+\sigma_{\theta r}=\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\Phi, \qquad \sigma_{\theta z}+\sigma_{z\theta}=\frac{\partial}{\partial z}\Phi. \tag{3.1}$$

The Eq. (2.23), when Eq. (3.1) is taken into account, is satisfied identically. Determining the stresses $\sigma_{r\theta}$ and $\sigma_{z\theta}$ from the Eq. (3.1) and substituting them into the Eq. $(2.2)_3$ we obtain v

$$\frac{\partial}{\partial r}\sigma_{\theta r}+\frac{\partial}{\partial z}\sigma_{\theta z}=\frac{\partial^2}{\partial r^2}\Phi+\frac{\partial^2}{\partial z^2}\Phi. \tag{3.2}$$

The equilibrium equation $(2.2)_3$ will be satisfied identically if a new function $\Psi(r, z)$ expressed on the basis of the Eq. (3.2) by means of the relations

$$\sigma_{\theta r}=\frac{\partial}{\partial r}\Phi+\frac{\partial}{\partial z}\Psi, \quad \sigma_{\theta z}=\frac{\partial}{\partial z}\Phi-\frac{\partial}{\partial r}\Psi \tag{3.3}$$

is introduced.

The Eqs. (3.1), after taking into account the Eqs. (3.3), take the form

$$\sigma_{r\theta}=-\frac{1}{r}\Phi-\frac{\partial}{\partial z}\Psi, \quad \sigma_{z\theta}=\frac{\partial}{\partial r}\Psi. \tag{3.4}$$

Next, let us assume the couple-stresses in a form of sum

$$\boldsymbol{\mu}=\boldsymbol{\mu}'+\boldsymbol{\mu}''. \tag{3.5}$$

The stresses $\boldsymbol{\mu}'$ are related to the functions Φ and Ψ in a following way:

$$\mu'_{rr}=\mu'_{\theta\theta}=\mu'_{zz}=2\Psi, \quad \mu'_{rz}=-\mu'_{zr}=\Phi. \tag{3.6}$$

Substituting the Eqs. (3.6) into the Eqs. $(2.2)_{1,2}$ we obtain, respectively

$$\frac{\partial}{\partial r}\mu''_{rr}+\frac{\partial}{\partial z}\mu''_{zr}+\frac{\mu''_{rr}-\mu''_{\theta\theta}}{r}=0, \tag{3.7}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\mu''_{rz})+\frac{\partial}{\partial z}\mu''_{zz}=0. \tag{3.8}$$

The equilibrium equation $(2.2)_1$ will be satisfied identically if two functions $\Omega(r, z)$ and $\Delta(r, z)$ determined on the basis of the Eq. (3.7) by means of the formulae

$$\mu''_{rr}=-\frac{\partial}{\partial z}\Omega, \quad \mu''_{zr}=\frac{\partial}{\partial r}\Omega-\frac{\Delta}{r}, \quad \mu''_{\theta\theta}=-\frac{\partial}{\partial z}\Omega-\frac{\partial}{\partial z}\Delta \tag{3.9}$$

are introduced.

To satisfy identically the equilibrium equation $(2.2)_2$. we introduce the function $\chi(r, z)$ which satisfies, on the basis of the Eq. (3.8), the following relations:

$$r\mu''_{rz}=\frac{\partial}{\partial z}\chi, \quad \mu''_{zz}=-\frac{1}{r}\frac{\partial}{\partial r}\chi. \tag{3.10}$$

From the Eqs. (3.5), (3.6), (3.9) and (3.10) we obtain finally the following representation for the couple stresses:

$$\mu_{rr}=2\Psi-\frac{\partial}{\partial z}\Omega, \quad \mu_{\theta\theta}=2\Psi-\frac{\partial}{\partial z}\Omega-\frac{\partial}{\partial z}\Delta, \quad \mu_{zz}=2\Psi-\frac{1}{r}\frac{\partial}{\partial r}\chi, \tag{3.11}$$

$$\mu_{zr}=-\Phi+\frac{\partial}{\partial r}\Omega-\frac{\Delta}{r}, \quad \mu_{rz}=\Phi+\frac{1}{r}\frac{\partial}{\partial z}\chi. \tag{3.12}$$

The stress functions introduced are not arbitrary functions but should satisfy the system of five differential equations obtained from the Eqs. (2.24) into which the relations (3.3), (3.4), (3.11) and (3.12) were substituted

$$\frac{1}{2\gamma(2\gamma+3\beta)}\left[4\gamma\Psi-(2\gamma+\beta)\frac{\partial}{\partial z}\Omega+\beta\frac{\partial}{\partial z}\Delta+\beta\frac{1}{r}\frac{\partial}{\partial r}\chi\right] +\frac{\partial}{\partial r}\left[\frac{\alpha+\mu}{4\alpha\mu}\frac{\partial}{\partial z}\Phi-\frac{1}{2\alpha}\frac{\partial}{\partial r}\Psi\right]=0,$$

$$\frac{1}{2\gamma(2\gamma+3\beta)}\left[4\gamma\Psi-(2\gamma+\beta)\frac{\partial}{\partial z}\Omega-2(\gamma+\beta)\frac{\partial}{\partial z}\Delta+\beta\frac{1}{r}\frac{\partial}{\partial r}\chi\right] +\frac{1}{r}\left[\frac{\alpha+\mu}{4\alpha\mu}\frac{\partial}{\partial z}\Phi-\frac{1}{2\alpha}\frac{\partial}{\partial r}\Psi\right]=0,$$

(3.13)
$$\frac{\partial}{\partial z}\left[\frac{\alpha+\mu}{4\alpha\mu}\frac{\partial}{\partial r}\Phi-\frac{\alpha-\mu}{4\alpha\mu}\frac{1}{r}\Phi+\frac{1}{2\alpha}\frac{\partial}{\partial z}\Psi\right]+\frac{1}{r}\left[\frac{\alpha-\mu}{4\alpha\mu}\frac{\partial}{\partial z}\Phi+\frac{1}{2\alpha}\frac{\partial}{\partial r}\Psi\right] -\frac{1}{2\gamma(2\gamma+3\beta)}\left[8\gamma\Psi-(2\gamma-\beta)\frac{\partial}{\partial z}\Omega-(2\gamma+\beta)\left(\frac{\partial}{\partial z}\Delta+\frac{1}{r}\frac{\partial}{\partial r}\chi\right)\right]=0,$$

$$\frac{\gamma+\varepsilon}{4\gamma\varepsilon}\left(\frac{\partial}{\partial r}\Omega-\frac{1}{r}\Delta\right)-\frac{\gamma-\varepsilon}{4\gamma\varepsilon}\frac{1}{r}\frac{\partial}{\partial z}\chi-\frac{1}{2\varepsilon}\Phi +\frac{\partial}{\partial z}\left[\frac{\alpha+\mu}{4\mu\alpha}\frac{\partial}{\partial z}\Phi-\frac{1}{2\alpha}\frac{\partial}{\partial r}\Psi\right]=0,$$

$$\frac{1}{2\mu}\left(\frac{\partial}{\partial r}\Phi-\frac{1}{r}\Phi\right)+r\frac{\partial}{\partial r}\left[\frac{\alpha+\mu}{4\mu\alpha}\frac{\partial}{\partial r}\Phi-\frac{\alpha-\mu}{4\alpha\mu}\frac{1}{r}\Phi+\frac{1}{2\alpha}\frac{\partial}{\partial z}\Psi\right] -r\left[\frac{1}{2\varepsilon}\Phi+\frac{\gamma+\varepsilon}{4\gamma\varepsilon}\frac{1}{r}\frac{\partial}{\partial z}\chi-\frac{\gamma-\varepsilon}{4\gamma\varepsilon}\left(\frac{\partial}{\partial r}\Omega-\frac{1}{r}\Delta\right)\right]=0.$$

The solution of the problem formulated in terms of stresses (Sec. 2) is reduced now to the determination from the Eqs. (3.13) of five stress functions which should satisfy the given stress boundary conditions.

4. Further equations for stress functions

The seperate higher order differential equations describing the stress functions may be determined from Eqs. (3.13). For simplicity of transformations let us replace the functions χ and Δ by the functions $\chi^*(r, z)$ and $\Delta^*(r, z)$, respectively, using the formulae

(4.1)
$$\chi = r\chi^*, \quad \Delta = r\frac{\partial}{\partial r}\Delta^*.$$

Introducing the Eqs. (4.1) into the system of the Eqs. (3.13) we obtain the following fourth-order differential equations for the functions Φ and Ψ

(4.2)
$$\nabla_0^2 D_0\Phi = 0, \quad \nabla^2 H\Psi = 0$$

and the eight-order equations for the function χ^* and the combination $(\Omega-\Delta^*)$ respectively

$$\nabla_0^2\nabla_0^2 H_0 D_0 \chi^* = 0, \quad \nabla^2\nabla^2 HD(\Omega-\Delta^*) = 0. \tag{4.3}$$

The operators introduced ∇^2, ∇_0^2, D, D_0, H, H_0 have a form

$$\nabla^2 = \partial^2/\partial r^2 + 1/r\partial/\partial r + \partial^2/\partial z^2, \quad \nabla_0^2 = \nabla^2 - 1/r^2, \quad D = l^2\nabla^2 - 1,$$

$$D_0 = l^2\nabla_0^2 - 1, \quad H = \nu^2\nabla^2 - 1, \quad H_0 = \nu^2\nabla_0^2 - 1.$$

The quantities l^2 and ν^2 denote the following constants

$$l^2 = (\gamma+\varepsilon)(\mu+\alpha)/4\mu\alpha, \quad \nu^2 = (2\gamma+\beta)/4\alpha.$$

Having the solution of the Eqs. (4.2) and (4.3), the expression $(\Omega-\Delta^*)$ may be separated by means of the formula

$$\frac{\partial}{\partial z}\Delta^* = -\frac{1}{r}\left[\frac{1}{\alpha_0}\frac{\partial}{\partial z}\Phi - \frac{\gamma}{\alpha}\frac{\partial}{\partial r}\Psi\right], \tag{4.4}$$

where

$$\alpha_0 = 2\alpha\mu/\gamma(\mu+\alpha).$$

5. Transition from the stress functions to the displacement functions

We shall show in this Section that the solution of the second axisymmetric problem may be reduced to the solution of two equations for two displacement functions $\Phi^*(r, z)$ and $\Psi^*(r, z)$ obtained from the stress functions discussed above. For this purpose let us substitute the Eq. (4.1) ihto the Eqs. (3.13) and treat the equations obtained as a system of equations for the unknowns

$$\frac{\partial}{\partial z}\chi^*, \frac{\partial}{\partial r}(\Omega-\Delta^*), \frac{1}{r}\frac{\partial}{\partial r}(r\chi^*), \frac{\partial}{\partial z}\Omega, r\frac{\partial^2}{\partial r\,\partial z}\Delta^*.$$

These unknowns we express in terms of the derivatives of the functions Φ and Ψ which, to distinguish them in further analysis, are denoted now by Φ^* and Ψ^*:

$$\begin{aligned}
&\frac{\partial}{\partial z}\chi^* = \frac{\gamma}{\alpha}\frac{\partial^2}{\partial r\,\partial z}\Psi^* - \frac{1}{\alpha_0}\frac{\partial^2}{\partial z^2}\Phi^* + (l^2\nabla_0^2 - 1)\Phi^*,\\
&\frac{\partial}{\partial r}(\Omega-\Delta^*) = \frac{\gamma}{\alpha}\frac{\partial^2}{\partial r\,\partial z}\Psi^* - \frac{1}{\alpha_0}\frac{\partial^2}{\partial z^2}\Phi^* + \left(\frac{\gamma-\varepsilon}{\gamma+\varepsilon}l^2\nabla_0^2 + 1\right)\Phi^*,\\
&\frac{1}{r}\frac{\partial}{\partial r}(r\chi^*) = -\frac{\beta}{2\alpha}\nabla^2\Psi^* - \frac{\gamma}{\alpha}\frac{\partial^2}{\partial z^2}\Psi^* + 2\Psi^* - \frac{1}{\alpha_0}\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\Phi^*)\right],\\
&\frac{\partial}{\partial z}\Omega = -\frac{\beta}{2\alpha}\nabla^2\Psi^* - \frac{\gamma}{\alpha}\frac{\partial^2}{\partial r^2}\Psi^* + 2\Psi^* + \frac{1}{\alpha_0}\frac{\partial^2}{\partial r\,\partial z}\Phi^*,\\
&r\frac{\partial^2}{\partial z\,\partial r}\Delta^* = \frac{\gamma}{\alpha}\left(\frac{\partial^2}{\partial r^2}\Psi^* - \frac{1}{r}\frac{\partial}{\partial r}\Psi^*\right) + \frac{1}{\alpha_0}\frac{\partial}{\partial z}\left(\frac{1}{r}\Phi^* - \frac{\partial}{\partial r}\Phi^*\right).
\end{aligned} \tag{5.1}$$

Substituting the relations (5.1) into the formulae [obtained from the Eqs. (3.11), (3.12) and (4.1)]

$$\text{(5.2)} \quad \mu_{rr} = 2\Psi^* - \frac{\partial}{\partial z}\Omega, \quad \mu_{\theta\theta} = 2\Psi^* - \frac{\partial}{\partial z}\Omega - r\frac{\partial^2}{\partial r\,\partial z}\Delta^*, \quad \mu_{zz} = 2\Psi^* - \frac{1}{r}\frac{\partial}{\partial r}(r\chi^*),$$

$$\text{(5.3)} \qquad \mu_{zr} = -\Phi^* + \frac{\partial}{\partial r}(\Omega - \Delta^*), \quad \mu_{rz} = \Phi^* + \frac{\partial}{\partial z}\chi^*,$$

we obtain that the couple stresses become dependent only on two functions Φ^* and Ψ^*. The formulae (3.3) and (4.4) determining the stresses after a change of notations take the form

$$\text{(5.4)} \qquad \sigma_{\theta r} = \frac{\partial}{\partial r}\Phi^* + \frac{\partial}{\partial z}\Psi^*, \quad \sigma_{\theta z} = \frac{\partial}{\partial z}\Phi^* - \frac{\partial}{\partial r}\Psi^*,$$

$$\text{(5.5)} \qquad \sigma_{r\theta} = -\frac{1}{r}\Phi^* - \frac{\partial}{\partial z}\Psi^*, \quad \sigma_{z\theta} = \frac{\partial}{\partial r}\Psi^*.$$

If the stresses and couple stresses expressed now in terms of the functions Φ^* and Ψ^* are substituted into the strain compatibility equations (2.23), (2.24) expressed in terms of stresses, these equations, as one can easily check, are satisfied identically. (One can also verify that the Eqs. (2.11)–(2.15) are satisfied identically). The third equilibrium equation $(2.2)_3$ is also satisfied identically. But the first two equations of the set of the Eqs. (2.2) impose on the functions Φ^* and Ψ^* the differential conditions

$$\text{(5.6)} \qquad 2(\nu^2\nabla_0^2 - 1)\frac{\partial}{\partial r}\Psi^* = \frac{\partial}{\partial z}(l^2\nabla_0^2 - 1)\Phi^*,$$

$$\text{(5.7)} \qquad 2(\nu^2\nabla^2 - 1)\frac{\partial}{\partial z}\Psi^* = -\left(\frac{1}{r} + \frac{\partial}{\partial r}\right)(l^2\nabla_0^2 - 1)\Phi^*.$$

From these relations the following separate equations for the functions Φ^* and Ψ^* [identical as the Eqs. (4.2)] may be determined

$$\text{(5.8)} \qquad \nabla_0^2(l^2\nabla_0^2 - 1)\Phi^* = 0, \quad \nabla^2(\nu^2\nabla^2 - 1)\Psi^* = 0.$$

Let us express now the displacement u_θ and the rotations φ_r, φ_z in terms of the functions Φ^* and Ψ^*. Such representation is obtained by means of integration of the equations relating the strains and the displacement and rotations (2.4), (2.5) and by use at the same time of the constitutive equations (2.20)–(2.22) and formulae (5.1)–(5.5)

$$\text{(5.9)} \qquad \begin{aligned} \varphi_r &= \frac{1}{2\alpha}\frac{\partial}{\partial r}\Psi^* - \frac{\mu+\alpha}{4\mu\alpha}\frac{\partial}{\partial z}\Phi^*, \\ \varphi_z &= \frac{1}{2\alpha}\frac{\partial}{\partial z}\Psi^* + \frac{\mu+\alpha}{4\mu\alpha}\frac{1}{r}\frac{\partial}{\partial r}(r\Phi^*), \\ u_\theta &= \frac{1}{2\mu}\Phi^*. \end{aligned}$$

The functions Φ^* and Ψ^* play now a role of the displacement functions. Assuming the representation (5.9) as a first step to the solution of the problem and substituting into the basic homogeneous equations of the field of displacement and rotations (5.10) $\left(\text{here } \varkappa = \frac{1}{r}\frac{\partial}{\partial r}(r\varphi_r)+\frac{\partial}{\partial z}\varphi_z\right)$, we satisfy the third equation identically, but the first two Eqs. (5.10) impose on the functions Φ^* and Ψ^* the already known differential equations (5.6) and (5.7).

The problem of determining of all the interesting us fields in a body was reduced to the knowledge of the functions Φ^* and ψ^* which are determined from the Eqs. (5.6) and (5.7) at given boundary conditions

$$[(\gamma+\varepsilon)(\nabla^2-1/r^2)-4\alpha]\varphi_r+(\beta+\gamma-\varepsilon)\frac{\partial}{\partial r}\varkappa-2\alpha\frac{\partial}{\partial z}u_\theta = 0,$$

$$(5.10) \qquad [(\gamma+\varepsilon)\nabla^2-4\alpha]\varphi_z+(\beta+\gamma-\varepsilon)\frac{\partial}{\partial z}\varkappa+2\alpha\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) = 0,$$

$$(\mu+\alpha)(\nabla^2-1/r^2)u_\theta+2\alpha\left(\frac{\partial}{\partial z}\varphi_r-\frac{\partial}{\partial r}\varphi_z\right) = 0.$$

6. Relations between the functions Φ^*, Ψ^* and displacement functions determined in [4]

W. NOWACKI in [4] elaborated the displacement method for the solution of the "second" axisymmetric problem by introducing two solving functions Φ, Ψ (elastic potentials) which are connected with the displacement u_θ and rotations φ_r, φ_z by formulae

$$\varphi_r = \frac{\partial}{\partial r}\Phi+\frac{\partial^2}{\partial r\,\partial z}\Psi,$$

$$(6.1) \qquad \varphi_z = \frac{\partial}{\partial z}\Phi-\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)\Psi,$$

$$u_\theta = -\frac{2\alpha}{\mu+\alpha}\frac{\partial}{\partial r}\Psi.$$

The representation (6.1) substituted into the basic equations expressed in terms of displacements and rotations (5.10) leads to the simple differential equations for the functions Φ and Ψ

$$(6.2) \qquad \nabla^2(l^2\nabla^2-1)\Psi = 0, \quad \nabla^2(\nu^2\nabla^2-1)\Phi = 0,$$

where Φ and Ψ are related by the equations

$$(6.3) \qquad (\nu^2\nabla^2-1)\Phi = -\frac{\mu}{\mu+\alpha}\frac{\partial}{\partial z}(l^2\nabla^2-1)\Psi,$$

$$(6.4) \qquad (\nu^2\nabla^2-1)\frac{\partial}{\partial z}\Phi = \frac{\mu}{\mu+\alpha}\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)(l^2\nabla^2-1)\Psi.$$

Determining in a body of all interesting us fields leads thus to the determination, from the Eqs. (6.2), of the functions Φ and Ψ satisfying the relations (6.3), (6.4) and given conditions on the boundary of a body.

The solution (6.1)–(6.4) may be obtained from the Eqs. (5.9) if the following relations between functions Φ^*, Ψ^* and functions Φ, Ψ are assumed:

$$\Phi^* = -\frac{4\alpha\mu}{\mu+\alpha}\frac{\partial}{\partial r}\Psi, \quad \Psi^* = 2\alpha\Phi. \tag{6.5}$$

7. Particular case ($\alpha = 0$)

The "second" axisymmetric problem in the case of $\alpha = 0$ leads to two problems:

7.1. Hipotetic medium — which has such properties that only rotations φ_r, φ_z and couple stresses μ_{rr}, $\mu_{\theta\theta}$, μ_{zz}, $\mu_{zr} \neq \mu_{rz}$ can exist in it.

The strain field is described by $\varkappa_{rr}$, $\varkappa_{\theta\theta}$, $\varkappa_{zz}$, $\varkappa_{zr} \neq \varkappa_{rz}$. The basic equations are formed from geometrical relations (2.5), constitutive equations (2.8)–(2.10), strain compatibility equations (2.11), (2.13) and eqilibrium equations

$$\begin{aligned} &\frac{\partial}{\partial r}\mu_{rr}+\frac{\partial}{\partial z}\mu_{zr}+\frac{\mu_{rr}-\mu_{\theta\theta}}{r} = 0,\\ &\frac{\partial}{\partial r}\mu_{rz}+\frac{\partial}{\partial z}\mu_{zz}+\frac{\mu_{rz}}{r} = 0. \end{aligned} \tag{7.1}$$

The compatibility equations (2.11), (2.13) together with constitutive equations (2.8)–(2.10) and equilibrium equations (7.1) may be presented in a form of five second-order partial differential equations (similar to Beltrami-Mitchell equations of classical elasticity):

$$\begin{aligned} &\nabla^2\mu_{zz}+k\frac{\partial^2}{\partial z^2}\Theta = 0,\\ &\nabla^2\mu_{\theta\theta}+\frac{2}{r^2}(\mu_{rr}-\mu_{\theta\theta})+k\frac{1}{r}\frac{\partial}{\partial r}\Theta = 0,\\ &\nabla^2\mu_{rr}-\frac{2}{r^2}(\mu_{rr}-\mu_{\theta\theta})+k\frac{\partial^2}{\partial r^2}\Theta = 0,\\ &\nabla^2\mu_{zr}-\frac{1}{r^2}\mu_{zr}+k\frac{\partial^2}{\partial r\partial z}\Theta = 0,\\ &\nabla^2\mu_{rz}-\frac{1}{r^2}\mu_{rz}+k\frac{\partial^2}{\partial r\partial z}\Theta = 0, \end{aligned} \tag{7.2}$$

where

$$k = \frac{2\gamma(\beta+\gamma-\varepsilon)}{(\gamma+\varepsilon)(2\gamma+3\beta)}, \quad \Theta = \mu_{rr}+\mu_{\theta\theta}+\mu_{zz}.$$

The Eqs. (7.2) together with equilibrium equations (7.1) and given boundary conditions expressed in terms of stresses constitute the basic equations of the stress field in a simply connected body. From the Eq. (7.2) the following separate equations for the components μ_{zz}, μ_{rz}, μ_{zr} and for the expression $(\mu_{rr}+\mu_{\theta\theta})$ may be obtained

$$\nabla^2\nabla^2\mu_{zz} = 0, \quad \nabla_0^2\nabla_0^2(\mu_{rz}, \mu_{zr}) = 0, \quad \nabla^2\nabla^2(\mu_{rr}+\mu_{\theta\theta}) = 0 \tag{7.3}$$

and

$$\nabla^2 \Theta = 0, \quad \nabla_0^2 (\mu_{rz} - \mu_{zr}) = 0. \tag{7.4}$$

Let us consider now the basic equations of the field of rotations. On the basis of the Eq. (5.10) they have a form

$$\begin{aligned} &(\gamma+\varepsilon)(\nabla^2 - 1/r^2)\varphi_r + (\beta+\gamma-\varepsilon)\frac{\partial}{\partial r}\varkappa = 0, \\ &(\gamma+\varepsilon)\nabla^2 \varphi_z + (\beta+\gamma-\varepsilon)\frac{\partial}{\partial z}\varkappa = 0. \end{aligned} \tag{7.5}$$

After separation of the system (7.5) we obtain

$$\nabla^2 \varkappa = 0, \quad \nabla^2\nabla^2 \varphi_z = 0, \quad \nabla_0^2 \nabla_0^2 \varphi_r = 0. \tag{7.6}$$

The solution of the problem expressed in terms of stresses or rotations with given boundary conditions may be reduced to the solution of the biharmonic equation for the function $\varDelta_0(r, z)$

$$\nabla^2\nabla^2 \varDelta_0 = 0, \tag{7.7}$$

where the function $\varDelta_0$ was introduced fully analogically as the Love function in the classical theory of elasticity (compare W. NOWACKI [14], p. 170). Thus the following representation is obtained

$$\varphi_r = -\frac{\partial^2}{\partial r \partial z}\varDelta_0, \quad \varphi_z = -\frac{\partial^2}{\partial z^2}\varDelta_0 + \frac{\beta+2\gamma}{\beta+\gamma-\varepsilon}\nabla^2 \varDelta_0. \tag{7.8}$$

Substituting formulae (7.8) to the equations of the field of rotations or the stresses expressed in terms of $\varDelta_0$ into the stress equations (7.1), (7.2) we notice that these equations can be satisfied identically if the function $\varDelta_0$ satisfies the Eq. (7.7). The representation (7.8) may also be obtained from three stress functions determined by the formulae (3.9), (3.10).

7.2. Classical axisymmetric problem — represented by the displacement vector $\mathbf{u} \equiv (0, u_\theta, 0)$ (compare I. SNEDDON [15], p. 558)

The fields of strains and stresses are represented respectively by the components

$$\gamma_{z\theta} = \gamma_{\theta z}, \quad \gamma_{r\theta} = \gamma_{\theta r}; \quad \sigma_{z\theta} = \sigma_{\theta z}, \quad \sigma_{r\theta} = \sigma_{\theta r}.$$

Geometrical equations, constitutive equations, equations of geometrical compatibility for strains are obtained from the formulae (2.4), (2.6), (2.7) and (2.16), respectively.

The displacement problem is described by the equation

$$(\nabla^2 - 1/r^2) u_\theta = 0. \tag{7.9}$$

In a stress formulation, besides the equilibrium equation

$$\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{z\theta} + \frac{2}{r}\sigma_{r\theta} = 0, \tag{7.10}$$

two Beltrami-Mitchell equations of a form

$$\nabla^2 \sigma_{z\theta} - \frac{1}{r^2}\sigma_{z\theta} = 0, \quad \nabla^2 \sigma_{r\theta} - \frac{4}{r^2}\sigma_{r\theta} = 0 \tag{7.11}$$

are obtained from the Eqs. (2.6), (2.7), (2.16) and (7.10) (compare A. I. ŁURIE [16], pp. 39 and 58).

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